Brillouin Scattering: A Tool for the Measurement of Elastic and Photoelastic Constants

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(Received 26 August 1970)

After an outline of the Brillouin effect and of elastic waves in crystals, a method for the determination of elastic and photoelastic constants is analyzed. The authors propose a set of conditions with a view to obtaining accurately the numerical values of elastic and photoelastic constants and to ascertain their sign. The Brillouin-line intensities for scattering angles of 90° and 180° are presented for all crystal systems, except for the monoclinic and triclinic ones, and for the low-symmetry classes of the rhombohedric system.

I. INTRODUCTION

The measurement of frequency shifts of Brillouin lines permits the determination of the propagation velocity of elastic waves; this method has recently been improved, so that technical applications, such as the determination of the elastic constants of crystals, can now be easily performed. The principle of the method was put forward by Krishnan in 1955.[1] Also, the photoelastic constants may be calculated from measurements of the Brillouin-line intensities. The first observation of the Brillouin-scattered lines was made by Gross,[2,3] and the first accurate experimental verifications were made by Benedek *et al.*,[4] Chiao and Stoicheff,[5] and Cecchi[6] in 1964. Measurements of elastic (see e.g., Refs. 7 and 8) and photoelastic constants have since been performed.[9,10] This method has the advantage of avoiding the creation of important disturbances in the medium; thermal fluctuations provide the necessary elastic waves of small amplitude, and the crystal is studied in conditions near the mechanical equilibrium. Measurements are not subject to boundaries conditions, i.e., the shape of the sample, insofar as the usual dimensions are concerned. The results are obtained at frequencies of about 10 GHz, permitting dispersion studies.[11] Besides, intensity measurements allow the determination of elasto-optic constants. However, the necessity of using a transparent material restricts the application of the technique (although measurement of elastic constants of a nontransparent crystal by means of the Brillouin effect has been recently performed by Sandercock[12]). Moreover, the usual difficulties accompanying ultrasonic pulse methods (i.e., choice of the directions leading to the most precise measurements and determination of the signs of some of the constants) are increased by the unobservably low intensity of some Brillouin doublets.

Our attempt here has been to select conditions that allow the most accurate determination of the elastic and photoelastic constants of all crystal systems (monoclinic and triclinic systems excepted). They have been applied to measurements of the elastic constants of crystals belonging to the cubic,[13] the tetragonal,[14] the trigonal,[15] and the orthorhombic[16] systems. Measurements of photoelastic constants are under way.

II. THEORETICAL CONSIDERATIONS

A. Propagation of Small Amplitude Waves in an Anisotropic Medium

The components of the displacement $\vec{U}$ of an elemental volume of density ρ satisfy the following differential equation,[17]

$$\rho \ddot{U}_i = C_{ijkl} \frac{\partial^2 U_l}{\partial x_j \partial x_k} , \tag{1}$$

with C_{ijkl} being the components of the tensor of the elastic constants. (The summation convention is always implied.)

With solutions in the form of plane sinusoidal waves, Eq. (1) becomes

$$\gamma U_i = \Gamma_{il} U_l , \tag{2}$$

with

$$\gamma = \rho V^2 , \tag{3}$$

where V is the phase velocity and

$$\Gamma_{il} = C_{ijkl} Q_j Q_k . \tag{4}$$

Γ_{il} is symmetric with respect to an interchange of the subscripts i and l, and Q_j, Q_k are the direction cosines of the unit vector normal to the wave plane. The three directions of vibration $\vec{u}^s$, eigenvectors of the matrix Γ, are mutually perpendicular, because of the symmetry properties of Γ, and are therefore associated with one given propagation direction. In general, they are neither purely longitudinal, nor purely transverse. The three real and positive associated eigenvalues γ^s can be obtained by solving the following equation:

$$\det|\Gamma_{il} - \delta_{il}\gamma| = 0 . \tag{5}$$

Formula (5) links the propagation velocity of elastic waves to the elastic constants of the crystal. The tensor Γ_{il} is centrosymmetric; the properties connected with the propagation of elastic waves are the same for all crystalline classes belonging to the same Laüe group.

B. Pure Modes

In the case of some propagation directions called "first-kind" pure-mode directions, one of the possible vibrations is purely longitudinal, and the corresponding waves are then pure-compressional waves; the two other possible vibrations are, in that case, purely transverse (corresponding to them are pure-shear waves). In the pure-mode directions of the "second-kind," one of the vibrations is purely transverse; the other two can be of any kind.

Some pure-mode directions are determined by considerations of symmetry.[18] As a matter of fact, a "first-kind" direction corresponds to every propagation direction whose vector $\vec{Q}$ is parallel to one of the axes of symmetry of the Laüe group. A degeneracy in mode frequencies occurs if the axis of symmetry is of a higher order than 2.[19,20,21]

On the other hand, a purely transverse "second-kind" vibration (whose direction is perpendicular to the plane of symmetry, the other two directions of vibration being contained in the plane) corresponds to every vector $\vec{Q}$ contained in the plane of symmetry. The utilization of these pure modes, determined through symmetry, for the determination of elastic constants is routine in the usual ultrasonic methods.

The general conditions for determining pure modes are obtained by writing in Eq. (2), $\vec{Q}=\vec{u}$ for the "first-kind" modes, and $\vec{Q}\cdot\vec{u}=0$ for the "second-kind" modes.[22,23] It is evident that the results contain not only the directions obtained through considerations of symmetry, but also directions which depend on the numerical value of the elastic constants, and are useful for determining the signs of some of these constants.

C. Brillouin Scattering

Brillouin scattering is usually compared to x-ray diffraction from crystals, because of the formal analogy between the conditions of interference (Bragg relation). The phenomenon is described as a selective reflection of the incident electromagnetic waves on lattice planes in case of x-ray diffraction, or on planes of maximum density in case of Brillouin scattering. In the latter case, a shift of frequency due to the Doppler effect is observed because the planes are in motion. The inelastic scattering of x rays by phonons is obviously the same phenomenon as Brillouin scattering.

One of the possible theoretical approaches describes light scattering as the radiation of polarization density created by the incident electromagnetic field.[6,7,24] If the size of the scattering volume is infinitely large with regard to the wavelength, there is only one "efficient" acoustic wave which is responsible for the scattering; its wave vector $\vec{\chi}_0$ is determined by momentum conservation. In fact, owing to the finite size of the scattering volume, all the acoustic waves whose wave vectors $\vec{\chi}$ are approximately equal to $\vec{\chi}_0$, contribute to the scattering[6]: Therefore, a broadening takes place and enhances the classical broadening due to the attenuation of hypersonic waves.[25]

We now define some of the notations to be used: Λ and N are the wavelength and the frequency of the "efficient" elastic wave; λ and ν are the wavelength in vacuum and the frequency of incident light, respectively; $\vec{q}$ and $\vec{q}'$ are unit vectors of the perpendiculars to the incident and scattered light-wave planes; $\vec{e}_1$ and $\vec{e}_2$ ($\vec{e}_1'$ and $\vec{e}_2'$) are unit vectors of the two vibrations which propagate without alteration along $\vec{q}$ ($\vec{q}'$), (we are only going to consider cases which will be sufficiently simple for $\vec{e}_2$ and $\vec{e}_2'$ to remain in the scattering plane, and therefore, $\vec{e}_1$ and $\vec{e}_1'$ are perpendicular to this plane); n_μ and $n'_{\mu'}$ are the indices of refraction of the medium for the light polarized along $\vec{e}_\mu$ and $\vec{e}'_{\mu'}$, respectively; and $\theta=(\vec{q},\vec{q}')$ is the scattering angle.

