Dynamical spin correlations in the Heisenberg ladder under a magnetic field and correlation functions in the SO(5) ladder

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The zero-temperature dynamical spin-spin correlation functions are calculated for the spin-$\frac{1}{2}$ two-leg antiferromagnetic Heisenberg ladder in a magnetic field above the lower critical field H_{c1}. The dynamical structure factors are calculated, which exhibit both massless modes and massive excitations. These modes appear in different sectors characterized by the parity in the rung direction and by the momentum in the direction of the chains. The structure factors have power-law singularities at the lower edges of their support. The results are also applicable to the spin-1 Heisenberg chain. The implications are briefly discussed for the various correlation functions and the π resonance in the SO(5) symmetric ladder model. [S0163-1829(99)14025-6]

I. INTRODUCTION

Systems located in between one dimension and two dimensions, ladders, have attracted great attention both theoretically and experimentally.[1] This line of research has been pursued both with the hope of finding some clues to the understanding of high-temperature superconductivity, and because of the experimental discoveries of new ladder compounds, such as a cuprate spin ladder[2] $SrCu_2O_3$ and a superconducting ladder[3] $Sr_xCa_{14-x}Cu_{24}O_{41}$. Another spin ladder system of current interest is the organic material $Cu_2(C_5H_{12}N_2)_2Cl_4$ whose spin-gap behavior was observed by the measurements of the spin susceptibility, the magnetization curve, and the NMR relaxation rate.[4,5] The nonzero magnetization is observed once the external magnetic field H exceeds the lower critical field H_{c1}, which is equal to the spin gap (in units where the product of the Bohr magneton and the g factor is set to unity). This is a quantum phase transition driven by the external magnetic field from the gapped spin liquid to the gapless Tomonaga-Luttinger liquid state. These experiments motivated further theoretical works on the various properties of the gapless regime: the magnetization process,[6,7] the spin-spin correlation functions,[8,9] and the spin-Peierls instability.[10,11]

In this paper we extend the theory of Shelton *et al.*[12] to the two-leg Heisenberg ladder in the gapless regime ($H > H_{c1}$) and calculate the dynamical spin-spin correlation functions and structure factors in the ground state.[13] We obtain the dynamical structure factors containing massive spin excitations as well as massless excitations. The latter contribution is characteristic of the Tomonaga-Luttinger liquid and is commonly found in the $S=\frac{1}{2}$ Heisenberg chain.[14,15] This component may be interpreted as coming from the Bose condensate of $S^z=1$ magnons.[13] On the other hand, the massive excitations we are interested in have $S^z=0$ magnons in their origin. Below H_{c1} the massive magnons give rise to δ-function peaks in the dynamical structure factor. In the gapless regime ($H>H_{c1}$) the δ-function peaks of the massive excitations turn into power-law singularities, because the massless excitations introduce algebraically decaying prefactors to the exponentially decaying correlation functions. An interesting feature of the antiferromagnetically coupled Heisenberg ladder is that these massless and massive modes appear in different sectors of the structure factors, characterized by the momentum q_y in the direction of the rungs (q_y can take only 0 and π) and by the momentum q in the direction of the chains.

Our study of the massive excitations in the gapless regime is also motivated to better understand the π-resonance mode[16] in the SO(5) symmetric ladder model introduced recently by Scalapino *et al.*[17] In this model there is a quantum phase transition driven by the chemical potential from the spin-gap Mott insulator to the d-wave-like superconducting phase. It was shown in the strong-coupling limit that the spin-gap magnon mode of the Mott insulator evolves continuously into the π-resonance mode of the superconducting phase.[17] A complementary weak-coupling approach[18–20] was taken to study a general two-leg ladder model of weakly interacting electrons. In particular, Lin *et al.*[20] showed that the model is renormalized to a fixed point where a global SO(8) symmetry is realized that contains the SO(5). They also obtained the ground-state phase diagram and low-energy excitation spectra at half filling which are qualitatively in agreement with the strong-coupling picture of the SO(5) symmetric ladder model. Yet the energy and the spectral weight of the π-resonance mode in the dynamical spin structure factor are not completely understood even for the ideally constructed SO(5) symmetric ladder. An extensive numerical exact-diagonalization study is performed to clarify this and related issues,[21] but clearly it is desirable to develop an analytic theory to discuss the power-law singularity which is expected to appear in the structure factor.[17] The field-driven quantum phase transition in the Heisenberg ladder we study in this paper is analogous to the quantum phase transition in the SO(5) symmetric ladder. A $S^z=1$ boson in the Heisenberg ladder corresponds to a hole pair in the latter model, and a $S^z=0$ boson to a magnon or the π-resonance mode. Therefore our Heisenberg spin ladder may be viewed as a toy model for the SO(5) symmetric ladder. Using the analogy, we can deduce the correlation functions for the d-wave superconductivity, the charge density wave, and the spin correlations in the SO(5) ladder from the spin-spin correlators of the Heisenberg ladder.

This paper is organized as follows. In Sec. II we introduce the model and analyze it in the limit of strong interchain coupling, where it is easier to get intuitive pictures of the physics. Detailed calculation of the dynamical spin-spin correlation functions and the structure factors are presented in Sec. III. Here we take the opposite limit of the weak interchain coupling, and employ the bosonization method. In Sec. IV we briefly discuss implications to the π resonance in the SO(5) symmetric ladder model and summarize the results.

II. STRONG-COUPLING APPROACH

The Hamiltonian for the two-leg Heisenberg ladder we study in this paper is

$$\mathcal{H}=J\sum_{\mu=1,2}\sum_{i=-\infty}^{\infty}\boldsymbol{S}_{\mu,i}\cdot\boldsymbol{S}_{\mu,i+1}+J_{\perp}\sum_{i=-\infty}^{\infty}\boldsymbol{S}_{1,i}\cdot\boldsymbol{S}_{2,i}-H\sum_{\mu=1,2}\sum_{i=-\infty}^{\infty}S^{z}_{\mu,i}, \tag{1}$$

where $\boldsymbol{S}_{\mu,i}$ is spin-$\frac{1}{2}$ operator, and the intrachain coupling J is positive (antiferromagnetic). The interchain coupling $J_{\perp}$ is also assumed to be positive, unless otherwise noted.

In this section we briefly discuss static correlations above the lower critical field H_{c1} in the strong-coupling limit, $J_{\perp}\gg J$. When $J/J_{\perp}=0$, the ladder is decomposed into independent rungs, each rung consisting of two spins $\boldsymbol{S}_{1,i}$ and $\boldsymbol{S}_{2,i}$. There are four eigenstates in each rung, the singlet $|S_i\rangle=(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)/\sqrt{2}$, and the triplet states $|T_{i,+}\rangle=|\uparrow\uparrow\rangle$, $|T_{i,0}\rangle=(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)/\sqrt{2}$, and $|T_{i-}\rangle=|\downarrow\downarrow\rangle$. Their energies are $E(S_i)=-3J_{\perp}/4$, $E(T_{i,+})=J_{\perp}/4-H$, $E(T_{i,0})=J_{\perp}/4$, and $E(T_{i,-})=J_{\perp}/4+H$, respectively. It is then natural to introduce boson operators $s_i^{\dagger}$, $t_{i,+}^{\dagger}$, $t_{i,0}^{\dagger}$, and $t_{i,-}^{\dagger}$, which create the singlet state and the triplet states at the ith rung.[22,23,9] They obey the constraint, $s_i^{\dagger}s_i+t_{i,+}^{\dagger}t_{i,+}+t_{i,0}^{\dagger}t_{i,0}+t_{i,-}^{\dagger}t_{i,-}=1$. With these hard-core bosons we represent the spin operators

$$S^{z}_{0,i}\equiv S^{z}_{1,i}+S^{z}_{2,i}=t^{\dagger}_{i,+}t_{i,+}-t^{\dagger}_{i,-}t_{i,-}, \tag{2a}$$

$$S^{z}_{\pi,i}\equiv S^{z}_{1,i}-S^{z}_{2,i}=t^{\dagger}_{i,0}s_i+s^{\dagger}_i t_{i,0}, \tag{2b}$$

$$S^{+}_{0,i}\equiv S^{+}_{1,i}+S^{+}_{2,i}=\sqrt{2}(t^{\dagger}_{i,+}t_{i,0}+t^{\dagger}_{i,0}t_{i,-}), \tag{2c}$$

$$S^{+}_{\pi,i}\equiv S^{+}_{1,i}-S^{+}_{2,i}=\sqrt{2}(-t^{\dagger}_{i,+}s_i+s^{\dagger}_i t_{i,-}), \tag{2d}$$

$$S^{-}_{0,i}\equiv S^{-}_{1,i}+S^{-}_{2,i}=\sqrt{2}(t^{\dagger}_{i,-}t_{i,0}+t^{\dagger}_{i,0}t_{i,+}), \tag{2e}$$

$$S^{-}_{\pi,i}\equiv S^{-}_{1,i}-S^{-}_{2,i}=\sqrt{2}(t^{\dagger}_{i,-}s_i-s^{\dagger}_i t_{i,+}). \tag{2f}$$

The suffixes 0 and π refer to the momentum q_y. Note that $S^{z}_{0,i}$ and $S^{\pm}_{\pi,i}$ do not contain the $t_{i,0}$ boson.

