Intrinsic Fluctuations in a Superconducting Ring Closed with a Josephson Junction

Juhani Kurkijärvi

Laboratory of Atomic and Solid State Physics, School of Applied and Engineering Physics, Cornell University, Ithaca, New York

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The distribution in the external flux at which a superconducting ring closed with a weak link admits a quantum of flux is determined assuming that the weak link can be treated as a Josephson junction. We find that this transition occurs at an appreciable fraction of the flux quantum from the theoretical critical external flux. To a first approximation the width of the distribution is proportional to the inductance of the ring and varies as $T^{2/3}i_c^{-1/3}$, where T is the temperature and i_c the critical current.

The understanding of the thermal fluctuations in Josephson junctions is rapidly becoming complete.[1,2] No theoretical solution, however, has been proposed for the one important circumstance in which individual phase-slip events are observable, namely, closure of a low-inductance superconducting loop with a Josephson junction. McCumber has previously treated a similar problem of a link of thin wire at temperatures close to T_c,[3] which has also been experimentally studied.[4] With the kind of ring used in quantum-flux detectors in mind, we take the phenomenological view that the weak link can be treated as a Josephson junction of critical current i_c shunted by a capacitance C and a resistance R representing the normal current and including a source of thermal Johnson noise. The results are not limited to the vicinity of the critical temperature, and we consider the interesting range where $Li_c \gtrsim \phi_0$, $i_c > k_B T/\phi_0$, where ϕ_0 is the flux quantum $h/2e = 2\times 10^{-15}$ Wb. Experimental investigation of the flux fluctuation considered in this paper is now in progress.[5] A correct but approximate determination of the fundamental noise limit of superconducting quantum-flux detectors[6] can be obtained by application of the present results and will be published elsewhere.

We consider fluctuations in the flux[7] ϕ at constant external flux ϕ_x. We seek to calculate the uncertainty in the magnitude of the external flux at which the flux through the ring jumps to a higher value as the external field is varied. We focus our attention on the energy barrier ΔU that holds ϕ in the lower metastable branch of the ϕ-vs-ϕ_x curve[8] (Fig. 1) against a fluctuation at constant ϕ_x. We determine ΔU as a function of $\Delta\phi_x = \phi_{xc} - \phi_x$, where ϕ_{xc} is the critical external flux at which a flux quantum enters the ring in the absence of fluctuations. For $\Delta\phi_x$ small we find $\Delta U \sim (\Delta\phi_x)^{3/2}$, whereas the equilibrium Gibb's energy is linear in $\Delta\phi_x$. If ϕ_x is increased slowly, the jump will occur long before the barrier ΔU has been reduced to a height $\sim k_B T$, thus easily at a distance of an appreciable fraction of the flux quantum from the critical external flux. This explains the experimental result[9] that the equilibrium ϕ-vs-ϕ_x curve cannot be followed too close to ϕ_{xc}. The mean flux at which the jump occurs depends roughly logarithmically on the sweep rate. The width $\propto Li_c(2\pi k_B T/i_c\phi_0)^{2/3}$, however, is almost independent of the rate, depending linearly on the inductance and to fractional powers on the critical current and the temperature.

The magnetic flux threading through a superconducting ring with one Josephson junction at absolute zero of temperature is rigidly determined by the external flux through[8]

$$\phi = \phi_x - Li_c \sin\frac{2\pi}{\phi_0}\,\phi \ . \tag{1}$$

There is, however, always a shunting capacitance, and at finite temperatures a normal current path across the junction. The currents through these elements in parallel with the ideal Josephson junction also contribute to the flux and Eq. (1) is relaxed to

$$\phi = \phi_x - Li_c \sin\frac{2\pi}{\phi_0}\,\phi - LC\frac{dV}{dt} - LI_n \ , \tag{2}$$

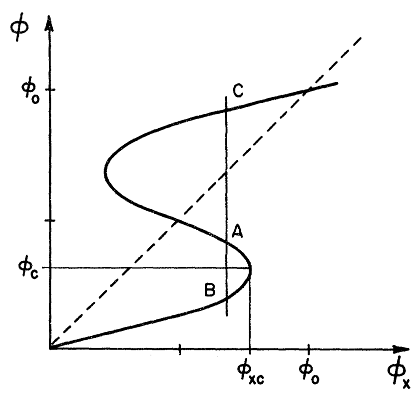

FIG. 1. Admitted flux vs applied flux for a ring with an ideal Josephson junction at $T=0$. Thermally activated transition occurs from the potential trough at B over potential maximum at A to a new state of lower potential at C. The illustrated case is $Li_c = \phi_0/2$.

