Distribution of time constants for tunneling through a one-dimensional disordered chain

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The dynamics of electronic tunneling through a disordered one-dimensional chain of finite length is considered. We calculate distributions of the transmission coefficient T, Wigner delay time τ_W, and the transport time $\tau_t = T\tau_W$. The central bodies of these distributions have a power-law form, which can be understood in terms of the resonant tunneling through localized states. [S0163-1829(99)07935-7]

During the past decade, transport properties of phase-coherent disordered low-dimensional conductors attracted much interest. The transmission properties of complex structures with a large number of internal degrees of freedom have been intensively studied[1] using random matrix theory,[2] and a universal distribution for transmission coefficients through metallic systems has been derived.[3] The one-dimensional (1D) localization problem is exactly solvable, and a complete description of the distribution of transmission coefficients, T, and localization properties of single-electron wave functions in disordered 1D and quasi-1D wires is now available.[4–9] In particular, it has been found that the inverse localization radius α of single-particle localized states by disorder in a 1D chain has a normal distribution with width inversely proportional to the chain length L:

$$P(\alpha) \sim \exp\left\{ -L \frac{(\alpha-\alpha_0)^2}{2\alpha_0} \right\}. \tag{1}$$

Here, α_0^{-1} is the most probable localization radius, which for a weakly disordered system is given by $\alpha_0^{-1} = 4l$, where $l \ll L$ is the mean free path. This is equivalent to a log-normal distribution of the transmission coefficients, $T \sim \exp(-\alpha L)$,

$$P(T) \sim \frac{1}{T} \exp\left\{ -\frac{[\ln(1/T) - 2L\alpha_0]^2}{8L\alpha_0} \right\}. \tag{2}$$

Recently, random matrix theory was applied to the problem of the dynamical electric response of mesoscopic conductors and disordered wires in the localized regime.[10,11] Here the problem involves understanding the distribution of the Wigner delay time, τ_W, which carries information about the lifetime of carriers in the resonant states responsible for transmission through a weakly couple quantum dot, or through a disordered wire in the localized regime. The Wigner delay time is related to the energy dependence of the phase shift $\theta(\epsilon)$ of a wave passing through a disordered wire or a quantum resonator, and is given by

$$\tau_W = \frac{d\theta(\epsilon)}{d\epsilon} = \frac{1}{2i} \frac{d}{d\epsilon} \ln \frac{G_R(\epsilon,L)}{G_A(\epsilon,L)}. \tag{3}$$

This equation also expresses the delay time in terms of retarded and advanced single-particle Green functions, $G_{R,A}(\epsilon,L) = \Sigma_\nu \psi_\nu^*(0)\psi_\nu(L)/[\epsilon - \epsilon_\nu \pm i0]$, which links τ_W to the density of states of a system, $\tau_W = \pi \int_0^L dx\, \nu(\epsilon,x)$. The Wigner delay time in zero-dimensional mesoscopic systems modeled using random matrix theory has been studied[12] within the zero-dimensional σ-model approach, and has been shown to have a universal distribution. In the present paper, we analyze a related dynamical characteristic of a disordered conductor, namely, the transport time τ_t defined as the delay time weighted by the transmission coefficient,

$$\tau_t = T\tau_W. \tag{4}$$

This quantity characterizes the ability of a resonant state to provide a dynamical response to an external ac electric field. Using the Landauer-Buttiker approach, the imaginary part of the dimensionless ac conductance $g = G(\omega)/(2e^2/h)$ of a single-channel mesoscopic wire calculated in the single-particle approximation,[13,14]

$$g = T(\omega) = v_F^2 G_R(\epsilon + \omega/2, L) G_A(\epsilon - \omega/2, L),$$

can be represented as

$$\mathrm{Im}\, g(\omega) = T\, \mathrm{Im}[1 - i\omega\tau_W + \cdots] = -\omega\tau_t,$$

so that the transport time τ_t is a directly measurable quantity, which can also be interpreted as the dielectric response function of an almost insulating 1D wire, when $\mathrm{Re}\, G \to 0$. In the present paper, we report the results of numerical studies and a qualitative asymptotic analysis of the transport-time distribution function $P(\tau_t)$ in the localized regime of 1D disordered wires. To anticipate a little, we find that the distribution of this quantity is affected by correlations between the value of the Wigner delay time and the transmission coefficient of resonances via localized states. Using information about the distribution of the localization radii in Eq. (1) and about the energetic widths of individual resonances, we show that the central body of the distribution of τ_t, which corresponds to $-2/3 < (1/z)\ln(\tau_t/\tau) < 1/3$, where $z = L\alpha_0 \gg 1$, is given by the power-law asymptotic,