If the incident and the scattered beams are polarized along $\vec{e}_\mu$ and $\vec{e}'_{\mu'}$, the "efficient" wave is defined by

$$\Lambda_{\mu\mu'}=\lambda(n_\mu^2+n_{\mu'}'^2-2n_\mu n'_{\mu'}\cos\theta)^{-1/2}, \tag{6}$$

$$\vec{Q}_{\mu\mu'}=\frac{n'_{\mu'}\vec{q}'-n_\mu\vec{q}}{|n'_{\mu'}\vec{q}'-n_\mu\vec{q}|}\;. \tag{7}$$

Three directions of vibration and three velocities of propagation correspond to each $\vec{Q}$ direction. The frequency shifts are then given by the expression,

$$\delta\nu^s_{\mu\mu'}/\nu=\pm(V^s_{\mu\mu'}/c)(n_\mu^2+n_{\mu'}'^2-2n_\mu n'_{\mu'}\cos\theta)^{1/2}\,. \tag{8}$$

In case the incident beam is not polarized and the polarization of the scattered beam is not measured, each of the six lines is, because of birefringence, composed of four components separated by a few MHz.[26]

The flux of light scattered by an elementary cubic volume with an edge a is given by (taking into account the losses due to reflection at the inlet and outlet faces of the crystal)

$$\Phi^s_{\mu\mu'}=\frac{8\pi^2kT}{\lambda^4}\,\frac{n_\mu^4}{(n_\mu+1)^2}\,\frac{n_{\mu'}'^4}{(n'_{\mu'}+1)^2}\,\beta^s_{\mu\mu'}\,\omega a^3\epsilon_\mu\,, \tag{9}$$

where k is the Boltzmann constant, T is the absolute temperature, ω is the scattered-light solid angle calculated externally to the crystal, and ϵ_μ is the illuminance of polarized incident light along $\vec{e}_\mu$. For all crystal systems, except the monoclinic and triclinic ones,

$$\beta^s_{\mu\mu'} = \frac{1}{n^4_\mu n'^4_{\mu'}} \; \frac{(n_i^2 n_j^2 e'_{\mu',j} B^s_{ij} e_{\mu,i})^2}{\rho\,(V^s_{\mu\mu'})^2} \,, \tag{10}$$

where

$$B^s_{ij} = p'_{ijkl} u^s_k Q_l \,, \tag{11}$$

n_i, n_j are the principal refractive indices, $e_{\mu,i}$ and $e'_{\mu',j}$ are the i and j components of the $\vec{e}_\mu$ and $\vec{e}'_{\mu'}$ vectors, respectively, and u^s_k is the k component of the $\vec{u}^s$ vector. p'_{ijkl} is the new photoelastic tensor which is defined by

$$(\delta\kappa^{-1})_{ij} = p'_{ijkl} \, \frac{\partial U_k}{\partial x_l} \,, \tag{12}$$

where $(\delta\kappa^{-1})$ is the variation of the inverse optical-dielectric constant and $\partial U_k/\partial x_l$ is the displacement gradient.

This tensor, introduced by Nelson and Lax in 1970,[27,9] allows one to calculate the change in the inverse dielectric tensor due to the strains and the rotations associated with the acoustic waves in an optically anisotropic medium. The rotational effect which takes place in shear waves had been omitted until then. It is convenient to put

$$p'_{ijkl} = p_{ijkl} + p_{ij(kl)} \,, \tag{13}$$

where p_{ijkl} is the Pöckels photoelastic tensor (symmetric with respect to interchange of k and l), and $p_{ij(kl)}$ is a tensor given by

$$p_{ij(kl)} = \tfrac{1}{2}\left[(\kappa^{-1})_{il}\delta_{kj} + (\kappa^{-1})_{lj}\delta_{ik} - (\kappa^{-1})_{ik}\delta_{lj} - (\kappa^{-1})_{kj}\delta_{il}\right] . \tag{14}$$

In all crystal systems where the principal axes of the dielectric tensor coincide with the crystallographic axes, we have

$$p_{ij(kl)} = \tfrac{1}{2}\,(\delta_{il}\delta_{kj} - \delta_{ik}\delta_{lj})\,(1/n_i^2 - 1/n_j^2) \,. \tag{15}$$

We shall use only the above expression since monoclinic and triclinic systems have been excluded from the scope of the present work.

III. POSITION OF THE PROBLEM

A. Elastic Constants

The solution of Eq. (5) enables us to establish a system of n equations, the unknown quantities of which are the n elastic constants, and in which the coefficients depend on $\vec{Q}$, the chosen directions. Therefore, the values of the elastic constants can be determined by measuring γ in a sufficient number of arbitrary directions p of $\vec{Q}$. The calculations may be carried out from the n values of γ and the $3p$ values of the components of $\vec{Q}$.[28–31]

The precision of the measurements of γ depends not only on the intensity of the corresponding components, but also on the resolving power of the spectrometer, on the intensity of the line without any frequency shift (due to the defects of the crystal and to stray light), and on the intensity and the frequency difference of the adjacent components, which can often provoke displacements of the maximum of a line through overlapping. The problem finally consists of determining $\vec{Q}$, $\vec{q}$, μ, and μ', so as to obtain the maximum accuracy of the C_{ij} values.

Putting such a complicated problem of optimization into equation form is difficult, and unnecessary in order to achieve good accuracy. We prefer to use a semiempirical method similar to the usual ultrasonic techniques.

The choice of the angle of scattering can be independently based on previous experiments. As a matter of fact, it is necessary to avoid spurious light as much as possible in order to carry out precise measurements on transverse components. Experience shows that it is preferable to use a 90° scattering angle instead of having recourse to backscattering. Moreover, calculation demonstrates that β, the scattering factor, is frequently zero for transverse components in case of backscattering. However, if the experimental arrangement for 90° scattering cannot be used for technical reasons, the backscattering may be attempted. This is the case for nontransparent materials[12] or when the samples are too small to permit the cutting of the necessary faces.

In the same manner as in the method of determination of elastic constants through propagation of ultrasound,[32] the $\vec{Q}$ vectors are chosen according to the following criteria: (a) The constant which is to be measured must appear in the characteristic equation (5) with largest possible coefficient; (b) this equation should be expressed very simply, so that the number of measurements necessary for determining this constant can be as small as possible, in order to achieve precision of calculations—such as, in particular, the case when Eq. (5) can be expressed in factored form.

On the other hand, there exists an infinity of possible directions of $\vec{q}$ and $\vec{q}'$ for a given $\vec{Q}$ direction, corresponding to different values of the intensity factor $\beta^s_{\mu\mu'}$ of each particular line. Since the tensor B^s_{ij} has been determined by (11) for each of the three acoustic modes expressed by $\vec{Q}$ and $\vec{u}^s$, the value of $\vec{q}$ should be chosen first (which thereby determines $\vec{q}'$ and the two possible values of $\vec{e}_\mu$ and $\vec{e}'_{\mu'}$) depending on the following criteria: The state of polarization of lines should permit their identification; the intensity should be maximal for the line under consideration.

B. Photoelastic Constants

In the same way, β, the scattering factor measured by comparison with that of benzene[33] or toluene[34] for a sufficient number of orientations, allows one to determine the values of photoelastic constants from (10) and (11). For the same reasons, 90° scattering will be chosen, with orientations that give simple relations between β and the photoelastic constants. Backscattering is used only when other measurements are necessary for ascertaining the sign of these constants.

IV. GENERAL STATEMENTS

Starting from these considerations, we have tried to provide a method for determining elastic constants for all systems, except the monoclinic, the triclinic, and the rhombohedric group R_2. The conditions are as follows: (a) $\vec{Q}$ must be parallel to one of the crystallographic axes[35] or the bisector of these axes (except in a few cases), and (b) for 90° scattering, the scattering angle must be equal to 90° inside the crystal.