In the presence of a high magnetic field $H\gtrsim J_{\perp}$, we may ignore the $t_{i,0}$ and $t_{i,-}$ bosons as they have higher energies than the $t_{i,+}$ boson. In lowest order in J the effective Hamiltonian thus becomes

$$\mathcal{H}_{\rm eff}=\frac{J}{2}\sum_i\left(t^{\dagger}_{i,+}s_i s^{\dagger}_{i+1}t_{i+1,+}+t^{\dagger}_{i+1,+}s_{i+1}s^{\dagger}_i t_{i,+}+t^{\dagger}_{i,+}t_{i,+}t^{\dagger}_{i+1,+}t_{i+1,+}\right)+(J_{\perp}-H)\sum_i t^{\dagger}_{i,+}t_{i,+}, \tag{3}$$

with the simplified constraint $s_i^{\dagger}s_i+t_{i,+}^{\dagger}t_{i,+}=1$. As pointed out in Ref. 24, this effective Hamiltonian may be written as

$$\mathcal{H}_{\rm eff}=J\sum_i\left(\tilde{S}^{x}_i\tilde{S}^{x}_{i+1}+\tilde{S}^{y}_i\tilde{S}^{y}_{i+1}+\frac{1}{2}\tilde{S}^{z}_i\tilde{S}^{z}_{i+1}\right)+\left(J_{\perp}+\frac{J}{2}-H\right)\sum_i\tilde{S}^{z}_i+\text{const}, \tag{4}$$

where $\tilde{S}_i$ is a spin-$\frac{1}{2}$ operator

$$\tilde{S}^{z}_i=t^{\dagger}_{i,+}t_{i,+}-\tfrac{1}{2}, \tag{5a}$$

$$\tilde{S}^{+}_i=t^{\dagger}_{i,+}s_i, \tag{5b}$$

$$\tilde{S}^{-}_i=s^{\dagger}_i t_{i,+}. \tag{5c}$$

We notice that Eq. (4) is just the Hamiltonian of the spin-$\frac{1}{2}$ XXZ chain with the external magnetic field $\tilde{H}=J_{\perp}+J/2-H$, whose properties are well understood. When $|\tilde{H}|<\frac{3}{2}J$, the XXZ chain is not fully polarized.[25] This means that the ladder has unsaturated magnetization $\langle S^{z}_{0,i}\rangle=\langle t^{\dagger}_{i,+}t_{i,+}\rangle=M$ $(0<M<1)$ when $J_{\perp}-J<H<J_{\perp}+2J$. We have thus obtained the lower and upper critical fields $H_{c1}=J_{\perp}-J$ and $H_{c2}=J_{\perp}+2J$ in lowest order in $J/J_{\perp}$.[26] The magnetization curve for $M\ll 1$ can also be obtained from the known result for the XXZ chain:[25]

$$M=\frac{1}{2}+\langle\tilde{S}^{z}_i\rangle=\sqrt{\frac{2(H-H_{c1})}{\pi^2 J}}. \tag{6}$$

A similar result holds for $1-M\ll 1$. The square-root behavior is a well-known universal behavior.[27–30,8] When $0<M<1$ the ground state is a superfluid of the $t_{i,+}$ bosons. Within our approximation the following equal-time spin correlation functions are readily obtained from those of the XXZ chain:[31]

$$\langle S^{z}_{0,n+1}S^{z}_{0,1}\rangle=\langle(\tilde{S}^{z}_{n+1}+1/2)(\tilde{S}^{z}_1+1/2)\rangle=M^2-\frac{c_1}{n^2}+\frac{c_2}{n^{\eta}}\cos(2\pi Mn), \tag{7a}$$

$$\langle S^x_{\pi,n+1}S^x_{\pi,1}\rangle = 2\langle \tilde{S}^x_{n+1}\tilde{S}^x_1\rangle = c_3\frac{(-1)^n}{n^{1/\eta}} - c_4\frac{(-1)^n}{n^{\eta+1/\eta}}\cos(2\pi Mn), \quad (7b)$$

where c's are numerical constants. For $M \ll 1$ the parameter η is given by[31]

$$\eta = 2 - \frac{4}{3}M + \mathcal{O}(M^2). \quad (8)$$

In general $\eta < 2$ for $0 < M < 1$, and $\eta \to 2$ when $M \to 0,1$.[32] Thus, the leading term in $\langle S^z_{0,n+1}S^z_{0,1}\rangle$, besides the constant M^2, is $n^{-\eta}\cos(2\pi Mn)$, which reflects the fact that the hard-core bosons $t_{i,+}$ have a tendency to form a density wave with a period equal to $1/M$. On the other hand, the correlation functions $\langle S^z_{\pi,i}S^z_{\pi,j}\rangle$ and $\langle S^x_{0,i}S^x_{0,j}\rangle$ involve the massive $t_{i,0}$ and $t_{i,-}$ bosons, and therefore decay exponentially for $|i-j| \gg 1$. We conclude that the correlators $\langle S^z_0 S^z_0\rangle$ and $\langle S^{\pm}_{\pi}S^{\mp}_{\pi}\rangle$ show quasi-long-range order while $\langle S^z_{\pi}S^z_{\pi}\rangle$ and $\langle S^{\pm}_0 S^{\mp}_0\rangle$ are short ranged. These correlation functions are discussed in more detail in the next section.

Before closing this section, we make a few comments. First, Eqs. (7a) and (7b) have the same form as the correlators $\langle S^z_{n+1}S^z_1\rangle$ and $\langle S^x_{n+1}S^x_1\rangle$ of the $S=1$ Heisenberg chain in a magnetic field larger than the Haldane gap.[28,29,33] An important difference is that $\eta \geqslant 2$ in the $S=1$ chain.[33] We will come back to this point in the next section. Second, from the knowledge of the exact propagator of hard-core bosons,[34,35] we may expect that the correlation function $\langle S^x_{\pi,n+1}S^x_{\pi,1}\rangle$ should have terms proportional to $\cos(2\pi lMn)$ with $l = 0,1,2,\ldots$. Third, as noted in the Introduction and to be discussed in Sec. IV, the $t_{i,+}$ boson and $t_{i,0}$ boson correspond to a pair of holes sitting on a rung and to a magnon in the SO(5) symmetric ladder, respectively. Thus, we may compare the hole-pair correlation and the spin correlation functions with $\langle S^x_{\pi,i}S^x_{\pi,j}\rangle$ and $\langle S^z_{\pi,i}S^z_{\pi,j}\rangle$ of our model.

III. WEAK-COUPLING APPROACH

In this section we calculate the dynamical spin-spin correlation functions in the weak-coupling limit, $J_\perp \ll J$. We use the Abelian bosonization method which has been successfully applied to the Heisenberg ladder.[36,37,12,8] In this approach we first bosonize two independent spin-$\frac{1}{2}$ Heisenberg chains and then treat the interchain coupling $J_\perp$ perturbatively. It is a relevant perturbation and is renormalized to a strong-coupling regime, generating a mass gap in the excitation spectrum. This behavior does not depend on the sign of $J_\perp$, and therefore the model describes both the antiferromagnetic Heisenberg ladder for $J_\perp > 0$ and the $S=1$ Heisenberg chain for $J_\perp < 0$.[12] The spin gap in the latter case is nothing but the Haldane gap.[38] Since the gapless phase in a magnetic field is a single phase for $0 < J_\perp < \infty$, we expect the correlation functions in the weak-coupling limit to have the same structure as in the strong-coupling limit.

We follow the formulation of Shelton *et al.*,[12] which we explain below to establish the notation. We begin with the bosonized Hamiltonian of the ladder in the continuum limit. It consists of three parts: $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- + \mathcal{H}_\perp$, where

$$\mathcal{H}_+ = \int dx \left\{ \frac{v}{2}\left[\left(\frac{d\phi_+}{dx}\right)^2 + \left(\frac{d\theta_+}{dx}\right)^2 \right] - \frac{m}{\pi a_0}\cos(\sqrt{4\pi}\phi_+) - \frac{H}{\sqrt{\pi}}\frac{d\phi_+}{dx} \right\}, \quad (9)$$

$$\mathcal{H}_- = \int dx \left\{ \frac{v}{2}\left[\left(\frac{d\phi_-}{dx}\right)^2 + \left(\frac{d\theta_-}{dx}\right)^2 \right] + \frac{2m}{\pi a_0}\cos(\sqrt{4\pi}\theta_-) + \frac{m}{\pi a_0}\cos(\sqrt{4\pi}\phi_-) \right\}, \quad (10)$$

$$\mathcal{H}_\perp = \frac{J_\perp}{2\pi^2 a_0}\int dx \cos(\sqrt{4\pi}\theta_-)[\cos(\sqrt{4\pi}\phi_+) - \cos(\sqrt{4\pi}\phi_-)] + \frac{J_\perp a_0}{4\pi}\int dx \left[\left(\frac{d\phi_+}{dx}\right)^2 - \left(\frac{d\phi_-}{dx}\right)^2 \right]. \quad (11)$$

Here v is the spin-wave velocity, a_0 is a lattice constant, and $m = J_\perp \lambda^2/2\pi$ with λ being a numerical constant. The bosonic fields $\phi_\pm$ and $\theta_\pm$ obey the commutation relations $[\phi_+(x),\theta_+(y)] = [\phi_-(x),\theta_-(y)] = i\Theta(y-x)$ and $[\phi_+(x),\phi_-(y)] = [\phi_+(x),\theta_-(y)] = [\phi_-(x),\theta_+(y)] = [\theta_+(x),\theta_-(y)] = 0$, where $\Theta(x)$ is the step function. It is important to note that H appears in $\mathcal{H}_+$ only. Thus, the external magnetic field changes the dynamics of the fields ϕ_+ and θ_+, while the other fields ϕ_- and θ_- are not directly influenced by the uniform field H. The excitations involving these latter fields remain gapful even above the lower critical field H_{c1}. The spin operators with $q_y = 0$ and π, defined in Eqs. (2a)–(2f), are written in terms of the bosonic fields as

$$S^z_0(x) = \frac{a_0}{\sqrt{\pi}}\frac{d\phi_+}{dx} - (-1)^{x/a_0}\frac{2\lambda}{\pi}\sin(\sqrt{\pi}\phi_+)\cos(\sqrt{\pi}\phi_-), \quad (12a)$$

$$S^z_\pi(x) = \frac{a_0}{\sqrt{\pi}}\frac{d\phi_-}{dx} - (-1)^{x/a_0}\frac{2\lambda}{\pi}\cos(\sqrt{\pi}\phi_+)\sin(\sqrt{\pi}\phi_-), \quad (12b)$$

$$S^+_0(x) = \frac{2}{\pi}e^{i\sqrt{\pi}\theta_+}[(-1)^{x/a_0}\lambda\cos(\sqrt{\pi}\theta_-) + \cos(\sqrt{\pi}\theta_-)\sin(\sqrt{\pi}\phi_+)\cos(\sqrt{\pi}\phi_-) + i\sin(\sqrt{\pi}\theta_-)\cos(\sqrt{\pi}\phi_+)\sin(\sqrt{\pi}\phi_-)], \quad (12c)$$

$$S_{\pi}^{+}(x)=\frac{2}{\pi}e^{i\sqrt{\pi}\theta_{+}}[i(-1)^{x/a_0}\lambda\sin(\sqrt{\pi}\theta_{-})+\cos(\sqrt{\pi}\theta_{-})\cos(\sqrt{\pi}\phi_{+})\sin(\sqrt{\pi}\phi_{-})+i\sin(\sqrt{\pi}\theta_{-})\sin(\sqrt{\pi}\phi_{+})\cos(\sqrt{\pi}\phi_{-})], \tag{12d}$$

where $S_{q_y}^{\alpha}(x)=S_{q_y,i}^{\alpha}$ for $x=ia_0\,(\alpha=x,y,z)$. Equations (12a)–(12d) directly follow from Eq. (30) and Appendix A of Ref. 12.