allowing fluctuations of ϕ at fixed ϕ_x. The voltage V across the junction from the Josephson equation,

$$V=\frac{\hbar}{2e}\left(\dot{\phi}\ \frac{2\pi}{\phi_0}\right)=\dot{\phi}\ , \tag{3}$$

is consistent with the voltage induced by the inductance of the ring. Approximating the normal current path by a constant phenomenological resistance R and rearranging, Eq. (2) will read like an equation of motion for the flux ϕ in a constant external flux ϕ_x[1]:

$$-\frac{d}{d\phi}U(\phi)=-\frac{d}{d\phi}\left(\frac{1}{2}\frac{(\phi-\phi_x)^2}{L}-i_c\frac{\phi_0}{2\pi}\cos\frac{2\pi}{\phi_0}\phi\right)$$
$$=\frac{C\dot{\phi}}{CR}+C\ddot{\phi}\ , \tag{4}$$

defining the potential $U(\phi)$. To complete the physical picture, we should add a Johnson noise term,[10] which, however, we will not need explicitly in the following. We simply note that such a noise gives the fluctuating flux a mean thermal energy k_BT.

The flux is in equilibrium when the left-hand side of Eq. (4) vanishes. This condition is expressed by Eq. (1), which is plotted in Fig. 1. For $Li_c > \phi_0/2\pi$, the equilibrium ϕ is a multivalued function of ϕ_x. On the lowest branch the second derivative of $U(\phi)$ is positive until it vanishes at the critical external flux ϕ_{xc}. On the returning branch it is negative, and positive again on the next. Up to the point ϕ_{xc} the curve runs in a potential valley guarded on the higher-flux side by a barrier whose crest lies along the unstable returning branch. The barrier grows smaller when ϕ_x increases toward ϕ_{xc}, where it vanishes. Determining the uncertainty in the external flux at which the jump in ϕ occurs is thus reduced to analyzing the potential barrier and the fluctuation process by which the flux escapes over this barrier.

Assuming that the jump occurs reasonably close to the critical flux ϕ_{xc}, we calculate the potential barrier to lowest order in $\Delta\phi_x=\phi_{xc}-\phi_x$. We first expand ϕ as a function of $\Delta\phi_x$ around the point ϕ_{xc} to lowest order in $\Delta\phi_x$,

$$\phi=\frac{\phi_0}{2\pi}\arccos\left(-\frac{\phi_0}{2\pi Li_c}\right)$$
$$\pm\frac{\phi_0}{2\pi}\left(\frac{2\Delta\phi_x}{Li_c[1-(\phi_0/2\pi Li_c)^2]^{1/2}}\right)^{1/2}+\cdots\ , \tag{5}$$

and then evaluate the potential difference between the points A and B in Fig. 1 $[\Delta U(\Delta\phi_x)=U(\phi_A)-U(\phi_B)]$:

$$\Delta U(\Delta\phi_x)$$
$$=\frac{4}{3L}\left(\frac{\phi_0}{2\pi}\right)^{5/2}\left(\frac{2}{Li_c[1-(\phi_0/2\pi Li_c)^2]^{1/2}}\right)^{1/2}\left(\frac{2\pi\Delta\phi_x}{\phi_0}\right)^{3/2}$$
$$=U_0\left(\frac{2\pi\Delta\phi_x}{\phi_0}\right)^{3/2}. \tag{6}$$

Adding higher-order terms in $\Delta\phi_x$ to Eq. (5) would not change this result to the order considered. We also calculate the curvature of the potential at the bottom of the valley at B and the top of the barrier at A:

$$\left.\frac{d^2U}{d\phi^2}\right|_{A,B}=\mp\left(\frac{2\pi}{\phi_0}\right)^{1/2}$$
$$\times\left\{\frac{2i_c}{L}\left[1-\left(\frac{\phi_0}{2\pi Li_c}\right)^2\right]^{1/2}\right\}^{1/2}\left(\frac{2\pi\Delta\phi_x}{\phi_0}\right)^{1/2}. \tag{7}$$