$$P(\tau_t) \sim \tau^{-1} e^{-z/2} \left(\frac{\tau}{\tau_t} \right)^{4/3}, \tag{5}$$

in complete agreement with our numerical simulations. The tail of short times τ_t, $(1/z)\ln(\tau_t/\tau)<-2/3$, decays in the logarithmically normal way,

$$P(\tau_t)\sim\tau^{-1}e^{-z/2}\left(\frac{\tau_t}{\tau}\right)^{-(3/2)+[1/(8z)]\ln(\tau/\tau_t)}, \tag{6}$$

whereas for $(1/z)\ln(\tau_t/\tau)>1/3$,

$$P(\tau_t)\sim\tau^{-1}e^{-z/2}\left(\frac{\tau}{\tau_t}\right)^{1+[1/(2z)]\ln(\tau_t/\tau)}. \tag{7}$$

Below, we show how Eqs. (8) and (7) can be obtained, and we describe the numerical procedure used to determine the distribution of τ_W and τ_t. We begin with an analysis of the Wigner delay time and introduce the numerically studied model. In agreement with the result of Ref. 15, we show, both analytically and numerically, that the body of the distribution function $P(\tau_W)$ is dominated by the inverse-square-law asymptotic,

$$P(\tau_W)\sim\tau/\tau_W^2 \quad \text{at} \quad \tau_W>\tau, \tag{8}$$

where τ is the mean free path time. These results provide a check of both the analytical method and the numerics and are followed by an analysis of the transport-time distribution, which is the central goal of this paper.

To obtain the distribution of τ_W, we note that in the absence of disorder in a 1D wire, $T=1$ and $\tau_W=\tau_t=L/v_F$, v_F being the electron Fermi velocity which determines the ballistic time of flight of an electron through the chain. For a weakly disordered chain characterized by a mean free path $l=v\tau$ or scattering time $\tau\ll 1/\epsilon$, the transmission T, τ_W, and τ_t are random variables. In a long wire $L\gg l$, where localization is strong, transmission can be viewed as being the result of tunneling through resonant levels, each characterized by its energy ϵ_0 and decay width $\gamma=\gamma_1+\gamma_2$ determined by the electron escape rates $\gamma_{1,2}$ into the left and right contacts. In the exponential localization regime $L\gg l$, the tunneling rates associated with a resonant state peaked at $x<L$ (calculated from the left end of the chain) are of order

$$\gamma_1=\tau^{-1}e^{-2\alpha_1 x} \quad \text{and} \quad \gamma_2=\tau^{-1}e^{-2\alpha_2(L-x)}, \tag{9}$$

where α_1 and α_2 are two independently fluctuating inverse localization radii of the wave-function tail on the left- and right-hand sides of the resonance. Note that since the resonance width falls exponentially with the wire length, whereas the mean level spacing $\Delta=1/\nu L$ is only inversely proportional to L, the assumption $L\gg l$ allows us to distinguish between resonances and to consider each resonant state as a slightly broadened discrete level.

Therefore, at each value of the energy, the ac transmission through the disordered 1D chain can be described using the Breit-Wigner formula,

$$T(\epsilon)=\frac{4\gamma_1\gamma_2}{(\epsilon-\epsilon_0)^2+\gamma^2},$$

parametrized by four independently fluctuating parameters: the energetic position ϵ_0 of the resonance which is closest to the energy ϵ, the location of the center of mass of the resonant state x, and two inverse localization radii α_1 and α_2. The associated Wigner delay time can be represented as

$$\tau_W=\frac{1}{\gamma}\,\frac{1}{1+[(\epsilon-\epsilon_0)/\gamma]^2}.$$

To analyze the distribution function $P(\tau_W)$, we assume that the center of the localized state and its energy have a homogeneous distribution, and that the probability to find some value of the inverse localization radius α_i in a segment of a wire with the length x_i ($x_1=x$ and $x_2=L-x$) is equal to

$$P(\alpha_i)=\sqrt{\frac{x_i}{2\pi\alpha_0}}\exp\left\{-\frac{x_i}{2\alpha_0}(\alpha_i-\alpha_0)^2\right\}. \tag{10}$$