Shelton *et al.* showed that $\mathcal{H}_{+}$ and $\mathcal{H}_{-}$ can be greatly simplified by fermionization. The Hamiltonian $\mathcal{H}_{-}$ then becomes that of free massive Majorana fermions $\xi_3(x)$ and $\rho(x)$ having the mass gap m and $-3m$, respectively. This is equivalent to the two-dimensional Ising model above or below the critical temperature. This observation allowed them to obtain the dynamical spin-spin correlation functions[39] from the known results of the Ising model.[40] Physically the ξ_3 fermion describes the $S^z=0$ magnon excitation, whereas the ρ corresponds to a singlet excitation with much higher energy.

On the other hand, the ϕ_{+} and θ_{+} fields represent the $S^z=\pm1$ magnon excitations. As the $S^z=1$ bosons condense above H_{c1}, these bosonic fields have massless excitations. We will first integrate out the massive Majorana fermions and concentrate on the massless bosonic fields. To proceed, here we introduce an approximation for dealing with $\mathcal{H}_{\perp}$. This interaction Hamiltonian has three components. The first component involving only ϕ_{-} and θ_{-}, i.e., $\cos(\sqrt{4\pi}\theta_{-})\cos(\sqrt{4\pi}\phi_{-})$ and $(d\phi_{-}/dx)^2$, has two major effects on the dynamics of $\mathcal{H}_{-}$. One effect is to renormalize the bare mass m to $m\ln(\Lambda/m)$, where Λ is a high-energy cutoff, as noted by Shelton *et al.*[12] This can be absorbed by redefining the mass. The other effect is a strong two-particle collision described by an $\mathcal{S}$ matrix having a superuniversal form, as recently discussed by Damle and Sachdev.[41] Since we are only concerned with processes in which at most one $S^z=0$ magnon is created, this strong scattering effect may be irrelevant for our discussion of the dynamical correlations at zero temperature. The second component is a coupling term, $\cos(\sqrt{4\pi}\theta_{-})\cos(\sqrt{4\pi}\phi_{+})$. When integrating out the θ_{-} field perturbatively, we find that the leading term $\langle\cos(\sqrt{4\pi}\theta_{-})\rangle_{-}\cos(\sqrt{4\pi}\phi_{+})\propto\ln(\Lambda/m)\cos(\sqrt{4\pi}\phi_{+})$ gives the renormalization of the mass of the ϕ_{+} and θ_{+} fields, $m\rightarrow m\ln(\Lambda/m)$, as expected from the SO(3) symmetry, where the average is taken in the ground state of $\mathcal{H}_{-}$. This is again taken care of by redefining the mass. The higher-order terms will generate, through gradient expansions, both irrelevant terms such as $\cos(\sqrt{4\pi}l\phi_{+})$ with $l>1$, which we can safely ignore, and a marginal operator $(\partial_x\phi_{+})^2$, which should be kept. The third component is the term $(d\phi_{+}/dx)^2$ already present in $\mathcal{H}_{\perp}$ and will be kept in the following calculation. Hence we reduce $\mathcal{H}_{\perp}$ to the form

$$\mathcal{H}_{\perp}\approx\frac{J_{\perp}a_0}{4\pi}\int dx\left(\frac{d\phi_{+}}{dx}\right)^2, \tag{13}$$

where the coupling constant $J_{\perp}a_0$ may be modified from its bare value because of the emergence of the operator $(\partial_x\phi_{+})^2$ in the higher-order terms discussed above. We believe, however, that the sign of the coupling constant is not changed by the renormalization.[42] Having made this approximation, we now integrate out the ϕ_{-} and θ_{-} fields to get the spin-spin correlation functions. Within our approximation the fields ϕ_{-} and θ_{-} are independent of ϕ_{+} and θ_{+}, and therefore the correlation functions of ϕ_{-} and θ_{-} are independent of H. We use Eq. (33) of Ref. 12 to represent $\cos(\sqrt{\pi}\phi_{-})$, $\sin(\sqrt{\pi}\phi_{-})$, $\cos(\sqrt{\pi}\theta_{-})$, and $\sin(\sqrt{\pi}\theta_{-})$ in terms of the order and disorder parameters of an Ising model. We then use Eqs. (38) and (39) of Ref. 12 to obtain their correlation functions. The correlation functions $\langle S_0^{\pm}S_0^{\mp}\rangle$ involve $\cos(\sqrt{\pi}\phi_{-})\cos(\sqrt{\pi}\theta_{-})$ and $\sin(\sqrt{\pi}\phi_{-})\sin(\sqrt{\pi}\theta_{-})$. They are equivalent to free massive Majorana fermions, ξ_3, whose correlators are easily obtained. Finally, the correlator $\langle\partial_x\phi_{-}(x)\partial_x\phi_{-}(0)\rangle$ decays exponentially [$\propto e^{-2mr/v}$ with $r=(x^2+v^2\tau^2)^{1/2}$] and is ignored. Hence, we arrive at the following expression of the dynamical spin-spin correlation functions:

$$\langle S_0^z(x,\tau)S_0^z(0,0)\rangle=\frac{a_0^2}{\pi}\langle\partial_x\phi_{+}(x,\tau)\partial_x\phi_{+}(0,0)\rangle_{+}, \tag{14a}$$

$$\langle S_{\pi}^z(x,\tau)S_{\pi}^z(0,0)\rangle=(-1)^{x/a_0}\left(\frac{2\lambda}{\pi}\right)^2\frac{A_1^2}{\pi}K_0(mr/v)\langle\cos[\sqrt{\pi}\phi_{+}(x,\tau)]\cos[\sqrt{\pi}\phi_{+}(0,0)]\rangle_{+}, \tag{14b}$$

$$\langle S_{\pi}^{+}(x,\tau)S_{\pi}^{-}(0,0)\rangle=(-1)^{x/a_0}\left(\frac{2\lambda}{\pi}\right)^2A_1^2\langle e^{i\sqrt{\pi}\theta_{+}(x,\tau)}e^{-i\sqrt{\pi}\theta_{+}(0,0)}\rangle_{+}, \tag{14c}$$

$$\langle S_0^+(x,\tau)S_0^-(0,0)\rangle = -\frac{a_0}{2\pi^2 v}\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{-i\sqrt{\pi}\phi_+(x,\tau)}e^{i\sqrt{\pi}\phi_+(0,0)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+ \partial_+ K_0(mr/v)$$

$$-\frac{a_0}{2\pi^2 v}\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{i\sqrt{\pi}\phi_+(x,\tau)}e^{-i\sqrt{\pi}\phi_+(0,0)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+ \partial_- K_0(mr/v)$$

$$+\frac{a_0 m}{2\pi^2 v}K_0(mr/v)[\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{-i\sqrt{\pi}\phi_+(x,\tau)}e^{-i\sqrt{\pi}\phi_+(0,0)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+$$

$$+\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{i\sqrt{\pi}\phi_+(x,\tau)}e^{i\sqrt{\pi}\phi_+(0,0)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+], \tag{14d}$$

where τ is the imaginary time, A_1 is a numerical constant, $\partial_\pm = \partial_\tau \pm iv\partial_x$, and $\langle\ \rangle_+$ is the average with respect to the Hamiltonian $\mathcal{H}_+ + \mathcal{H}_\perp$. In Eqs. (14a)–(14d) we have ignored the terms decaying much faster than $e^{-mr/v}$. Therefore we discarded the contribution from processes involving more than one massive magnon. These equations are valid for $J_\perp > 0$. When $J_\perp < 0$, on the other hand, the strongly renormalized interchain coupling combines the spins $\boldsymbol{S}_{1,i}$ and $\boldsymbol{S}_{2,i}$ into a single spin, and the ladder behaves as a $S=1$ Heisenberg chain. As explained in Ref. 12, when taking average over the massive Majorana fermions ξ_3 and ρ, we only need to exchange the order and disorder parameter of the Ising model. Using the correlators of ϕ_- and θ_- for $J_\perp < 0$ in Ref. 12, we find

$$\langle S_0^z(x,\tau)S_0^z(0,0)\rangle\big|_{J_\perp<0} = \langle S_0^z(x,\tau)S_0^z(0,0)\rangle\big|_{J_\perp>0} + \langle S_\pi^z(x,\tau)S_\pi^z(0,0)\rangle\big|_{J_\perp>0}, \tag{15a}$$

$$\langle S_0^+(x,\tau)S_0^-(0,0)\rangle\big|_{J_\perp<0} = \langle S_0^+(x,\tau)S_0^-(0,0)\rangle\big|_{J_\perp>0} + \langle S_\pi^+(x,\tau)S_\pi^-(0,0)\rangle\big|_{J_\perp>0}, \tag{15b}$$

where the correlators in the right-hand side are those in Eqs. (14a)–(14d). When $J_\perp < 0$, the $q_y = \pi$ correlators decay much faster than the $q_y = 0$ correlator and are thus negligible. Hence, the dynamical spin-spin correlation functions of the $S=1$ Haldane chain are linear combinations of those $q_y = 0$ and π correlators of the Heisenberg ladder ($J_\perp > 0$).