The flux fluctuates at the frequency

$$\omega=\left(\frac{d^2U}{d\phi^2}\Big/C\right)^{1/2}=\omega_0\left(\frac{2\pi\Delta\phi_x}{\phi_0}\right)^{1/4}\ ,$$

where ω_0 is the order of $(1/LC)^{1/2}$. In the case of high damping, $\eta=1/RC$ large, the time scale is set by ω_0^2/η,[11] which is the order of R/L. We assume that the changes in $\Delta\phi_x$ are much slower. Then the escape problem is equivalent to the well-known case of a classical particle with a mean thermal energy of k_BT trying to get over a barrier $\Delta U(\phi_x)$ large compared with k_BT. The curvature of the barrier at the bottom which determines an attempt frequency and the curvature at the top are given by Eq. (7). Although the general solution is available[12] we limit ourselves to the case of low capacitance or high damping. The lifetime associated with the flux remaining in the lower state is then

$$\frac{1}{\tau}=\frac{\omega^2}{2\pi\eta}e^{-\Delta U/k_BT}\ . \tag{8}$$

Determining the distribution of applied flux at which the jump occurs depends on how ϕ_x evolves in time. The probability that a decay has not taken place is given by

$$W(t)=\exp\left(-\int_{-\infty}^{t}\frac{ds}{\tau(\Delta\phi_x(s))}\right)$$
$$=\exp\left(-\int_{-\infty}^{t}\frac{\omega^2}{2\pi\eta}e^{-\Delta U/k_BT}\,ds\right)\ . \tag{9}$$

As an example, imagine the external flux being swept at a constant rate defined through

$$-\frac{d}{dt}\left(\frac{2\pi\Delta\phi_x}{\phi_0}\right)=-\frac{d}{dt}\Delta\varphi_x=\omega_x\ .$$

Then the integral in the exponent of Eq. (9) can be carried out and we can express the probability that the decay has not taken place up to a given flux. After a change of variables we have

$$W(u)=\exp(-Xe^{-u^{3/2}})\ , \tag{10}$$

where $u=(U_0/k_BT)^{2/3}\,\Delta\varphi_x$, U_0 is defined in Eq. (6), and

$$X=\frac{2}{3\omega_x}\,\frac{\omega_0^2}{2\pi\eta}\left(\frac{k_BT}{U_0}\right). \tag{11}$$

For a typical flux sensor,[6] $L\sim 10^{-9}$ H and $Li_c \simeq \phi_0$ and $X\simeq (10^8/\omega_x)$/sec. In Fig. 2 we display the distribution $dW(u)/du$ for several values of X. The distribution moves toward an earlier jump in ϕ_x when X increases (rate of sweep decreases), while the asymmetric shape and the width $\sigma_u = \langle (u-\bar{u})^2\rangle^{1/2}$ do not change appreciably. The width in terms of ϕ_x, $\sigma_{\phi x} = \langle(\phi_x - \bar{\phi}_x)^2\rangle^{1/2}$ is

$$\sigma_{\phi x}=\sigma_u\left\{Li_c\left(\frac{k_BT2\pi}{\phi_0 i_c}\right)^{2/3}\left(\frac{3}{4\sqrt{2}}\right)^{2/3}\left[1-\left(\frac{\phi_0}{2\pi Li_c}\right)^2\right]^{1/6}\right\}, \tag{12}$$

where σ_u is typically 04., as indicated in Fig. 2. For the ring described above, $\sigma_{\phi x}$ from Eq. (12) is about $\phi_0/25$ at a few degrees Kelvin.

It is clear from these results that, for other than an extremely rapid sweep, the transition should indeed occur much before the theoretical critical flux, corroborating the conjecture of Sullivan *et al.*[13] This is a consequence of the high attempt frequency and the rather slow growth, $\sim(\Delta\phi_x)^{3/2}$, of the energy barrier for small $\Delta\phi_x$. When the sweep is very slow, so that $\langle\Delta\phi_x\rangle \sim \frac{1}{2}Li_c[1-(\phi_0/2\pi Li_c)^2]^{1/2}$, the higher terms in the potential $\Delta U(\Delta\phi_x)$ become important. An extremely rapid sweep can, on the other hand, begin to compete with the factor $\omega_0^2/2\pi\eta$. Then our barrier escape treatment is no longer accurate. At even higher rates of sweep, the process of quantum transition would no longer be faster than the sweep. For simplicity we have not considered the case of very large L and small critical current where reverse transitions could occur, as then a description of ΔU would be needed for all $\Delta\phi_x$.