For typographic reasons, the tables are presented in the following order: Backscattering, and then 90° scattering.

A. Backscattering

Samples. Samples having faces perpendicular to the crystallographic axes and bisectors are required. In order to avoid stray light, which is generally strong, samples with oblique faces with respect to the crystallographic axes are often used, allowing the use of Brewster's angle to remove specular reflections. Unfortunately, this technique requires a high number of samples. Alternatively, a device for filtering the unshifted line may be used (see, e.g., Refs. 12 and 36–38).

Results. For each Laüe group, the values of γ, $\vec{Q}$, $\vec{u}$, $\vec{e}_\mu$, $\vec{e}'_{\mu'}$, and the Brillouin intensities are presented in table form. When necessary, the values of γ and $\vec{u}$ are reassembled in a distinct table. The notation refers to one acoustic mode, i.e., to a given $\vec{Q}$, $\vec{u}^s$ couple.

B. 90° Scattering

Experimental process. What we call a "scattering case" is determined by a $\vec{Q}$ direction, a scattering plane, and a polarization state. Given the condition $\vec{q}\cdot\vec{q}'=0$, the components of $\vec{q}$ and $\vec{q}'$ may be calculated by applying (7) in each scattering case. Four samples are required for the most general case (orthorhombic group). These four samples are cut in the form of rectangular parallelepipeds with the following faces: (a) (100), (010), (001); (b) (110), ($\bar{1}$10), (001); (c) (011), (0$\bar{1}$1), (100); and (d) (101), (10$\bar{1}$), (010).

Only two samples are necessary in the case of the cubic system. If $\vec{q}$ and $\vec{q}'$ are perpendicular to the inlet and outlet faces, the crystal will be positioned on a spectrometer adjusted for an angle of 90°. This is the case when $n_\mu = n'_{\mu'}$, or when the birefringence effect can be neglected within the accuracy of the experiments. Otherwise, $\vec{q}$, $\vec{q}'$ and the refractive indices being known, the incidence angles i and i' of the incident and scattered rays are calculated, and consequently, the angle θ' between the incident and scattered rays outside of the crystal is also calculated. After having properly oriented the laser beam, the crystal is arranged so as to obtain the angle i between the incident beam and the direction perpendicular to the face.

Results. Table IV shows the $\vec{q}$ and $\vec{q}'$ values for any scattering case of an optically biaxial crystal. The indices refer to the scattering cases. The table is used for optically uniaxial crystals with $n_2=n_1$. For each Laüe group, a table has been provided showing the $\vec{e}_\mu$ and $\vec{e}'_{\mu'}$ values and the scattering factor (the same notation of acoustic modes for a Laüe group is used in the backscattering and 90° scattering tables).

C. Notations

When dealing with the elastic and Pöckels photoelastic constants below, we will make use of the usual contracted notation of two indices running from 1 to 6.

The elements $p_{ij(kl)}$ are written $p_{m(n)}$ with the correspondences:

$$1,1\to 1\ ,\quad 2,1\to \bar{6}\ ,\quad 3,1\to 5\ ,$$
$$1,2\to 6\ ,\quad 2,2\to 2\ ,\quad 3,2\to \bar{4}\ ,$$
$$1,3\to \bar{5}\ ,\quad 2,3\to 4\ ,\quad 3,3\to 3\ .$$

With the same convention, the elements p'_{ijkl} are written p'_{mn}.

The intensity factors relevant to one acoustic mode have been calculated for two perpendicular scattering planes. These planes are called a and b. The components of $\vec{q}$ are called q_i^α, where α refers to the scattering case. The scattering factors are called $\beta^M_{\mu\mu'}(\theta)$ where the index M refers to the acoustic mode and the scattering plane, and the indices μ and μ' refer to the polarization directions. The state of polarization of the incident and scattered beams polarized in the directions $\vec{e}_\mu$ and $\vec{e}'_{\mu'}$ will be designated as polarization (μ, μ'). For simplicity, the index which refers to the acoustic modes is omitted for the notation of $\vec{u}$.

V. DETERMINATION OF ELASTIC CONSTANTS BY MEANS OF 90° SCATTERING EXPERIMENTS

A. Orthorhombic Group (Classes 222, *mm*2, and *mmm*)

A good starting point for this study is the orthorhombic group 0 (Tables II and III). Equation (5) is completely factored for the (100), (010), and (001) directions and partially so for any $\vec{Q}$ vectors lying in the coordinate planes. C_{11} is thus determined with the help of measurements made for the modes $1a$ and $1b$ in polarization (1, 1) and C_{22} and C_{33} by measurements similar to the ones made for the modes

4 and 7, respectively. The value of C_{44} can be deduced from measurements made for the modes 5*b* and 9*a*, in polarization (1, 2) or (2, 1), and the values of C_{55} and C_{66} by measurements similar to the ones made for modes 3*a*, 8*b* and 2*b*, 6*a*, respectively. If some transverse intensity factors are too weak to allow these last measurements, the modes 12, 15, and 18 can provide a solution. Each of these six constants is obtained from one measurement of γ only: In this way maximum precision is achieved.

The measurements for the modes 10, 11, 13, 14, 16, and 17 supply the experimental values of the sums $S_1 = (C_{12} + C_{66})$, $S_2 = (C_{13} + C_{55})$, and $S_3 = (C_{23} + C_{44})$. On the other hand, the signs of these sums cannot be directly determined by the measurement of elastic velocities, whatever the direction of propagation may be. Moreover, it is not always possible to lift the indeterminacy when starting from conditions of stability (positive energy of deformation).

The method described by Fisher and McSkimin[39] does not apply unless one can distinguish *a priori* the longitudinal mode from the mixed one, which is possible by the pulse-echo method. The problem has been solved by observing the fact that the propagation velocity in the planes of symmetry does not depend on the sign of the sums S_1, S_2, and S_3. For example in the plane $X0Y$, the values of γ^s are roots of the following equation:

$$(C_{55} Q_1^2 + C_{44} Q_2^2 - \gamma)[\gamma^2 - (C_{11} Q_1^2 + C_{22} Q_2^2 + C_{66})\gamma + (C_{11} Q_1^2 + C_{66} Q_2^2)(C_{66} Q_1^2 + C_{22} Q_2^2) - (C_{12} + C_{66})^2 Q_1^2 Q_2^2] = 0 . \quad (16)$$

On the other hand, the propagation direction of pure modes of the first kind exists in this plane and is expressed by

$$\frac{Q_2}{Q_1} = \pm\left(\frac{C_{11} - 2C_{66} - C_{12}}{C_{22} - 2C_{66} - C_{12}}\right)^{1/2}, \quad Q_3 = 0 . \quad (17)$$

The corresponding value of γ for the pure-longitudinal mode L is given by

$$\gamma^L = \frac{(C_{22} - 2C_{66} - C_{12})\, C_{11} + (C_{11} - 2C_{66} - C_{12})(2C_{66} + C_{12})}{C_{11} + C_{22} - 4C_{66} - 2C_{12}} . \quad (18)$$

We should calculate the presumed direction of a pure mode and the value of γ for each one of the possible values of C_{12}. Comparison with the roots of Eq. (12) lifts the indeterminacy. The signs of S_2 and S_3 are determined in the same way.