Now our task is to calculate the correlators of ϕ_+ and θ_+ in the presence of the magnetic field. The Hamiltonian $\mathcal{H}_+$ is identical to the one used to study the commensurate-incommensurate transition in classical two-dimensional systems.[43–46] In fact some of the correlation functions in Eqs. (14a)–(14d) have been discussed in this context.[43–46] In particular, the leading term of the correlation function corresponding to $\langle e^{i\sqrt{\pi}\phi_+(x,\tau)}e^{-i\sqrt{\pi}\phi_+(0,0)}\rangle$ is obtained by Schulz[45] including its universal exponent in the limit $M \to 0$. These results are used to calculate the spin-spin correlation functions by Chitra and Giamarchi,[8] who unfortunately seem to have overlooked some terms including the leading term [$\propto \cos(2\pi M x)$] in $\langle S_0^z(x)S_0^z(0)\rangle$. We think therefore that it is still worthwhile to describe the calculation of the correlations of ϕ_+ and θ_+ in Eqs. (14a)–(14d) in some detail, despite the fact that the Hamiltonian $\mathcal{H}_+$ has been analyzed in many literatures.

Following Ref. 12, we fermionize $\mathcal{H}_+$:

$$\mathcal{H}_+ = \int dx\left[iv\left(\psi_L^\dagger \frac{d}{dx}\psi_L - \psi_R^\dagger \frac{d}{dx}\psi_R\right) - im(\psi_R^\dagger\psi_L - \psi_L^\dagger\psi_R) - H(\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R)\right]$$

$$= \int_{-\infty}^{\infty} dk[vk(c_{R,k}^\dagger c_{R,k} - c_{L,k}^\dagger c_{L,k}) - im(c_{R,k}^\dagger c_{L,k} - c_{L,k}^\dagger c_{R,k}) - H(c_{R,k}^\dagger c_{R,k} + c_{L,k}^\dagger c_{L,k})], \tag{16}$$

where ψ_L (ψ_R) is the right-going (left-going) complex fermion field, and $\psi_{R(L)}(x) = \int (dk/\sqrt{2\pi}) e^{ikx} c_{R(L),k}$. The fermion fields are related to the bosons by the standard relations:

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a_0}} e^{i\sqrt{\pi}[\phi_+(x) - \theta_+(x)]},$$

$$:\psi_R^\dagger(x)\psi_R(x): = \frac{1}{2\sqrt{\pi}}\frac{d}{dx}[\phi_+(x) - \theta_+(x)], \tag{17a}$$

$$\psi_L(x) = \frac{1}{\sqrt{2\pi a_0}} e^{-i\sqrt{\pi}[\phi_+(x) + \theta_+(x)]},$$

$$:\psi_L^\dagger(x)\psi_L(x): = \frac{1}{2\sqrt{\pi}}\frac{d}{dx}[\phi_+(x) + \theta_+(x)]. \tag{17b}$$

It is important to note that the normal ordering in the above equations is defined with respect to the ground state of $H = 0$. The fermionized Hamiltonian $\mathcal{H}_+$ is easily diagonalized:

$$\mathcal{H}_+ = \int_{-\infty}^{\infty} dk[(\sqrt{v^2k^2 + m^2} - H)a_k^\dagger a_k$$

$$-(\sqrt{v^2k^2 + m^2} + H)\tilde{a}_k^\dagger \tilde{a}_k], \tag{18}$$

where

$$\begin{pmatrix} a_k \\ \tilde{a}_k \end{pmatrix} = \begin{pmatrix} \cos(\varphi_k/2) & -i\sin(\varphi_k/2) \\ -i\sin(\varphi_k/2) & \cos(\varphi_k/2) \end{pmatrix} \begin{pmatrix} c_{R,k} \\ c_{L,k} \end{pmatrix}, \tag{19}$$

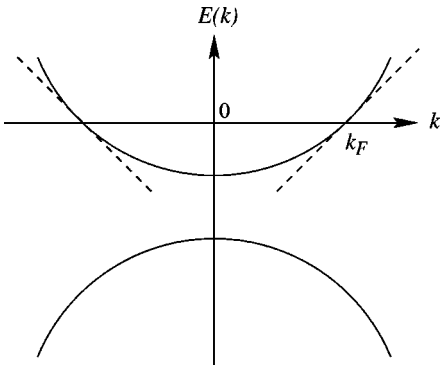

FIG. 1. Schematic picture of the upper and lower bands: $E(k) = \pm\sqrt{v^2k^2+m^2}-H$. The negative-energy states are filled. The long-distance behavior of the correlation functions are determined by the low-energy excitations around $|k|=k_F$, where the dispersion is linearized.

with $\tan\varphi_k = m/vk$. The magnetic field plays a role of the chemical potential to the fermions. We are concerned with the case where H is slightly above the lower critical field $H_{c1}(=m)$ such that $0<M\ll 1$. The ground state is obtained by filling the upper band (a_k) up to the Fermi points ($|k|<k_F$) and the lower band ($\tilde{a}_k$) completely; see Fig. 1. The Fermi wave number k_F is related to the magnetization $k_F=\pi M/a_0$. This follows from

$$\frac{M}{a_0}=\frac{1}{\sqrt{\pi}L}\int dx\langle\partial_x\phi_+\rangle_+=\frac{1}{L}\int dx\langle:\psi_R^\dagger\psi_R+\psi_L^\dagger\psi_L:\rangle_+$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\langle a_k^\dagger a_k\rangle_+, \tag{20}$$

where L is the length of the ladder and $\langle a_k^\dagger a_k\rangle_+=\Theta(k_F-|k|)$. In calculating long-distance correlations, we can safely ignore the lower band and keep only the low-energy excitations around the Fermi points in the upper band. In the fermion representation the interaction term $\mathcal{H}_\perp$ reads

$$\mathcal{H}_\perp\approx\frac{J_\perp a_0}{4}\int dx(\psi^\dagger\psi)^2, \tag{21}$$

where $\psi(x)=\int(dk/\sqrt{2\pi})e^{ikx}a_k$ and we have dropped the contribution from the lower band. The total Hamiltonian for the fermions in the upper band, $\mathcal{H}_a=\mathcal{H}_+ +\mathcal{H}_\perp$, consists of the kinetic energy, Eq. (18), and the short-range scattering term, Eq. (21). The coupling constant of the latter term is proportional to $J_\perp$ in lowest order. Thus, the interaction is repulsive for the antiferromagnetically coupled ladder, while it is attractive for the $S=1$ chain.[42] Obviously the scattering term has only negligible effects in both limits $M\to 0$ and $M\to 1$, where we will get the correlation functions of free fermions.[32]

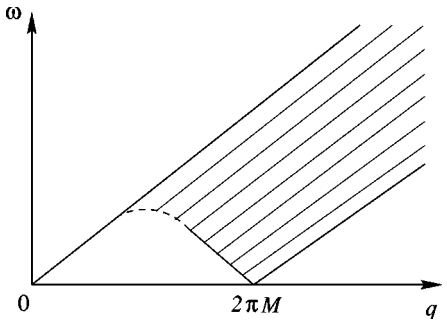

FIG. 2. Support of $S_0^{zz}(q,\omega)$. The shaded region shows where $S_0^{zz}(q,\omega)$ is nonzero.

The low-energy physics of $\mathcal{H}_a$ can be easily solved by the Abelian bosonization.[47,48] We first linearize the dispersion around $k=\pm k_F$ (Fig. 1). We then bosonize the fermions in the upper band:

$$\psi(x)\approx\frac{1}{\sqrt{2\pi a_0}}(e^{i\pi Mx/a_0+i\sqrt{\pi}[\phi(x)-\theta(x)]}$$
$$+e^{-i\pi Mx/a_0-i\sqrt{\pi}[\phi(x)+\theta(x)]}), \tag{22}$$

where the bosonic fields $\phi(x)$ and $\theta(x)$ obey $[\phi(x),\theta(y)]=i\Theta(y-x)$. Using these fields, we write the Hamiltonian $\mathcal{H}_a$ as

$$\mathcal{H}_a=\frac{\tilde{v}}{2}\int dx\left[\frac{1}{g}\left(\frac{d\phi}{dx}\right)^2+g\left(\frac{d\theta}{dx}\right)^2\right], \tag{23}$$

where $\tilde{v}$ is the Fermi velocity, and g is a parameter determined by the interaction $g<1$ ($g>1$) when $J_\perp>0$ ($J_\perp<0$), and $g\to 1$ as $M\to 0,1$. Incidentally, g is related to the compactification radius R of the field ϕ by $g=1/(4\pi R^2)$. We now need to express ϕ_+ and θ_+ in terms of ϕ and θ. Once this is done, it is straightforward to calculate the correlation functions since $\mathcal{H}_a$ is a free-boson Hamiltonian. First we note that for states near the Fermi surface we have

$$c_{R,k}^\dagger c_{R,p}+c_{L,k}^\dagger c_{L,p}\approx a_k^\dagger a_p+\tilde{a}_k^\dagger\tilde{a}_p, \tag{24}$$

where we used the approximation $\varphi_k\approx\varphi_p\approx\varphi_{k_F}$. Using Eqs. (22) and (24) and discarding the $\tilde{a}_k$ fermions, we find

$$\frac{1}{\sqrt{\pi}}\frac{d\phi_+}{dx}=:\psi_R^\dagger(x)\psi_R(x)+\psi_L^\dagger(x)\psi_L(x):\approx\psi^\dagger(x)\psi(x)=\frac{M}{a_0}+\frac{1}{\sqrt{\pi}}\frac{d\phi}{dx}+\frac{1}{\pi a_0}\cos[2\pi Mx+\sqrt{4\pi}\phi(x)]. \tag{25}$$