Our results depend in no way on the value of the capacitance of the junction as long as $(2\pi i_c/\phi_0 C)^{1/2} RC \ll 1$, which is the condition for the high-damping limit to apply. The smallest capacitance at which our analysis is correct is about 10^{-14} F, by the neglect of other than Johnson noise, which imposes the condition $eV \ll k_BT$ while the "kinetic energy" is $CV^2 \sim k_BT$. The above analysis is as easy to carry out for the case of finite damping and the results would be very much the same.[14] This is so because the distribution of jumps is mainly determined by the exponential dependence of the lifetime for decay on $(\Delta\phi_x)^{3/2}$, the prefactor being relatively unimportant. It should be mentioned that the observed multiple quantum transitions at high[9] Li_c are not surprising.[15] The potential at the first jump then includes a large term $(\phi-\phi_x)^2/2L$, which imposes a slope toward large ϕ. The cosine term introduces hollows on that slope in which the flux may or may not stop on its way, according to how well the high-damping criterion is satisfied. This situation is analogous to the onset of voltage in the case of a Josephson junction driven from a current source.[16]

Finally, a few words should be said about approximating the normal current with a constant resistance.[17] By the fluctuation-dissipation theorem, the picture of associating the flux with a particle of thermal energy k_BT remains strictly correct as long as the normal current is not nonlinear in V. As to nonlinear dependence, we believe that all

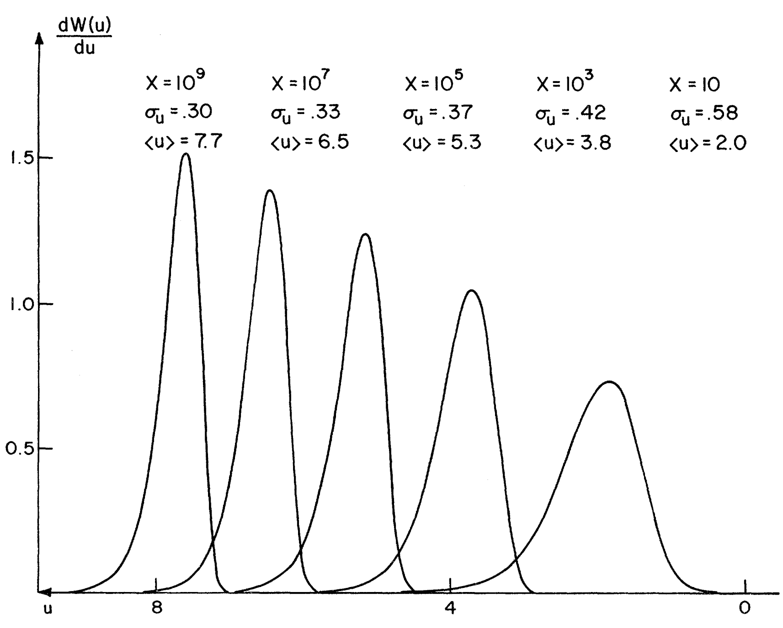

FIG. 2. Distribution $dW(u)/du$ for several values of X. The corresponding standard variations and mean values are indicated above the curves.

real weak links are asymmetric enough to eliminate the predicted logarithmic vanishing of R at small V for identical superconductors on the two sides of the junction. In any case, the value of the resistance does not appear in our results in any crucial fasion.

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Pressure Effect on Superconducting $NbSe_2$ and NbS_2†

R. E. Jones, Jr.,* H. R. Shanks, and D. K. Finnemore
Ames Laboratory-U. S. Atomic Energy Commission and Department of Physics, Iowa State University, Ames, Iowa 50010

and

B. Morosin
Sandia Laboratories, Albuquerque, New Mexico 87115

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The pressure dependence of the superconducting transition temperature T_c has been measured for NbS_2 and $NbSe_2$ in order to determine whether tunneling between laminar layers is a dominant factor controlling T_c. Measurements of the lattice constants by x-ray diffraction indicate that both materials are very compressible along the c axis, but T_c measurements do not correlate with the c-axis lattice constant. $NbSe_2$ shows a rapid change in T_c with pressure, whereas NbS_2 shows practically no change at all. T_c measurements correlate with intralaminar changes much better than they correlate with interlaminar spacings.

INTRODUCTION

Superconductivity in layer-structure compounds has special features associated with the extreme anisotropy of the material.[1-4] For these substances, the chemical binding within the trigonal prismatic laminar plane is very strong and transport properties parallel to the laminar are similar to those of ordinary metals. Transport properties perpendicular to the laminar, however, show a high impedance to particle motion because the chemical binding has a weak van der Waals character. In fact, the coupling between lamina is so weak that one is tempted to think of the 6-Å-thick lamina as independent layers and to discuss the material as though it were a two-dimensional su-