B. Quadratic Groups T_1 (4*mm*, $\bar{4}2m$, 422, and 4/*mmm* Classes) and Hexagonal H_1 (6*mm*, $\bar{6}2m$, 622, and 6/*mmm* Classes)

The results for the quadratic group T_1 (Tables VII and IX) stated below can be applied to the hexagonal group H_1 by taking into account the supplementary relations relevant to H_1,

$$C_{66} = \tfrac{1}{2}(C_{11} - C_{12}) , \quad (19)$$

$$p_{66} = \tfrac{1}{2}(p_{11} - p_{12}) . \quad (20)$$

The only modes indicated in Table VII—as well as in the following tables, are those that lead to intensities different from zero. C_{11}, C_{33}, C_{44}, and C_{66} are readily measured (modes 1*a* and *b*; 7; 3*a*, 8, 12; 2*b*; respectively, with 15 for cross checking). C_{12} is calculated from the measurements of γ^{10}. The measurements of γ^{13} or γ^{14} give two possible values for C_{13}; the above described method (Sec. V A) may be applied for choosing the correct value. If necessary, the value of $\frac{1}{2}(C_{11} - C_{12})$ for T_1 can be measured by the mode 11, e.g., taking the plane $(\bar{1}10)$ for the scattering plane. The calculation of intensities is tedious and will not be given here. It should be noted that for H_1 the mode 10 does not supply any new information as, in fact, $\gamma = C_{11}$.

C. Hexagonal H_2 Group (Classes 6/*m*, 6, and $\bar{6}$)

There is no need to distinguish between the hexagonal group H_1 and H_2 (Table VIII) when studying the propagation of elastic waves; in some cases of scattering only the intensities are modified.

D. Quadratic Group T_2 (Classes 4/*m*, 4, and $\bar{4}$)

The method herein proposed is an adaptation of the method put forward by Alton and Barlow.[31] It is possible to apply it if the velocities of longitudinal and mixed modes for a given direction can be measured (this case is frequently found in Brillouin scattering[7]). The calculations for this group are given in Tables XI and XII. C_{33} and C_{44} are readily measured. C_{11}, C_{12}, C_{16}, and C_{66} are calculated from measurements of γ^s for the following values of $\vec{Q}$: (1, 0, 0), $2^{-1/2}(1, 1, 0)$, and $(\sqrt{3}/2, \frac{1}{2}, 0)$. As a matter of fact, Eq. (5) is factorable if $\vec{Q}$ lies in the (001) plane. Thus

$$(C_{44} - \gamma)\{\gamma^2 - \gamma(C_{11} + C_{66}) - [C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66}) Q_1 Q_2]^2 + (C_{11} Q_1^2 + C_{66} Q_2^2 + 2C_{16} Q_1 Q_2)(C_{66} Q_1^2 + C_{11} Q_2^2 - 2C_{16} Q_1 Q_2)\} = 0 . \quad (21)$$

γ^L and γ^M (roots of the quadratic equation) are the values of ρV^2 for the longitudinal and mixed modes, respectively. The sum $(\gamma^L + \gamma^M)$ remains constant and a first relation is written,

$$S = \gamma^L + \gamma^M = C_{11} + C_{66} . \quad (22)$$

If $\vec{Q} = (1, 0, 0)$,

$$(\gamma^L)^2 + (\gamma^M)^2 = A \;, \tag{23}$$

where

$$A = C_{11}^2 + C_{66}^2 + 2C_{16}^2 \;. \tag{24}$$

If $\vec{Q} = 2^{-1/2}(1, 1, 0)$,

$$(\gamma^L)^2 + (\gamma^M)^2 = A + \tfrac{1}{2} B \;, \tag{25}$$

where

$$B = C_{12}^2 - C_{11}^2 + 2C_{66}(C_{11} + C_{12}) \;. \tag{26}$$

Finally, if $\vec{Q} = (\sqrt{3}/2, \frac{1}{2}, 0)$,

$$(\gamma^L)^2 + (\gamma^M)^2 = A + (3/4)B - (\sqrt{3}/2)D \;, \tag{27}$$

where

$$D = C_{16}(C_{11} + C_{12}) \;. \tag{28}$$

After elimination of terms other than C_{11} between (22), (23), (25), and (27), a quartic equation is obtained. To each given value of C_{11} there corresponds one single value of C_{12}, C_{16}, and C_{66}. Stability conditions for T_2 are given by[40]

$$C_{11}, C_{33}, C_{44}, \quad C_{66} > 0 \;, \qquad C_{11} > C_{12}, \quad C_{11}C_{66} > C_{16}^2, \quad C_{11}C_{33} > C_{13}^2 \;. \tag{29}$$

It is thus generally possible to exclude some of the values found; however the method proposed in Sec. V A must be used to complete the determination. As a matter of fact, the quadratic factor in (21) may be written

$$\gamma^2 - S\gamma + 2DQ_1Q_2(Q_2^2 - Q_1^2) - BQ_1^2Q_2^2 + \tfrac{1}{2}(S^2 - A) = 0. \tag{30}$$

The values of γ^L and γ^M are also known for any $\vec{Q}$ vector lying in the (001) plane. To each possible value of the constants, the directions of pure modes, in this plane, are given by

$$m = Q_2/Q_1$$
$$= -b + 2^{1/2}[1 + b^2 \mp b(1 + b^2)^{1/2}]^{1/2} \pm (1 + b^2)^{1/2} \;, \tag{31}$$

where

$$b = (C_{11} - 2C_{66} - C_{12})/4C_{16} \;. \tag{32}$$

The corresponding value of γ^L is

$$\gamma^L = (1 + m^2)^{-1}[C_{11} + m^2(2C_{66} + C_{12}) + m(3 - m^2)C_{16}] \;. \tag{33}$$

A comparison between these values and the roots of Eq. (30) lifts the ambiguity.

For the measurement of C_{13}, we use the following property: The planes containing $0Z$ and one of the first-kind pure-mode directions (given by m) are planes containing second-kind pure-mode directions. Let us consider such a $\vec{Q}$ direction given by

$$Q_1 = (1 + m^2 + k^2)^{-1/2} \;,$$

$$Q_2 = m(1 + m^2 + k^2)^{-1/2} \;,$$

$$Q_3 = k(1 + m^2 + k^2)^{-1/2} \;,$$

where k is arbitrary. With the following choice of a right-handed rectangular-coordinate system: $0X'$ parallel to $\vec{Q}_1$ $0Y'$ lying in the crystallographic (001) plane, Eq. (5) is written[41]

$$(C'_{66} - \gamma)[(C'_{11} - \gamma)(C'_{55} - \gamma) - C'^2_{15}] = 0 \;, \tag{34}$$

with

$$C'_{11} = (Q_1^4 + Q_2^4)C_{11} + 2Q_1^2Q_2^2C_{12} + 2Q_3^2(Q_1^2 + Q_2^2)(C_{13} + 2C_{44}) + Q_3^4C_{33} + 4Q_1^2Q_2^2C_{66} + 4Q_1Q_2(Q_1^2 - Q_2^2)C_{16} \;, \tag{35}$$

$$C'_{55} = Q_3^2(Q_1^2 + Q_2^2)^{-1}(Q_1^4 + Q_2^4)C_{11} + 2Q_2^2Q_3^2(Q_1^2 + Q_2^2)^{-1}(C_{12} + 2C_{66}) - Q_3^2(Q_1^2 + Q_2^2)(2C_{13} - C_{33}) + (1 - 2Q_3^2)^2C_{44} + 4Q_1Q_2Q_3^2(Q_1^2 + Q_2^2)^{-1}(Q_1^2 - Q_2^2)C_{16} \;, \tag{36}$$

$$C'_{66} = 2Q_1^2Q_2^2(Q_1^2 + Q_2^2)^{-1}(C_{11} - C_{12}) + Q_3^2C_{44} + (Q_1^2 + Q_2^2)^{-1}(Q_1^2 - Q_2^2)^2C_{66} - 4Q_1Q_2(Q_1^2 + Q_2^2)^{-1}(Q_1^2 - Q_2^2)C_{16} \;, \tag{37}$$

$$C'_{15} = -Q_3(Q_1^2 + Q_2^2)^{-1/2}(Q_1^4 + Q_2^4)C_{11} - 2Q_1^2Q_2^2Q_3(Q_1^2 + Q_2^2)^{-1/2}C_{12} + Q_3(Q_1^2 + Q_2^2)^{1/2}(1 - 2Q_3^2)(C_{13} + 2C_{44}) + Q_3^3(Q_1^2 + Q_2^2)^{1/2}C_{33} - 4Q_1^2Q_2^2Q_3(Q_1^2 + Q_2^2)^{-1/2}C_{66} - 4Q_1Q_2Q_3(Q_1^2 + Q_2^2)^{-1/2}(Q_1^2 - Q_2^2)C_{16} \;. \tag{38}$$