It is convenient to use random variables $p=(\alpha_1x_1+\alpha_2x_2)/z$ and $q=(\alpha_1x_1-\alpha_2x_2)/z$, instead of α_1 and α_2, where $z=L\alpha_0=L/4l\gg 1$, which can be described using the joint distribution function

$$P(p,q)=\frac{1}{2\pi}\,\frac{z}{\sqrt{1-y^2}}\exp\left\{-\frac{z}{2}\left[(p-1)^2-\frac{(q-yp)^2}{(1-y^2)}\right]\right\}, \tag{11}$$

where $y=(x/L)-2$ is uniformly distributed within the interval $[-1,1]$. In the same parametrization, one can represent the Wigner delay time as

$$\tau_W=\frac{\tau}{1+(\epsilon/\gamma)^2}\,\frac{\exp(zp)}{\cosh(zq)}$$

and the corresponding probability density as the conditional probability integral

$$P(\tau_W)=\tau^{-1}\int_{-1}^{1}dy\int dp\,dq\,P(p,q)\times\int\frac{d\epsilon}{\Delta}\delta\left(\frac{1}{1+(\epsilon/\gamma)^2}\,\frac{\exp(zp)}{\cosh(zq)}-\frac{\tau_W}{\tau}\right). \tag{12}$$

To analyze Eq. (12), we evaluate the integral over statistical variables p,q using the saddle-point method. The use of the saddle-point method is justified when $z\gg 1$, that is, when the exponential factor in the joint distribution function in Eq. (11) is large, and for $\tau_W<\tau e^z$ we find that

$$P(\tau_W)\sim\frac{\tau}{\tau_W^2}. \tag{13}$$

Note that the length of the chain does not appear in this result, which indicates that this intermediate asymptotic of the delay-time distribution, which is related to the distribution of the spectral width of the resonant states, is dominated by the electron escape rate from the resonant state into the nearest reservoir and for $L\to\infty$ is exact for any delay time. However, the finite length L determines a cutoff, $\tau_W\sim\tau e^z$, for this universal behavior, and for $\tau_W>\tau\exp(z)$ we find

$$P(\tau_W)\sim\frac{\exp(-z/2)}{\tau}\left(\frac{\tau}{\tau_W}\right)^{1+[1/(2z)]\ln(\tau_W/\tau)}.$$

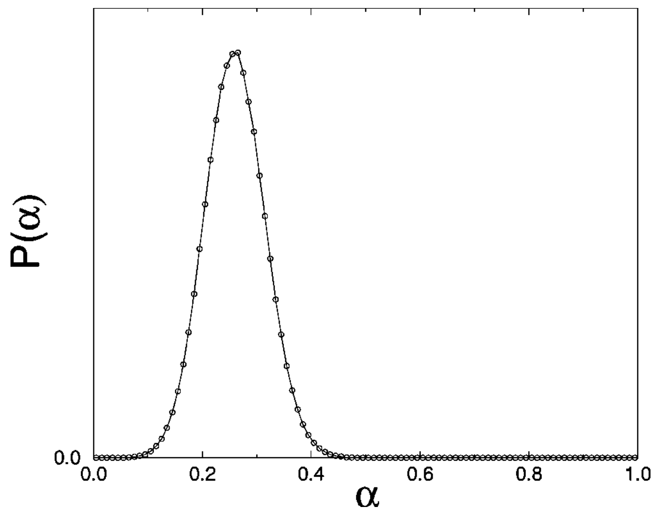

FIG. 1. Distribution $P(\alpha)$ versus α for a chain length of 200.

in agreement with Melnikov.[16]

To illustrate the validity of the above estimates, we compute the scattering matrix of a series of equal strength, randomly spaced δ-function scatterers. The spacings between scatterers possesses a Poisson distribution, and the system is described by the Hamiltonian

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U_0\sum_{i=1}^{L}\delta(x-x_i). \tag{14}$$