It follows that

$$\frac{1}{\pi}\langle\partial_x\phi_+(x,\tau)\partial_x\phi_+(0,0)\rangle_+=\frac{M^2}{a_0^2}+\frac{1}{\pi}\langle\partial_x\phi(x,\tau)\partial_x\phi(0,0)\rangle_a+\frac{1}{(\pi a_0)^2}\cos(2\pi Mx)\langle\cos[\sqrt{4\pi}\phi(x,\tau)]\cos[\sqrt{4\pi}\phi(0,0)]\rangle_a, \tag{26}$$

where $\langle\ \rangle_a$ represents the average taken in the ground state of $\mathcal{H}_a$. The averages are found to be

$$\langle \partial_x\phi(x,\tau)\partial_x\phi(0,0)\rangle_a = -\frac{g}{4\pi}\left(\frac{1}{(x+i\tilde{v}\tau)^2}+\frac{1}{(x-i\tilde{v}\tau)^2}\right), \tag{27}$$

$$\langle \cos[\sqrt{4\pi}\phi(x,\tau)]\cos[\sqrt{4\pi}\phi(0,0)]\rangle_a = \frac{1}{2}\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^g, \tag{28}$$

where $\tilde{a}_0$ is a short-distance cutoff of order a_0. Integration of Eq. (25) then yields

$$\phi_+(x,\tau) = \frac{\sqrt{\pi}Mx}{a_0}+\phi(x,\tau). \tag{29}$$

We have neglected the contribution from the oscillating term. We thus get

$$\langle \cos[\sqrt{\pi}\phi_+(x,\tau)]\cos[\sqrt{\pi}\phi_+(0,0)]\rangle_+ = \cos(\pi Mx/a_0)\langle \cos[\sqrt{\pi}\phi(x,\tau)]\cos[\sqrt{\pi}\phi(0,0)]\rangle_a = \frac{1}{2}\cos(\pi Mx/a_0)\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^{g/4}. \tag{30}$$

We next consider $e^{-i\sqrt{\pi}\theta_+(x)}$. From Eqs. (17a) and (17b) we can express it as

$$e^{-i\sqrt{\pi}\theta_+(x)} = \sqrt{\frac{\pi a_0}{2}}[e^{i\pi/4}e^{-i\sqrt{\pi}\phi_+(x)}\psi_R(x)+e^{-i\pi/4}e^{i\sqrt{\pi}\phi_+(x)}\psi_L(x)]. \tag{31}$$

Using the same approximation as in the derivation of Eqs. (24) and (25), we get $\psi_R(x)\approx\psi(x)/\sqrt{2}$ and $\psi_L(x)\approx i\psi(x)/\sqrt{2}$. Here we have made a further approximation $\varphi_{k_F}\approx\pi/2$ which is valid for $0<M\ll 1$. From Eqs. (22), (29), and (31) we find

$$e^{-i\sqrt{\pi}\theta_+(x)} = \left\{\frac{1}{\sqrt{2}}+\sin[2\pi Mx/a_0+\sqrt{4\pi}\phi(x)+(\pi/4)]\right\}e^{-i\sqrt{\pi}\theta(x)+i\pi/4}. \tag{32}$$

From Eq. (32) we may write the correlation functions in Eqs. (14c) and (14d) as

$$\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+ = \frac{1}{2}\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a+\frac{1}{4}\sum_{\epsilon=\pm1}e^{2\pi i\epsilon Mx/a_0}\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{i\epsilon\sqrt{4\pi}\phi(x,\tau)}e^{-i\epsilon\sqrt{4\pi}\phi(0,0)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a,$$

$$\begin{aligned}\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{-i\sqrt{\pi}\phi_+(x,\tau)}e^{i\sqrt{\pi}\phi_+(0,0)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+ &= \frac{e^{-i\pi Mx/a_0}}{2}\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{-i\sqrt{\pi}\phi(x,\tau)}e^{i\sqrt{\pi}\phi(0,0)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a\\ &+\frac{e^{i\pi Mx/a_0}}{4}\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{i\sqrt{\pi}\phi(x,\tau)}e^{-i\sqrt{\pi}\phi(0,0)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a\\ &+\frac{e^{-3\pi iMx/a_0}}{4}\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{-i\sqrt{9\pi}\phi(x,\tau)}e^{i\sqrt{9\pi}\phi(0,0)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a,\end{aligned}$$

and

$$\sum_{\epsilon=\pm1}\langle e^{i\sqrt{\pi}\theta_+(x,\tau)}e^{i\epsilon\sqrt{\pi}\phi_+(x,\tau)}e^{i\epsilon\sqrt{\pi}\phi_+(0,0)}e^{-i\sqrt{\pi}\theta_+(0,0)}\rangle_+ = \frac{1}{2}\sum_{\epsilon=\pm1}e^{i\epsilon\pi Mx/a_0}\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{i\epsilon\sqrt{\pi}\phi(x,\tau)}e^{-i\epsilon\sqrt{\pi}\phi(0,0)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a.$$

The averages in the above equations are given by

$$\langle e^{i\sqrt{\pi}\theta(x,\tau)}e^{in\sqrt{\pi}\phi(x,\tau)}e^{-in\sqrt{\pi}\phi(0,0)}e^{-i\sqrt{\pi}\theta(0,0)}\rangle_a = e^{-i\pi n/2}\left(\frac{\tilde{a}_0}{x+i\tilde{v}\tau}\right)^{(1/\sqrt{g}-n\sqrt{g})^2/4}\left(\frac{\tilde{a}_0}{x-i\tilde{v}\tau}\right)^{(1/\sqrt{g}+n\sqrt{g})^2/4}. \tag{33}$$

Combining these results together, we finally obtain the dynamical spin-spin correlation functions:

$$\langle S_0^z(x,\tau)S_0^z(0,0)\rangle = M^2-\frac{g}{4\pi^2}\left[\frac{1}{(x+i\tilde{v}\tau)^2}+\frac{1}{(x-i\tilde{v}\tau)^2}\right]+C_1\cos(2\pi Mx)\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^g, \tag{34}$$

$$\langle S^z_\pi(x,\tau)S^z_\pi(0,0)\rangle = C_2(-1)^x\cos(\pi Mx)K_0(mr/v)\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^{g/4}, \tag{35}$$

$$\langle S^+_\pi(x,\tau)S^-_\pi(0,0)\rangle = (-1)^x\left\{C_3\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^{1/4g} - C_4\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^{(1/2\sqrt{g}-\sqrt{g})^2}\left[\frac{\tilde{a}_0^2 e^{2\pi iMx}}{(x+i\tilde{v}\tau)^2}+\frac{\tilde{a}_0^2 e^{-2\pi iMx}}{(x-i\tilde{v}\tau)^2}\right]\right\}, \tag{36}$$

$$\langle S^\pm_0(x,\tau)S^\mp_0(0,0)\rangle = \frac{i}{8\pi^2 v}\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^{(1/\sqrt{g}-\sqrt{g})^2/4}\left[\left(\frac{2\tilde{a}_0 e^{\pm i\pi Mx}}{x-i\tilde{v}\tau}-\frac{\tilde{a}_0 e^{\mp i\pi Mx}}{x+i\tilde{v}\tau}\right)\partial_- K_0(mr/v)+(x\to -x)\right]$$

$$\pm\frac{im}{4\pi^2 v}K_0(mr/v)\left(\frac{\tilde{a}_0^2}{x^2+\tilde{v}^2\tau^2}\right)^{(1/\sqrt{g}-\sqrt{g})^2/4}\left(\frac{\tilde{a}_0 e^{\mp i\pi Mx}}{x+i\tilde{v}\tau}-\frac{\tilde{a}_0 e^{\pm i\pi Mx}}{x-i\tilde{v}\tau}\right), \tag{37}$$

where C's are positive numerical constants, and we have set $a_0=1$. The correlation function $\langle S^-_\pi(x,\tau)S^+_\pi(0,0)\rangle$ is obtained by replacing M with $-M$ in Eq. (36). In Eq. (37) we have discarded the term proportional to $e^{\pm 3\pi iMx}$ decaying much faster than the kept terms. We should therefore regard Eqs. (34)–(37) as listing only the leading terms. As noted in Sec. II, we may expect that $\langle S^\pm_\pi(x,\tau)S^\mp_\pi(0,0)\rangle$ should contain algebraically decaying terms that are proportional to $\cos(2\pi lMx)$ with any integer l. From Ref. 46, we expect that the correlator $\langle S^z_0(x)S^z_0(0)\rangle$ should also have terms proportional to $\cos(2\pi lMx)$ which decay as x^{-2l^2g}. The appearance of the term proportional to $e^{-3\pi iMx}$ suggests that $\langle S^\pm_0(x,\tau)S^\mp_0(0,0)\rangle$ should have terms proportional to $K_0(mr/v)\cos[\pi(2l+1)Mx]$. We note that the equal-time correlations $\langle S^z_0(x,0)S^z_0(0,0)\rangle$ and $\langle S^\pm_\pi(x,0)S^\mp_\pi(0,0)\rangle$ agree with Eqs. (7a) and (7b) if we identify η with $2g$. As is well known, the strongest correlation is $\langle S^\pm_\pi(x,0)S^\mp_\pi(0,0)\rangle\sim(-1)^x x^{-1/\eta}$. Note also that the value of the exponent is consistent between the weak- and strong-coupling approach: $g<1$ ($g>1$) for $J_\perp>0$ ($J_\perp<0$) and $g\to 1$ as $M\to 0,1$. Another interesting finding is that the exponentially decaying equal-time correlation functions have different phases by $\pi/2$: $\langle S^z_\pi(x,0)S^z_\pi(0,0)\rangle\propto\cos(\pi Mx)$ and $\langle S^+_0(x,0)S^-_0(0,0)\rangle\propto\sin(\pi Mx)$.