The measurement of γ^L and γ^M for this direction gives two possible values for C_{13}. The ambiguity must be lifted by the method proposed in Sec. V A, from the first-kind pure-mode direction given by

$$Q_2/Q_1 = m \;, \tag{39}$$

$$\frac{Q_3}{Q_1} = l = \left(\frac{C_{11} - 2C_{44} - C_{13}}{C_{33} - 2C_{44} - C_{13}}(1 + m^2) + \frac{2(C_{11} - 2C_{66} - C_{12})m^2(m^2 + 1)}{(C_{33} - 2C_{44} - C_{13})(m^4 - 6m^2 + 1)}\right)^{1/2} . \tag{40}$$

In this direction one finds

$$\gamma^L = (1 + m^2 + l^2)^{-1}[C_{11} + m^2(2C_{66} + C_{12}) + l^2(2C_{44} + C_{13}) + m(3 - m^2)C_{16}] \;. \tag{41}$$

For this class, it is necessary to cut two samples so as to obtain

$$\vec{Q} = (\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}, 0) \quad \text{and} \quad \vec{Q} \parallel (1, m, k) \;.$$

E. Cubic C_1 (43m, 432, and $m3m$ Classes), Cubic C_2 (23 and $m3$ Classes), and Isotropic Groups

There is no necessity to distinguish between cubic groups C_1 and C_2 (Table XIV) for the propagation

TABLE I. Orthorhombic group 0. $\theta = 180°$. The values of γ and $\vec{u}$ are given in Table III.

Not.	γ	$\vec{Q}$	$\vec{u}$	Type of mode	$\vec{e}_1 = \vec{e}_1'$	$\vec{e}_2 = \vec{e}_2'$	β_{11}	$\beta_{12} = \beta_{21}$	β_{22}
1	C_{11}	(1,0,0)	(1,0,0)	L	(0,1,0)	(0,0,1)	p_{21}^2/C_{11}	0	p_{31}^2/C_{11}
2	C_{66}	(1,0,0)	(0,1,0)	T	(0,1,0)	(0,0,1)	0	0	0
3	C_{55}	(1,0,0)	(0,0,1)	T	(0,1,0)	(0,0,1)	0	0	0
4	C_{22}	(0,1,0)	(0,1,0)	L	(0,0,1)	(1,0,0)	p_{32}^2/C_{22}	0	p_{12}^2/C_{22}
5	C_{44}	(0,1,0)	(0,0,1)	T	(0,0,1)	(1,0,0)	0	0	0
6	C_{66}	(0,1,0)	(1,0,0)	T	(0,0,1)	(1,0,0)	0	0	0
7	C_{33}	(0,0,1)	(0,0,1)	L	(1,0,0)	(0,1,0)	p_{13}^2/C_{33}	0	p_{23}^2/C_{33}
8	C_{55}	(0,0,1)	(1,0,0)	T	(1,0,0)	(0,1,0)	0	0	0
9	C_{44}	(0,0,1)	(0,1,0)	T	(1,0,0)	(0,1,0)	0	0	0
10	γ^{10}	$2^{-\frac{1}{2}}(1,1,0)$	$(u_1^{10},u_2^{10},0)$	L	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	$(p_{31}u_1 + p_{32}u_2)^2/2\gamma$	0	$\frac{(n_1^2+n_2^2)^4}{128\, n_1^8 n_2^8\, \gamma}\left(n_1^4(p_{11}u_1 + p_{12}u_2) + n_2^4(p_{21}u_1 + p_{22}u_2) - 2\, n_1^2 n_2^2(p'_{66}u_1 + p'_{6\bar{6}}u_2)\right)^2$
11	γ^{11}	$2^{-\frac{1}{2}}(1,1,0)$	$(u_1^{11},u_2^{11},0)$	T	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	$(p_{31}u_1 + p_{32}u_2)^2/2\gamma$	0	$\frac{(n_1^2+n_2^2)^4}{128\, n_1^8 n_2^8\, \gamma}\left(n_1^4(p_{11}u_1 + p_{12}u_2) + n_2^4(p_{21}u_1 + p_{22}u_2) - 2\, n_1^2 n_2^2(p'_{66}u_1 + p'_{6\bar{6}}u_2)\right)^2$
12	$\frac{1}{2}(C_{44} + C_{55})$	$2^{-\frac{1}{2}}(1,1,0)$	(0,0,1)	T	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	0	$\frac{(n_1^2+n_2^2)^2}{8\, n_1^4 n_2^4\, \gamma}(n_1^2 p'_{55} - n_2^2 p'_{4\bar{4}})^2$	0
13	γ^{13}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{13},u_3^{13})$	L	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$	0	$\frac{(n_2^2+n_3^2)^4}{128\, n_2^8 n_3^8\, \gamma}\left(n_2^4(p_{22}u_2 + p_{23}u_3) + n_3^4(p_{32}u_2 + p_{33}u_3) - 2\, n_2^2 n_3^2(p'_{44}u_2 + p'_{4\bar{4}}u_3)\right)^2$
14	γ^{14}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{14},u_3^{14})$	M	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$	0	$\frac{(n_2^2+n_3^2)^4}{128\, n_2^8 n_3^8\, \gamma}\left(n_2^4(p_{22}u_2 + p_{23}u_3) + n_3^4(p_{32}u_2 + p_{33}u_3) - 2\, n_2^2 n_3^2(p'_{44}u_2 + p'_{4\bar{4}}u_3)\right)^2$
15	$\frac{1}{2}(C_{55} + C_{66})$	$2^{-\frac{1}{2}}(0,1,1)$	(1,0,0)	T	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	0	$\frac{(n_2^2+n_3^2)^2}{8\, n_2^4 n_3^4\, \gamma}(n_2^2 p'_{66} - n_3^2 p'_{5\bar{5}})^2$	0
16	γ^{16}	$2^{-\frac{1}{2}}(1,0,1)$	$(u_1^{16},0,u_3^{16})$	L	(0,1,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$(p_{23}u_3 + p_{21}u_1)^2/2\gamma$	0	$\frac{(n_1^2+n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}\left(n_1^4(p_{11}u_1 + p_{13}u_3) + n_3^4(p_{31}u_1 + p_{33}u_3) - 2\, n_1^2 n_3^2(p'_{5\bar{5}}u_1 + p'_{55}u_3)\right)^2$
17	γ^{17}	$2^{-\frac{1}{2}}(1,0,1)$	$(u_1^{17},0,u_3^{17})$	M	(0,1,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$(p_{23}u_3 + p_{21}u_1)^2/2\gamma$	0	$\frac{(n_1^2+n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}\left(n_1^4(p_{11}u_1 + p_{13}u_3) + n_3^4(p_{31}u_1 + p_{33}u_3) - 2\, n_1^2 n_3^2(p'_{5\bar{5}}u_1 + p'_{55}u_3)\right)^2$
18	$\frac{1}{2}(C_{44} + C_{66})$	$2^{-\frac{1}{2}}(1,0,1)$	(0,1,0)	T	(0,1,0)	$2^{-\frac{1}{2}}(-1,0,1)$	0	$\frac{(n_1^2+n_3^2)^2}{8\, n_1^4 n_3^4\, \gamma}(n_2^2 p'_{44} - n_3^2 p'_{6\bar{6}})^2$	0