In the region $x_{j-1}<x<x_j$, an eigenstate of energy ϵ is of the form

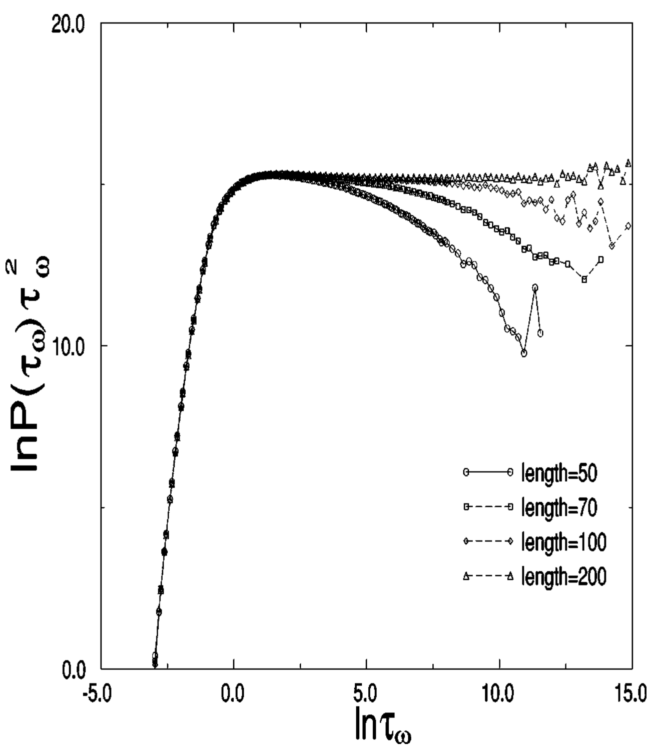

FIG. 2. Plots of $\ln P(\tau_W)\tau_W^2$ versus $\ln\tau_W$, for various lengths ranging from $L=50$ (upper curve) to $L=200$ (lower curve). The size of the ensemble is 10^8.

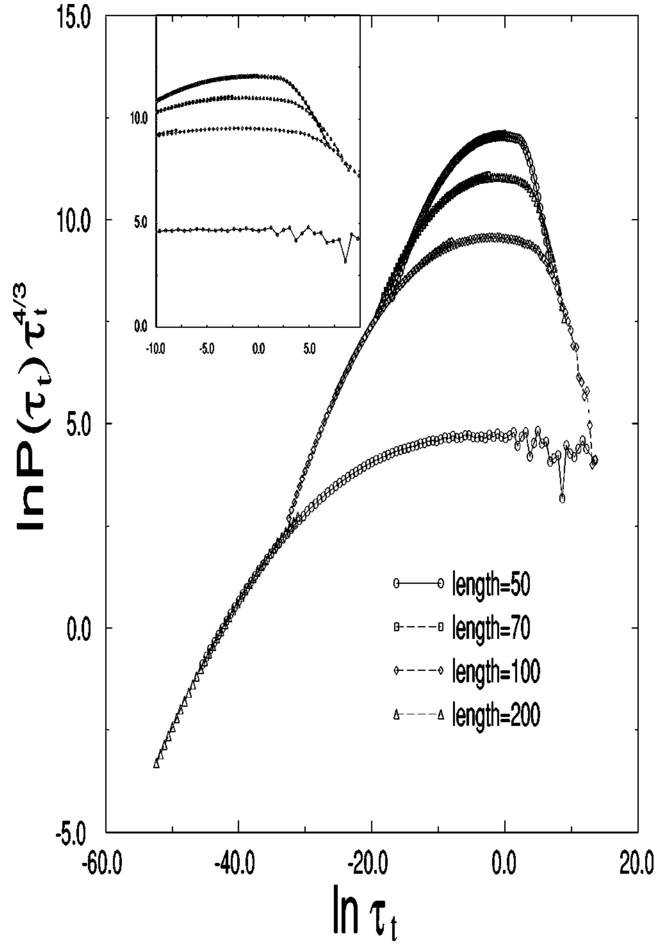

FIG. 3. Plots of $\ln P(\tau_t)\tau_t^{4/3}$ versus $\ln\tau_t$, for various lengths, ranging from $L=50$ (upper curve) to $L=200$ (lower curve). The inset shows the central portions of the distributions. The size of the ensemble is 10^8.

$$\psi_\epsilon(x)=A_je^{ikx}+B_je^{-ikx},\quad x_{j-1}<x<x_j,$$

where $k=\sqrt{2m\epsilon/\hbar^2}$ and the wave amplitudes on either side of the scatterer j satisfy

$$\begin{pmatrix}A_{j+1}\\B_{j+1}\end{pmatrix}=T_j\begin{pmatrix}A_j\\B_j\end{pmatrix}. \tag{15}$$

In this expression, T_j is the transfer matrix,

$$T_j=\begin{pmatrix}1-i\beta & -i\beta e^{-2ikx_j}\\ i\beta e^{2ikx_j} & 1+i\beta\end{pmatrix}, \tag{16}$$

where $\beta=m\mu/\hbar^2k$.