Now we are in a position to calculate the dynamical spin structure factors defined by

$$S^{\alpha\beta}_{q_y}(q,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\langle S^\alpha_{q_y}(x,\tau=it+0^+)S^\beta_{q_y}(0,0)\rangle e^{-iqx+i\omega t}, \tag{38a}$$

where t is a real time and the correlation functions in the real time are obtained by replacing $\tau\to it+0^+$. We may also calculate it from

$$S^{\alpha\beta}_{q_y}(q,\omega)=\frac{1}{\pi}\mathrm{Im}\lim_{i\tilde{\omega}\to\omega+i0^+}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\tau\langle S^\alpha_{q_y}(x,\tau)S^\beta_{q_y}(0,0)\rangle e^{-iqx+i\tilde{\omega}\tau} \tag{38b}$$

for $\omega>0$. From the obvious relation $S^{\alpha\beta}_{q_y}(q,\omega)=S^{\alpha\beta}_{q_y}(-q,\omega)=S^{\alpha\beta}_{q_y}(q+2\pi,\omega)$, we assume $0\le q\le\pi$ in the following discussion. We first consider the correlation functions showing the quasi-long-range order. Using Eq. (A1) in Appendix, we get from Eqs. (34) and (36)

$$S^{zz}_0(q,\omega)=2\pi M^2\delta(q)\delta(\omega)+\frac{g\omega}{2\pi\tilde{v}}\Theta(\omega)\delta(\omega-\tilde{v}q)$$

$$+\frac{\pi C_1\tilde{a}_0^2}{2\tilde{v}[\Gamma(g)]^2}\Theta(\omega-\tilde{v}(q-2\pi M))\Theta(\omega+\tilde{v}(q-2\pi M))\left(\frac{4\tilde{v}^2/\tilde{a}_0^2}{\omega^2-\tilde{v}^2(q-2\pi M)^2}\right)^{1-g}, \tag{39}$$

$$S_{\pi}^{+-}(q,\omega)=\frac{\pi C_3\tilde{a}_0^2}{\tilde{v}[\Gamma(1/4g)]^2}\Theta(\omega+\tilde{v}(q-\pi))\Theta(\omega-\tilde{v}(q-\pi))\left(\frac{4\tilde{v}^2/\tilde{a}_0^2}{\omega^2-\tilde{v}^2(q-\pi)}\right)^{1-1/4g}$$
$$+\frac{\pi C_4\tilde{a}_0^2}{\tilde{v}\Gamma(\eta_+)\Gamma(\eta_-)}\Bigg[\Theta(\omega-\tilde{v}[q-\pi(1-2M)])\Theta(\omega+\tilde{v}[q-\pi(1-2M)])$$
$$\times\left(\frac{\omega-\tilde{v}[q-\pi(1-2M)]}{2\tilde{v}/\tilde{a}_0}\right)^{\eta_+-1}\left(\frac{2\tilde{v}/\tilde{a}_0}{\omega+\tilde{v}[q-\pi(1-2M)]}\right)^{1-\eta_-}$$
$$+\Theta(\omega-\tilde{v}[q-\pi(1+2M)])\Theta(\omega+\tilde{v}[q+\pi(1+2M)])$$
$$\times\left(\frac{\omega+\tilde{v}[q-\pi(1+2M)]}{2\tilde{v}/\tilde{a}_0}\right)^{\eta_+-1}\left(\frac{2\tilde{v}/\tilde{a}_0}{\omega-\tilde{v}[q-\pi(1+2M)]}\right)^{1-\eta_-}\Bigg], \tag{40}$$

where $\eta_{\pm}=[(1/2\sqrt{g})\pm\sqrt{g}\,]^2$. The structure factor $S_{\pi}^{-+}(q,\omega)$ can be obtained from $S_{\pi}^{+-}(q,\omega)$ by $M\to -M$. Being obtained from the long-distance asymptotic expansions (34) and (36), each term in Eqs. (39) and (40) describe the behavior of the structure factors correctly only near its low-energy threshold. For example, the last term ($\propto C_1$) in Eq. (39) is valid only for $\omega\ll\pi\tilde{v}M$ ($|q-2\pi M|\ll\pi M$) and cannot be extended to $|q|\lesssim\pi M$. The supports of these structure factors are shown in Figs. 2–4. They are essentially the same as those of the $S=\frac{1}{2}$ XXZ chain[14] except that the small M in the ladder corresponds to the nearly polarized state in the $S=\frac{1}{2}$ chain through the relation $M=\frac{1}{2}+\langle\tilde{S}^z\rangle$. The strongest divergence is at $q=\pi$ of $S_{\pi}^{\pm\mp}(q,\omega)$: $S_{\pi}^{\pm\mp}(q,\omega)\propto[\omega-\tilde{v}(q-\pi)]^{-1+1/4g}$. The exponent approaches $-3/4$ as $M\to 0,1$. We note that the boundaries of the supports of these structure factors became all straight lines because of our linearization of the dispersion relation in the continuum limit. This is an artifact of the approximation, and the true boundary lines should be given by some nonlinear functions. Furthermore, some of the boundary lines may be parts of a single curve.

We next consider the massive components. Using Eqs. (35), (38b), and (A3), we get

$$S_{\pi}^{zz}(q,\omega)=\left(\frac{\tilde{a}_0}{2\tilde{v}}\right)^{g/2}\frac{\pi v\tilde{v}C_2}{[\Gamma(g/4)]^2}\int_{-\infty}^{\infty}dk\frac{\Theta(\omega-\varepsilon(q-k-\pi+\pi M)-\tilde{v}|k|)}{\varepsilon(q-k-\pi+\pi M)\{[\omega-\varepsilon(q-k-\pi+\pi M)]^2-\tilde{v}^2k^2\}^{1-g/4}}+(M\to -M), \tag{41}$$

where $\varepsilon(q)=\sqrt{v^2q^2+m^2}$. The minimum energy above which $S_{\pi}^{zz}(q,\omega)>0$ is $\omega=\varepsilon(q-\pi(1-M))$ around $q=\pi(1-M)$. Near this threshold energy the structure factor reduces to

$$S_{\pi}^{zz}(q,\omega)=\frac{\pi\tilde{a}_0vC_2}{2m\tilde{v}\Gamma(g/2)}\Theta(\omega-\varepsilon(q-\pi+\pi M))\left(\frac{\tilde{v}/\tilde{a}_0}{\omega-\varepsilon(q-\pi+\pi M)}\right)^{1-g/2}+(M\to -M), \tag{42}$$

where $v|q-\pi(1\mp M)|\ll m\tilde{v}/v$ and $|\omega-\varepsilon(q-\pi(1\mp M))|\ll m(\tilde{v}/v)^2$ are assumed. The support of $S_{\pi}^{zz}(q,\omega)$ is shown in Fig. 5. We see that $S_{\pi}^{zz}(q,\omega)$ diverges at the low-energy threshold as $S_{\pi}^{zz}(q,\omega)\propto[\omega-\varepsilon(q-\pi+\pi M)]^{-1+g/2}$. Since the two thresholds $\omega=\varepsilon(q-\pi\pm\pi M)$ intersect at $q=\pi$, we expect to have a peak at $q=\pi$ and $\omega=\varepsilon(\pi M)=H$. The exponent approaches $-1/2$ as $M\to 0,1$, and the singularity is even stronger for $0<M<1$, where $g<1$. Note that the exponent jumps from -1 to $-1/2$ when H crosses H_{c1} from below. In the strong-coupling limit of the ladder, the square-root divergence may be understood in the following way. The correlation function $\langle S_{\pi,i}^zS_{\pi,j}^z\rangle$ is a propagator of the $t_{i,0}$ bosons. If we ignored the interaction with the $t_{i,+}$ bosons, we would get the massive free-particle propagator, $K_0(mr/v)$. Due to the interaction the motion of the $t_{i,0}$ boson is necessarily accompanied by a superfluid flow of the $t_{i,+}$ bosons. Its main effect in the low-density limit amounts both to multiplying the free propagator by that of the hard-core $t_{i,+}$ bosons $\propto(x^2+\tilde{v}^2\tau^2)^{-1/4}$ and to shifting the momentum by πM. The Fourier transform of the product has the square-root divergence at the threshold.

Finally we consider $S_0^{\pm\mp}(q,\omega)$. This can be obtained from Eqs. (37) and (38b) as described in Appendix. The result is

$$S_0^{\pm\mp}(q,\omega)=\frac{\tilde{a}_0}{4\pi\eta_0[\Gamma(\eta_0)]^2}\left(\frac{\tilde{a}_0}{2\tilde{v}}\right)^{2\eta_0}\int_{-\infty}^{\infty}dk\frac{\Theta(\omega-\varepsilon(q-k\mp\pi M)-\tilde{v}|k|)[\omega-\varepsilon(q-k\mp\pi M)-\tilde{v}k]^{\eta_0}}{\varepsilon(q-k\mp\pi M)[\omega-\varepsilon(q-k\mp\pi M)+\tilde{v}k]^{1-\eta_0}}$$
$$\times\left[\frac{3}{2}\varepsilon(q-k\mp\pi M)-\frac{1}{2}v(q-k\mp\pi M)\pm m\right]+(q\to -q), \tag{43}$$

where $\eta_0=[(1/\sqrt{g})-\sqrt{g}\,]^2/4$. Near the lower edge $0<\omega-\varepsilon(q\mp\pi M)\ll m$, it may be approximated by

$$S_0^{+-}(q,\omega)=\frac{5\tilde{a}_0}{8\pi\tilde{v}\Gamma(2\eta_0+1)}\Theta(\omega-\varepsilon(q-\pi M))\left(\frac{\omega-\varepsilon(q-\pi M)}{\tilde{v}/\tilde{a}_0}\right)^{2\eta_0}+(M\to -M). \tag{44}$$

In general the exponent $2\eta_0\geqslant 0$ and approaches 0 as $M\to 0,1$. The support of $S_0^{+-}(q,\omega)$ is shown in Fig. 6. Like $S_\pi^{zz}(q,\omega)$, $S_0^{+-}(q,\omega)$ has a peak at $(q,\omega)=(0,H)$, where the two thresholds $\omega=\varepsilon(q\pm\pi M)$ cross. The structure factor $S_0^{-+}(q,\omega)$ is approximately equal to $S_0^{+-}(q,\omega)/5$ near the low-energy threshold, $\omega\gtrsim\varepsilon(q\mp\pi M)$ and $q\approx\pm\pi M$. That $S_0^{-+}(q,\omega)$ is much smaller than $S_0^{+-}(q,\omega)$ is consistent with the result in the strong-coupling limit where the $t_{i,0}$ bosons are absent in the ground state [see Eqs. (2c) and (2e)].