TABLE II. Orthorhombic group 0. $\theta = 90°$. The values of γ and

$\vec{\sigma}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}\,'_{\mu'}$	Longitudinal mode Not.	γ	$R_{\mu\mu'}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	(0,0,1)	1a	C_{11}	p_{31}^2/C_{11}
(1,0,0)	$(q_1^1,q_2^1,0)$	(0,0,1)	$(-q_1^1,-q_2^1,0)$			0
(1,0,0)	$(q_1^2,q_2^2,0)$	$(q_2^2,-q_1^2,0)$	(0,0,1)			0
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$			$\frac{(n_1^2+n_2^2)^4}{64\,n_1^8\,n_2^8\,C_{11}}(n_1^4p_{11}-n_2^4p_{21})^2$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	(0,-1,0)	(0,-1,0)	1b	C_{11}	p_{21}^2/C_{11}
(1,0,0)	$(q_1^3,0,q_3^3)$	(0,-1,0)	$(-q_1^3,0,-q_3^3)$			0
(1,0,0)	$(q_1^4,0,q_3^4)$	$(q_3^4,0,-q_1^4)$	(0,-1,0)			0
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(1,0,-1)$			$\frac{(n_1^2+n_3^2)^4}{64\,n_1^8\,n_3^8\,C_{11}}(n_1^4p_{11}-n_3^4p_{31})^2$
(0,1,0)	$2^{-\frac{1}{2}}(0,-1,1)$	(1,0,0)	(1,0,0)	4a	C_{22}	p_{12}^2/C_{22}
(0,1,0)	$(0,q_2^5,q_3^5)$	(1,0,0)	$(0,-q_2^5,-q_3^5)$			0
(0,1,0)	$(0,q_2^6,q_3^6)$	$(0,q_3^6,-q_2^6)$	(1,0,0)			0
(0,1,0)	$2^{-\frac{1}{2}}(0,-1,1)$	$2^{-\frac{1}{2}}(0,1,1)$	$2^{-\frac{1}{2}}(0,1,-1)$			$\frac{(n_2^2+n_3^2)^4}{64\,n_2^8\,n_3^8\,C_{22}}(n_2^4p_{22}-n_3^4p_{32})^2$
(0,1,0)	$2^{-\frac{1}{2}}(1,-1,0)$	(0,0,-1)	(0,0,-1)	4b	C_{22}	p_{32}^2/C_{22}
(0,1,0)	$(q_1^7,q_2^7,0)$	(0,0,-1)	$(-q_1^7,-q_2^7,0)$			0
(0,1,0)	$(q_1^8,q_2^8,0)$	$(-q_2^8,q_1^8,0)$	(0,0,-1)			0
(0,1,0)	$2^{-\frac{1}{2}}(1,-1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(-1,1,0)$			$\frac{(n_1^2+n_2^2)^4}{64\,n_1^8\,n_2^8\,C_{22}}(n_1^4p_{12}-n_2^4p_{22})^2$
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	(0,1,0)	(0,1,0)	7a	C_{33}	p_{23}^2/C_{33}
(0,0,1)	$(q_1^9,0,q_3^9)$	(0,1,0)	$(-q_1^9,0,-q_3^9)$			0
(0,0,1)	$(q_1^{10},0,q_3^{10})$	$(-q_3^{10},0,q_1^{10})$	(0,1,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(-1,0,1)$			$\frac{(n_1^2+n_3^2)^4}{64\,n_1^8\,n_3^8\,C_{33}}(n_1^4p_{13}-n_3^4p_{33})^2$
(0,0,1)	$2^{-\frac{1}{2}}(0,1,-1)$	(-1,0,0)	(-1,0,0)	7b	C_{33}	p_{13}^2/C_{33}
(0,0,1)	$(0,q_2^{11},q_3^{11})$	$(0,-q_3^{11},q_2^{11})$	(-1,0,0)			0
(0,0,1)	$(0,q_2^{12},q_3^{12})$	$(0,-q_3^{12},q_2^{12})$	(-1,0,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(0,1,-1)$	$2^{-\frac{1}{2}}(0,1,1)$	$2^{-\frac{1}{2}}(0,-1,1)$			$\frac{(n_2^2+n_3^2)^4}{64\,n_2^8\,n_3^8\,C_{33}}(n_2^4p_{23}-n_3^4p_{33})^2$

$\vec{u}$ are given in Table III; the values of $\vec{q}$ are given in Table IV.

pure shear or mixed mode			pure shear mode		
Not.	γ	$\beta_{\mu\mu'}$	Not.	γ	$\beta_{\mu\mu'}$
2a	C_{66}	0 0 0 0	3a	C_{55}	0 $\frac{((n_2q_1^1)^2 + (n_1q_2^1)^2)^2}{n_2^4\, C_{55}} (q_1^1 p'_{55})^2$ $\frac{((n_1q_1^2)^2 + (n_2q_2^2)^2)^2}{n_2^4\, C_{55}} (q_2^2 p'_{55})^2$ 0
2b	C_{66}	0 $\frac{((n_3q_1^3)^2 + (n_1q_3^3)^2)^2}{n_3^4\, C_{66}} (q_1^3 p'_{66})^2$ $\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_3^4\, C_{66}} (q_3^4 p'_{66})^2$ 0	3b	C_{55}	0 0 0 0
5a	C_{44}	0 0 0 0	6a	C_{66}	0 $\frac{((n_3q_2^5)^2 + (n_2q_3^5)^2)^2}{n_3^4\, C_{66}} (q_2^5 p'_{66})^2$ $\frac{((n_2q_2^6)^2 + (n_3q_3^6)^2)^2}{n_3^4\, C_{66}} (q_3^6 p'_{66})^2$ 0
5b	C_{44}	0 $\frac{((n_2q_1^7)^2 + (n_1q_2^7)^2)^2}{n_1^4\, C_{44}} (q_2^7 p'_{44})^2$ $\frac{((n_1q_1^8)^2 + (n_2q_2^8)^2)^2}{n_1^4\, C_{44}} (q_1^8 p'_{44})^2$ 0	6b	C_{66}	0 0 0 0
8a	C_{55}	0 0 0 0	9a	C_{44}	0 $\frac{((n_3q_1^9)^2 + (n_1q_3^9)^2)^2}{n_1^4\, C_{44}} (q_3^9 p'_{44})^2$ $\frac{((n_1q_1^{10})^2 + (n_3q_3^{10})^2)^2}{n_1^4\, C_{44}} (q_1^{10} p'_{44})^2$ 0
8b	C_{55}	0 $\frac{((n_3q_2^{11})^2 + (n_2q_3^{11})^2)^2}{n_2^4\, C_{55}} (q_3^{11} p'_{55})^2$ $\frac{((n_2q_2^{12})^2 + (n_3q_3^{12})^2)^2}{n_2^4\, C_{55}} (q_2^{12} p'_{55})^2$ 0	9b	C_{44}	0 0 0 0

TABLE II.