The transfer matrix for a series of N scatterers has the form

$$\tilde{T}=\begin{pmatrix}\tilde{T}_{11} & \tilde{T}_{12}\\ \tilde{T}_{21} & \tilde{T}_{22}\end{pmatrix} \tag{17}$$

and is given by the product of the transfer matrices of each individual scatterer,

$$\tilde{T}=T_NT_{N-1}\cdots T_1,$$

from which one obtains the scattering matrix

$$\mathbf{S}=\begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \tag{18}$$

via the relation

$$\mathbf{S}=\begin{pmatrix} -T_{21}T_{22}{}^{-1} & T_{22}{}^{-1} \\ T_{11}^{*}{}^{-1} & T_{12}T_{22}{}^{-1} \end{pmatrix}. \tag{19}$$

For a chain of length L, the conductance and the inverse localization radius are determined by

$$g=|t|^2 \quad \text{and} \quad \alpha(L)\sim\frac{\ln|t|^2}{L}.$$

As an example of Eq. (1), and to emphasize that we are simulating a chain of weak scatterers for which α_0^{-1} is greater than the mean spacing, Fig. 1 shows $P(\alpha)$ for a chain of scatterers of concentration unity, strength $U_0=3$, and energy $\epsilon=4.1\pi^2$. These same parameter values were used in all numerical simulations described below.

Figure 2 shows the corresponding plot of $\ln P(\tau_W)\tau_W^2$ for the chain lengths $L=50,70,100,200$. These results were obtained by evaluating a finite difference $\tau_W=[\theta(\epsilon+\delta)-\theta(\epsilon)]/\delta$, and were found to be stable with respect to any choice of δ within the range $10^{-9}<\delta<10^{-6}$. For $\ln\tau>0$, all curves exhibit a plateau with a slope that tends to zero with increasing length, in good agreement with the analytically estimated asymptotic behavior in Eq. (8).

Having obtained agreement with known results for $P(\alpha)$ and $P(\tau_W)$, we now present an analysis of the transport time distribution, $P(\tau_t)$, which represents the central new result of this paper. Using the resonant tunneling description of Sec. II, the transport time for a particle with a given energy ϵ is given by

$$\tau_t=T\tau_W=\frac{4\gamma_1\gamma_2\gamma^{-3}}{\{1+[(\epsilon-\epsilon_0)/\gamma]^2\}^2}. \tag{20}$$

This allows us to parametrize the transport time using the position x of the resonant state center and two (left and right) inverse localization radii, $\alpha_{1,2}$ as in Sec. II,

$$\tau_t=\frac{4\tau}{[1+(\epsilon/\gamma)^2]^2}\,\frac{\exp(zp)}{\cosh^3(zq)}.$$

The probability to find a given value of τ_t can again be expressed as a conditional probability,

$$P(\tau_t)=\tau^{-1}\int_{-1}^{1}dy\int dp\,dq\,P(p,q)\times\int\frac{d\epsilon}{\Delta}\,\delta\!\left(\frac{4}{[1+(\epsilon/\gamma)^2]^2}\,\frac{\exp(zp)}{\cosh^3(zq)}-\frac{\tau_t}{\tau}\right), \tag{21}$$

and evaluation of the integral in Eq. (21) using the saddle-point method yields the result of Eqs. (5)–(7). Note that the power-law asymptotic $P(\tau_t)\sim\tau^{-1}e^{-z/2}(\tau/\tau_t)^{4/3}$, valid for the finite length wires within the parametric interval $-2/3<(1/z)\ln(\tau_t/\tau)<1/3$, formally transforms into the universal central body of the distribution in the thermodynamic limit $L\to\infty$.

To illustrate the validity of this result, Fig. 3 shows plots of the function $\ln P(\tau_t)\tau_t^{4/3}$ versus $\ln\tau_t$ for various lengths. These numerical simulations show that at large τ_t, all curves exhibit a plateau with a slope that tends to zero with increasing length, demonstrating that the tail in the distribution of τ_t, varies as $\tau_t^{-4/3}$.

In summary, we have shown how earlier results for the Wigner-delay time τ_W, based on a picture of resonant transport through localized states, can be extended to yield the distribution of the transport time $\tau_t=T\tau_W$. In contrast with the distribution of τ_W, which exhibits a universal $1/\tau_W^2$ tail, the corresponding intermediate asymptotic of the distribution of τ_t exhibits a universal $1/\tau_t^{4/3}$ behavior.

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