IV. DISCUSSION

We shall discuss implications of the results we obtained for the Heisenberg ladder to the π-resonance mode in the SO(5) symmetric ladder model. As pointed out in Ref. 17, there is an analogy between the quantum phase transition driven by the chemical potential in the SO(5) symmetric ladder model and the field-induced phase transition in the Heisenberg ladder. Obviously, the chemical potential plays the role of the magnetic field. The analogy is most clearly seen in the strong-coupling limit.[17] At half filling the ground state of the E_0 phase or the Mott insulating phase discussed in Ref. 17 is a state in which all the rungs are in the spin singlet state. When the chemical potential is zero, there are fivefold degenerate low-lying massive modes above the ground state. The five modes consist of a $S=1$ magnon triplet, a hole pair state where two holes are placed on a single rung, and a state where two additional electrons are put on a rung. When the chemical potential is turned on, the energy of the hole-pair (electron-pair) excitation decreases (increases) while the magnon triplet is not directly affected by the chemical potential. Thus, we see that the hole pair corresponds to the $S^z=1$ magnon or the $t_{i,+}$ boson in the Heisenberg ladder. The triplet magnon in the SO(5) model is an analog of the $S^z=0$ magnon ($t_{i,0}$ boson) in the Heisenberg ladder. Furthermore, the low-energy effective Hamiltonian for the hole-pair excitations in the strong-coupling limit is similar to the effective Hamiltonian for the $t_{i,+}$ bosons. That is, hole pairs may be viewed as hard-core bosons which repel each other when two hole pairs sit on neighboring rungs.[17,21] Let us find operators playing the role of the spin operators in the Heisenberg ladder. First, the operator corresponding to $S^z_{\pi,i}$ should change a singlet rung $|\Omega\rangle=(1/\sqrt{2})(c^\dagger_{i,\uparrow}d^\dagger_{i,\downarrow}-c^\dagger_{i,\downarrow}d^\dagger_{i,\uparrow})|0\rangle$ into a hole pair or $|0\rangle$. Here $c^\dagger_{i,\sigma}$ and $d^\dagger_{i,\sigma}$ are creation operators of an electron with spin σ on the ith rung of upper (c) and lower (d) chains. Obviously the d-wave pair operator Δ_i is such an operator: $\Delta_i=(c_{i,\uparrow}d_{i,\downarrow}-c_{i,\downarrow}d_{i,\uparrow})/\sqrt{2}$. Second, the operator corresponding to $S^z_{0,i}\approx t^\dagger_{i,+}t_{i,+}$ should be a number operator of hole pairs. It is given by $N_i\equiv 1-(1/2)\Sigma_\sigma(c^\dagger_{i,\sigma}c_{i,\sigma}+d^\dagger_{i,\sigma}d_{i,\sigma})$. Finally, from the relation

$$(S^+_{i,c}-S^+_{i,d})|\Omega\rangle\equiv(c^\dagger_{i,\uparrow}c_{i,\downarrow}-d^\dagger_{i,\uparrow}d_{i,\downarrow})|\Omega\rangle=-\sqrt{2}c^\dagger_{i,\uparrow}d^\dagger_{i,\uparrow}|0\rangle,$$

we find that $\boldsymbol{S}_{i,c}-\boldsymbol{S}_{i,d}$ creates a triplet magnon from a rung singlet. Thus we conclude that the spin operator $\boldsymbol{S}_{i,c}-\boldsymbol{S}_{i,d}$ is an analog of $S^z_{\pi,i}$.

When the chemical potential is increased beyond the charge gap which equals the spin gap in the SO(5) symmetric model, the ladder is doped with the charge carrier (holes) and becomes superconducting with the d-wave-like symmetry. For spatial dimension greater than or equal to two, the superconducting order is long-ranged in the ground state, and this gives rise to a δ-function peak or the π resonance in the dynamic spin structure factor. In one dimension, however, the order is quasi-long-ranged, and therefore the peak is expected to be replaced by a power-law singularity.[17,20] The threshold energy at $q=\pi$ is also shown to be equal to the chemical potential.[17] These features are readily reproduced from our results for the Heisenberg ladder model.

From the approximate mapping we discussed above, the d-wave pair correlation function is expected to show the quasi-long-range order corresponding to the XY order in the Heisenberg ladder, Eq. (7b):

$$\langle\Delta^\dagger_i\Delta_j\rangle\propto\frac{1}{|i-j|^{1/\tilde{\eta}}}, \tag{45}$$

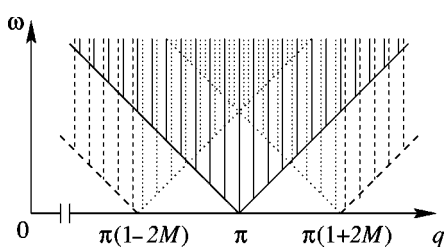

FIG. 3. Support of $S_\pi^{+-}(q,\omega)$. The shaded regions show where $S_\pi^{+-}(q,\omega)$ is nonzero. The strongest divergence is at $\omega=\pm\tilde{v}(q-\pi)$. The next strongest singularity is at $\omega=\pm\tilde{v}[q-\pi(1\pm 2M)]$.

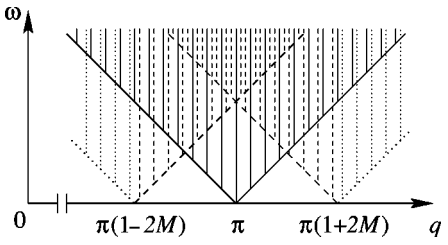

FIG. 4. Support of $S_\pi^{-+}(q,\omega)$. The shaded regions show where $S_\pi^{-+}(q,\omega)$ is nonzero. The strongest divergence is at $\omega=\pm\tilde{v}(q-\pi)$. The next strongest singularity is at $\omega=\mp\tilde{v}[q-\pi(1\pm 2M)]$. Note the difference from Fig. 3.

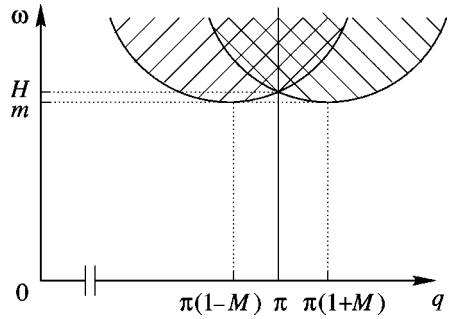

FIG. 5. Support of $S_{\pi}^{zz}(q,\omega)$. The shaded regions show where $S_{\pi}^{zz}(q,\omega)$ is nonzero.

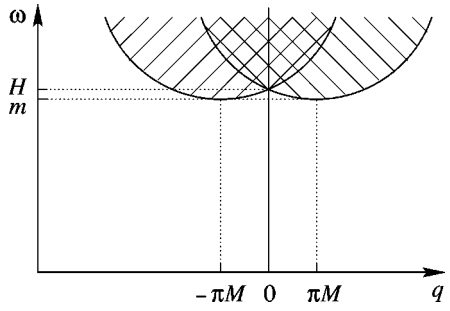

FIG. 6. Support of $S_0^{+-}(q,\omega)$ and $S_0^{-+}(q,\omega)$. The shaded regions show where $S_0^{\pm\mp}(q,\omega)$ are nonzero.

where the exponent $\tilde{\eta}$ is presumably smaller than 2 and approaches 2 in the limit where the hole density δ goes to zero. The correlator has no $(-1)^{i-j}$ factor because the hole-pair mode has a minimum energy at $q=0$. The charge density correlation is related to $\langle S_{0,i}^z S_{0,j}^z\rangle$, Eq. (7a), and is also quasi-long ranged:

$$\langle N_i N_j\rangle - \delta^2 \propto \frac{\cos(2\pi\delta|i-j|)}{|i-j|^{\tilde{\eta}}} = \frac{\cos(4k_F|i-j|)}{|i-j|^{\tilde{\eta}}}, \quad (46)$$

where we have used the relation between the hole density and the Fermi wave number, $\pi(1-\delta)=2k_F$. The result that the correlations of the d-wave superconductivity and $4k_F$ charge density wave show power-law decay with the exponents whose product is 1 was also obtained by Nagaosa for a generic two-chain model.[49] Finally the spin correlation of the SO(5) ladder is expected to be

$$\langle (S_{i,c}^{\alpha} - S_{i,d}^{\alpha})(S_{j,c}^{\alpha} - S_{j,d}^{\alpha})\rangle$$
$$\propto (-1)^{i-j}\cos(\pi\delta|i-j|)\frac{K_0(|i-j|/\xi)}{|i-j|^{\tilde{\eta}/4}}, \quad (47)$$

where ξ is the correlation length determined by the spin gap. The spin structure factor is then

$$S(q,\omega) \propto [\omega - \tilde{\varepsilon}(q-\pi\pm\pi\delta)]^{-1+\tilde{\eta}/4}, \quad (48)$$

where $\tilde{\varepsilon}(q)$ is the magnon dispersion at $\delta=0$. From Fig. 5 we see that the threshold energy at $q=\pi$ is determined by the chemical potential, as expected. Although the exponent $\tilde{\eta}$ depends on the detail of the model, we can generally conclude that it is 2 in the low-density limit of holes ($\delta\to 0$), where we may regard the hard-core bosons as free fermions ($g=1$). This universal exponent was independently found by Ivanov and Lee[50] and by Schulz[51] for the t-J ladder and was also obtained by Konik *et al.* for the SO(8) Gross-Neveu model.[52] We thus find that the spin structure factor has a universal square-root divergence at the critical point. When the superfluid density is finite, the interaction between bosons becomes important and modifies the exponent, as we saw in the Heisenberg ladder model. We expect that the square-root singularity is a universal feature for the spectral weight of a gapped excitation generated by injecting a massive particle to a superfluid in the low-density limit.