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}'_{\mu'}$	Longitudinal mode: Not.	γ	$\beta_{\mu\mu'}$
$2^{-\frac{1}{2}}(1,1,0)$	$(-1,0,0)$	$(0,0,1)$	$(0,0,1)$	10	γ^{10}	$(p_{31}u_1 + p_{32}u_2)^2/2\gamma$
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{13},q_2^{13},0)$	$(0,0,1)$	$(-q_1^{13},-q_2^{13},0)$			0
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{14},q_2^{14},0)$	$(q_2^{14},-q_1^{14},0)$	$(0,0,1)$			0
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{15},q_2^{15},0)$	$(q_2^{15},-q_1^{15},0)$	$(-q_1^{15},-q_2^{15},0)$			$\frac{((n_1q_1^{15})^2 + (n_2q_2^{15})^2)^2((n_2q_1^{15})^2 + (n_1q_2^{15})^2)^2}{2\,n_1^8n_2^8\,\gamma}\Big\{q_1^{15}q_2^{15}(n_2^4(p_{21}u_1 + p_{22}u_2) - n_1^4(p_{11}u_1 + p_{12}u_2)) + n_1^2n_2^2(p'_{66}u_1 + p'_{66}u_2)((q_1^{15})^2 - (q_2^{15})^2)\Big\}^2$
$2^{-\frac{1}{2}}(0,1,1,)$	$(0,-1,0)$	$(1,0,0)$	$(1,0,0)$	13	γ^{13}	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{16},q_3^{16})$	$(1,0,0)$	$(0,-q_2^{16},-q_3^{16})$			0
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{17},q_3^{17})$	$(0,q_3^{17},-q_2^{17})$	$(1,0,0)$			0
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{18},q_3^{18})$	$(0,q_3^{18},-q_2^{18})$	$(0,-q_2^{18},-q_3^{18})$			$\frac{((n_2q_2^{18})^2 + (n_3q_3^{18})^2)^2((n_3q_2^{18})^2 + (n_2q_3^{18})^2)^2}{2\,n_2^8n_3^8\,\gamma}\Big\{q_2^{18}q_3^{18}(n_3^4(p_{32}u_2 + p_{33}u_3) - n_2^4(p_{22}u_2 + p_{23}u_3)) + n_2^2n_3^2\,(p'_{44}u_2 + p'_{44}u_3)((q_2^{18})^2 - (q_3^{18})^2)\Big\}^2$
$2^{-\frac{1}{2}}(1,0,1)$	$(0,0,-1)$	$(0,1,0)$	$(0,1,0)$	16	γ^{16}	$(p_{23}u_3 + p_{21}u_1)^2/2\gamma$
$2^{-\frac{1}{2}}(1,0,1)$	$(q_1^{19},0,q_3^{19})$	$(0,1,0)$	$(-q_1^{19},0,q_3^{19})$			0
$2^{-\frac{1}{2}}(1,0,1)$	$(q_1^{20},0,q_3^{20})$	$(-q_3^{20},0,q_1^{20})$	$(0,1,0)$			0
$2^{-\frac{1}{2}}(1,0,1)$	$(q_1^{21},0,q_3^{21})$	$(-q_3^{21},0,q_1^{21})$	$(-q_1^{21},0,q_3^{21})$			$\frac{((n_3q_3^{21})^2 + (n_1q_1^{21})^2)^2((n_1q_3^{21})^2 + (n_3q_1^{21})^2)^2}{2\,n_3^8n_1^8\,\gamma}\Big\{q_3^{21}q_1^{21}(n_1^4(p_{13}u_3 + p_{11}u_1) - n_3^4(p_{33}u_3 + p_{31}u_1)) + n_3^2n_1^2(p'_{55}u_3 + p'_{55}u_1)((q_3^{21})^2 - (q_1^{21})^2)\Big\}^2$

(Continued)

Pure shear or mixed mode			Pure shear mode		
Not.	γ	$\beta_{\mu\mu'}$	Not.	γ	$\beta_{\mu\mu'}$
11	γ^{11}	See mode 10	12	$\frac{1}{2}(C_{44} + C_{55})$	0
					$\frac{((n_2q_1^{13})^2 + (n_1q_2^{13})^2)^2}{n_1^4n_2^4(C_{44} + C_{55})}(n_1^2q_1^{13}p'_{55} + n_2^2q_2^{13}p'_{4\bar{4}})^2$
					$\frac{((n_1q_1^{14})^2 + (n_2q_2^{14})^2)^2}{n_1^4n_2^4(C_{44} + C_{55})}(n_1^2c_2^{14}p'_{55} - n_2^2q_1^{14}p'_{4\bar{4}})^2$
					0
14	γ^{14}	See mode 13	15	$\frac{1}{2}(C_{55} + C_{66})$	0
					$\frac{((n_3q_2^{16})^2 + (n_2q_3^{16})^2)^2}{n_2^4n_3^4(C_{55} + C_{66})}(n_2^2q_2^{16}p'_{66} + n_3^2q_3^{16}p'_{5\bar{5}})^2$
					$\frac{((n_2q_2^{17})^2 + (n_3q_3^{17})^2)^2}{n_2^4n_3^4(C_{55} + C_{66})}(n_2^2q_3^{17}p'_{66} - n_3^2q_2^{17}p'_{5\bar{5}})^2$
					0
17	γ^{17}	See mode 16	18	$\frac{1}{2}(C_{66} + C_{44})$	0
					$\frac{((n_1q_3^{19})^2 + (n_3q_1^{19})^2)^2}{n_3^4n_1^4(C_{66} + C_{44})}(n_3^2q_3^{19}p'_{44} + n_1^2q_1^{19}p'_{6\bar{6}})^2$
					$\frac{((n_3q_3^{20})^2 + (n_1q_1^{20})^2)^2}{n_3^4n_1^4(C_{66} + C_{44})}(n_3^2q_1^{20}p'_{44} - n_1^2q_3^{20}p'_{6\bar{6}})^2$
					0

TABLE III. Orthorhombic group 0.

Not.	Values of γ	Not.	Values of γ
10	$\frac{1}{4}\left\{C_{11} + C_{22} + 2C_{66} + \left((C_{11} - C_{22})^2 + 4(C_{12} + C_{66})^2\right)^{\frac{1}{2}}\right\}$	14	$\frac{1}{4}\left\{C_{22} + C_{33} + 2C_{44} - \left((C_{22} - C_{33})^2 + 4(C_{23} + C_{44})^2\right)^{\frac{1}{2}}\right\}$
11	$\frac{1}{4}\left\{C_{11} + C_{22} + 2C_{66} - \left((C_{11} - C_{22})^2 + 4(C_{12} + C_{66})^2\right)^{\frac{1}{2}}\right\}$	16	$\frac{1}{4}\left\{C_{33} + C_{11} + 2C_{55} + \left((C_{33} - C_{11})^2 + 4(C_{13} + C_{55})^2\right)^{\frac{1}{2}}\right\}$
13	$\frac{1}{4}\left\{C_{22} + C_{33} + 2C_{44} + \left((C_{22} - C_{33})^2 + 4(C_{23} + C_{44})^2\right)^{\frac{1}{2}}\right\}$	17	$\frac{1}{4}\left\{C_{33} + C_{11} + 2C_{55} - \left((C_{33} - C_{11})^2 + 4(C_{13} + C_{55})^2\right)^{\frac{1}{2}}\right\}$