Although our argument above is based on the analogy and approximate mapping, our results should be valid as long as the weak-coupling and strong-coupling limits are in the same phase. We notice that Eq. (47) is the same as the ''mean-field'' result given in Sec. VII of Ref. 20. The validity of this result is, however, questioned by Lin *et al.* as it misses the existence of the bound states such as the Cooper pair-magnon bound states found in the SO(8) Gross-Neveu model. On the other hand, we did not find such a bound state in our weak-coupling calculation. It is not clear at the moment whether this is due to the approximation we made, for example, concerning the interaction term in $\mathcal{H}_{\perp}$. It was shown by Damle and Sachdev[41] that this term can indeed lead to a bound state of two magnons when $H=0$. The fate of the bound state in the gapless phase is an open question.

Finally we conclude this paper by summarizing our results on the spin correlations in the gapless phase of the two-leg Heisenberg ladder. We have obtained the dynamical spin-spin correlation functions and the structure factors, extending the bosonization theory of Shelton *et al.* to the gapless regime. The correlation functions are classified into two categories: algebraically decaying ones, $\langle S_0^z(x,\tau)S_0^z(0,0)\rangle$ and $\langle S_{\pi}^{\pm}(x,\tau)S_{\pi}^{\mp}(0,0)\rangle$, and exponentially decaying ones, $\langle S_{\pi}^z(x,\tau)S_{\pi}^z(0,0)\rangle$ and $\langle S_0^{\pm}(x,\tau)S_0^{\mp}(0,0)\rangle$. We have also found that the terms $\propto\cos(2\pi lMx)$ (l: integer) are quasi-long-ranged, while the terms $\propto\cos[(2l+1)\pi Mx]$ are short-ranged. The exponents of the correlation functions are controlled by the single parameter g, which is smaller (larger) than 1 for $J_{\perp}>0$ ($J_{\perp}<0$). The parameter g approaches 1 in the limits $M\to 0,1$. The structure factors have power-law singularities at the lower edges, and the strongest divergence is at $\omega=\pm\tilde{v}(q-\pi)$ in $S_{\pi}^{\pm\mp}(q,\omega)$ due to the dominant XY spin correlation. The next strongest singularity is found at the lower edge of $S_{\pi}^{zz}(q,\omega)$: $\omega=\{v^2[q-\pi(1\pm M)]^2+m^2\}^{1/2}$. The exponent is universally given by $-1/2$ in the limits $M\to 0,1$.

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APPENDIX: INTEGRALS

In this appendix we list integral formulas we used to calculate the dynamical structure factors.
For the gapless modes, we need the following integral:

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\frac{e^{-iqx+i\omega t}}{(x+\tilde{v}t-i0^{+})^{\gamma_{+}}(x-\tilde{v}t+i0^{+})^{\gamma_{-}}}=\Theta(\omega-\tilde{v}q)\Theta(\omega+\tilde{v}q)\frac{2\pi^{2}e^{i\pi(\gamma_{+}-\gamma_{-})/2}}{\tilde{v}\Gamma(\gamma_{+})\Gamma(\gamma_{-})}\left(\frac{2\tilde{v}}{\omega-\tilde{v}q}\right)^{1-\gamma_{+}}\left(\frac{2\tilde{v}}{\omega+\tilde{v}q}\right)^{1-\gamma_{-}}. \tag{A1}$$

For the structure factor $S_{\pi}^{zz}(q,\omega)$ we first take the Fourier transform of the correlation function in the imaginary time:

$$I_{\pi}^{zz}(q,i\tilde{\omega})\equiv\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\tau\frac{K_{0}(m\sqrt{x^{2}+v^{2}\tau^{2}}/v)}{(x^{2}+\tilde{v}^{2}\tau^{2})^{\gamma}}e^{-iqx+i\tilde{\omega}\tau}=\frac{v}{(2\tilde{v})^{2\gamma-1}}\frac{\Gamma(1-\gamma)}{\Gamma(\gamma)}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}d\nu\frac{(\nu^{2}+\tilde{v}^{2}k^{2})^{\gamma-1}}{(\omega-\nu)^{2}+\varepsilon^{2}(q-k)}$$

$$=\frac{v}{(2\tilde{v})^{2\gamma-1}}\frac{\Gamma(1-\gamma)}{\Gamma(\gamma)}\int_{-\infty}^{\infty}dk\left(\frac{\pi}{\varepsilon(q-k)\{[\tilde{\omega}+i\varepsilon(q-k)]^{2}+\tilde{v}^{2}k^{2}\}^{1-\gamma}}+\int_{\tilde{v}|k|}^{\infty}d\nu\frac{2\sin(\pi\gamma)(\nu^{2}-\tilde{v}^{2}k^{2})^{\gamma-1}}{(\tilde{\omega}-i\nu)^{2}+\varepsilon^{2}(q-k)}\right). \tag{A2}$$

After the analytic continuation we take the imaginary part to find

$$\mathrm{Im}I_{\pi}^{zz}(q,\omega+i0^{+})=\frac{v}{(2\tilde{v})^{2\gamma-1}}\left(\frac{\pi}{\Gamma(\gamma)}\right)^{2}\int_{-\infty}^{\infty}dk\frac{\Theta(\omega-\varepsilon(q-k)-\tilde{v}|k|)}{\varepsilon(q-k)\{[\omega-\varepsilon(q-k)]^{2}-\tilde{v}^{2}k^{2}\}^{1-\gamma}} \tag{A3}$$

for $\omega>0$. When $v|q|\ll m\tilde{v}/v$, $\varepsilon(q-k)+\tilde{v}|k|\approx\varepsilon(q)+\tilde{v}|k|$. In this case we may approximate the last integral as

$$\frac{\Theta(\omega-\varepsilon(q))}{m}\int dk\frac{\Theta(\omega-\varepsilon(q)-\tilde{v}|k|)}{\{[\omega-\varepsilon(q)]^{2}-\tilde{v}^{2}k^{2}\}^{1-\gamma}}=\Theta(\omega-\varepsilon(q))\frac{B(\gamma,1/2)}{\tilde{v}m}[\omega-\varepsilon(q)]^{2\gamma-1}, \tag{A4}$$

where $B(a,b)$ is the beta function. From Eqs. (A3) and (A4) we finally obtain

$$\mathrm{Im}I_{\pi}^{zz}(q,\omega+i0^{+})=\Theta(\omega-\varepsilon(q))\frac{\pi^{2}}{\Gamma(2\gamma)}\frac{v}{m\tilde{v}}\left(\frac{\tilde{v}}{\omega-\varepsilon(q)}\right)^{1-2\gamma}, \tag{A5}$$

which is valid for $\omega-\varepsilon(q)\ll m$.

We next consider $S_{0}^{\pm\mp}(q,\omega)$. According to Eq. (36), we need the following Fourier transform:

$$I_{0}^{+-}(q,i\tilde{\omega})\equiv-i\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\tau\frac{K_{0}(m\sqrt{x^{2}+v^{2}\tau^{2}}/v)}{(x^{2}+\tilde{v}^{2}\tau^{2})^{\gamma}(x-i\tilde{v}\tau)}e^{-iqx+i\tilde{\omega}\tau}=i\frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)}\frac{v}{(2\tilde{v})^{2\gamma}}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}d\nu\frac{(\nu+i\tilde{v}k)^{\gamma}(\nu-i\tilde{v}k)^{\gamma-1}}{(\omega-\nu)^{2}+\varepsilon^{2}(q-k)}$$

$$=\frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)}\frac{v}{(2\tilde{v})^{2\gamma}}\int_{-\infty}^{\infty}dk\left(\frac{i\pi[\omega+i\tilde{v}k+i\varepsilon(q-k)]^{\gamma}}{\varepsilon(q-k)[\omega-i\tilde{v}k+i\varepsilon(q-k)]^{1-\gamma}}-2\sin(\pi\gamma)\int_{\tilde{v}|k|}^{\infty}d\nu\frac{(\nu+\tilde{v}k)^{\gamma}(\nu-\tilde{v}k)^{\gamma-1}}{(\omega-i\nu)^{2}+\varepsilon^{2}(q-k)}\right). \tag{A6}$$

After the analytic continuation we obtain

$$\mathrm{Im}I_{0}^{+-}(q,\omega+i0^{+})=\frac{v}{\gamma(2\tilde{v})^{2\gamma}}\left(\frac{\pi}{\Gamma(\gamma)}\right)^{2}\int_{-\infty}^{\infty}dk\frac{\Theta(\omega-\varepsilon(q-k)-\tilde{v}|k|)[\omega-\varepsilon(q-k)-\tilde{v}k]^{\gamma}}{\varepsilon(q-k)[\omega-\varepsilon(q-k)+\tilde{v}k]^{1-\gamma}}. \tag{A7}$$

Using the same approximation as in Eq. (A4), we obtain

$$\mathrm{Im}I_{0}^{+-}(q,\omega+i0^{+})=\Theta(\omega-\varepsilon(q))\frac{\pi^{2}}{\Gamma(1+2\gamma)}\frac{v}{m\tilde{v}}\left(\frac{\omega-\varepsilon(q)}{\tilde{v}}\right)^{2\gamma} \tag{A8}$$

for $0<\omega-\varepsilon(q)\ll m$ and $v|q|\ll m\tilde{v}/v$. In the same way we get

$$\mathrm{Im}I_{0}^{-+}(q,\omega+i0^{+})\equiv\mathrm{Im}\lim_{i\tilde{\omega}\to\omega+i0^{+}}i\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\tau\frac{K_{0}(m\sqrt{x^{2}+v^{2}\tau^{2}}/v)}{(x^{2}+\tilde{v}^{2}\tau^{2})^{\gamma}(x+i\tilde{v}\tau)}e^{-iqx+i\tilde{\omega}\tau}$$

$$=\frac{v}{\gamma(2\tilde{v})^{2\gamma}}\left(\frac{\pi}{\Gamma(\gamma)}\right)^{2}\int_{-\infty}^{\infty}dk\frac{\Theta(\omega-\varepsilon(q-k)-\tilde{v}|k|)[\omega-\varepsilon(q-k)+\tilde{v}k]^{\gamma}}{\varepsilon(q-k)[\omega-\varepsilon(q-k)-\tilde{v}k]^{1-\gamma}}, \tag{A9}$$

which reduces to Eq. (A8) for $0<\omega-\varepsilon(q)\ll m$ and $v|q|\ll m\tilde{v}/v$.

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