Values of $\vec{u}$

	u_1	u_2	u_3
10	$(C_{12} + C_{66})\left((C_{12} + C_{66})^2 + (2\gamma^{10} - C_{11} - C_{66})^2\right)^{-\frac{1}{2}}$	$(2\gamma^{10} - C_{11} - C_{66})\left((C_{12} + C_{66})^2 + (2\gamma^{10} - C_{11} - C_{66})^2\right)^{-\frac{1}{2}}$	0
11	$(2\gamma^{10} - C_{11} - C_{66})\left((C_{12} + C_{66})^2 + (2\gamma^{10} - C_{11} - C_{66})^2\right)^{-\frac{1}{2}}$	$-(C_{12} + C_{66})\left((C_{12} + C_{66})^2 + (2\gamma^{10} - C_{11} - C_{66})^2\right)^{-\frac{1}{2}}$	0
13	0	$(C_{23} + C_{44})\left((C_{23} + C_{44})^2 + (2\gamma^{13} - C_{22} - C_{44})^2\right)^{-\frac{1}{2}}$	$(2\gamma^{13} - C_{22} - C_{44})\left((C_{23} + C_{44})^2 + (2\gamma^{13} - C_{22} - C_{44})^2\right)^{-\frac{1}{2}}$
14	0	$(2\gamma^{13} - C_{22} - C_{44})\left((C_{23} + C_{44})^2 + (2\gamma^{13} - C_{22} - C_{44})^2\right)^{-\frac{1}{2}}$	$-(C_{23} + C_{44})\left((C_{23} + C_{44})^2 + (2\gamma^{13} - C_{22} - C_{44})^2\right)^{-\frac{1}{2}}$
16	$(2\gamma^{16} - C_{33} - C_{55})\left((C_{13} + C_{55})^2 + (2\gamma^{16} - C_{33} - C_{55})^2\right)^{-\frac{1}{2}}$	0	$(C_{13} + C_{55})\left((C_{13} + C_{55})^2 + (2\gamma^{16} - C_{33} - C_{55})^2\right)^{-\frac{1}{2}}$
17	$-(C_{13} + C_{55})\left((C_{13} + C_{55})^2 + (2\gamma^{16} - C_{33} - C_{55})^2\right)^{-\frac{1}{2}}$	0	$(2\gamma^{16} - C_{33} - C_{55})\left((C_{13} + C_{55})^2 + (2\gamma^{16} - C_{33} - C_{55})^2\right)^{-\frac{1}{2}}$

TABLE IV. Directions of incident light, where $\epsilon_1 = (n_2+n_3-2n_1)/4n_1$, $\epsilon_2 = (n_1+n_3-2n_2)/4n_2$, $\epsilon_3 = (n_1+n_2-2n_3)/4n_3$.

Not.	q_1	q_2	q_3
$\vec{q}^{1}$	$-2^{-\frac{1}{2}}(1-\epsilon_3)$	$2^{-\frac{1}{2}}(1+\epsilon_3)$	0
$\vec{q}^{2}$	$-2^{-\frac{1}{2}}(1+\epsilon_3)$	$2^{-\frac{1}{2}}(1-\epsilon_3)$	0
$\vec{q}^{3}$	$-2^{-\frac{1}{2}}(1-\epsilon_2)$	0	$2^{-\frac{1}{2}}(1+\epsilon_2)$
$\vec{q}^{4}$	$-2^{-\frac{1}{2}}(1+\epsilon_2)$	0	$2^{-\frac{1}{2}}(1-\epsilon_2)$
$\vec{q}^{5}$	0	$-2^{-\frac{1}{2}}(1-\epsilon_1)$	$2^{-\frac{1}{2}}(1+\epsilon_1)$
$\vec{q}^{6}$	0	$-2^{-\frac{1}{2}}(1+\epsilon_1)$	$2^{-\frac{1}{2}}(1-\epsilon_1)$
$\vec{q}^{7}$	$2^{-\frac{1}{2}}(1+\epsilon_3)$	$-2^{-\frac{1}{2}}(1-\epsilon_3)$	0
$\vec{q}^{8}$	$2^{-\frac{1}{2}}(1-\epsilon_3)$	$-2^{-\frac{1}{2}}(1+\epsilon_3)$	0
$\vec{q}^{9}$	$2^{-\frac{1}{2}}(1+\epsilon_2)$	0	$-2^{-\frac{1}{2}}(1-\epsilon_2)$
$\vec{q}^{10}$	$2^{-\frac{1}{2}}(1-\epsilon_2)$	0	$-2^{-\frac{1}{2}}(1+\epsilon_2)$
$\vec{q}^{11}$	0	$2^{-\frac{1}{2}}(1+\epsilon_1)$	$-2^{-\frac{1}{2}}(1-\epsilon_1)$
$\vec{q}^{12}$	0	$2^{-\frac{1}{2}}(1-\epsilon_1)$	$-2^{-\frac{1}{2}}(1+\epsilon_1)$
$\vec{q}^{13}$	$-2^{-\frac{1}{2}}(n_1+n_3)(n_1^2+n_3^2)^{-\frac{1}{2}}$	$2^{-\frac{1}{2}}(n_1-n_3)(n_1^2+n_3^2)^{-\frac{1}{2}}$	0
$\vec{q}^{14}$	$-2^{-\frac{1}{2}}(n_2+n_3)(n_2^2+n_3^2)^{-\frac{1}{2}}$	$2^{-\frac{1}{2}}(n_2-n_3)(n_2^2+n_3^2)^{-\frac{1}{2}}$	0
$\vec{q}^{15}$	$-1+\frac{(n_1^2-n_2^2)^2}{32\,n_1^4}$	$\frac{n_1^2-n_2^2}{4\,n_1^2}$	0
$\vec{q}^{16}$	0	$-2^{-\frac{1}{2}}(n_1+n_2)(n_1^2+n_2^2)^{-\frac{1}{2}}$	$+2^{-\frac{1}{2}}(n_2-n_1)(n_1^2+n_2^2)^{-\frac{1}{2}}$
$\vec{q}^{17}$	0	$-2^{-\frac{1}{2}}(n_1+n_3)(n_1^2+n_3^2)^{-\frac{1}{2}}$	$2^{-\frac{1}{2}}(n_3-n_1)(n_1^2+n_3^2)^{-\frac{1}{2}}$
$\vec{q}^{18}$	0	$-1+\frac{(n_2^2-n_3^2)^2}{32\,n_2^4}$	$\frac{n_2^2-n_3^2}{4\,n_2^2}$
$\vec{q}^{19}$	$2^{-\frac{1}{2}}(n_3-n_2)(n_2^2+n_3^2)^{-\frac{1}{2}}$	0	$-2^{-\frac{1}{2}}(n_2+n_3)(n_2^2+n_3^2)^{-\frac{1}{2}}$
$\vec{q}^{20}$	$2^{-\frac{1}{2}}(n_1-n_2)(n_1^2+n_2^2)^{-\frac{1}{2}}$	0	$-2^{-\frac{1}{2}}(n_1+n_2)(n_1^2+n_2^2)^{-\frac{1}{2}}$
$\vec{q}^{21}$	$\frac{n_3^2-n_1^2}{4\,n_3^2}$	0	$-1+\frac{(n_1^2-n_3^2)^2}{32\,n_3^4}$
$\vec{q}^{22}$	$-2^{-\frac{1}{2}}(Q_1+Q_2)$	$2^{-\frac{1}{2}}(Q_1-Q_2)$	0
$\vec{q}^{23}$	$-\frac{n_1Q_2+n_3Q_1}{(n_1^2+n_3^2)^{1/2}}$	$\frac{n_1Q_1-n_3Q_2}{(n_1^2+n_3^2)^{1/2}}$	0
$\vec{q}^{24}$	$-\frac{n_1Q_1+n_3Q_2}{(n_1^2+n_3^2)^{1/2}}$	$\frac{n_3Q_1-n_1Q_2}{(n_1^2+n_3^2)^{1/2}}$	0
$\vec{q}^{25}$	0	$2^{-\frac{1}{2}}(n_1+n_3)(n_1^2+n_3^2)^{-\frac{1}{2}}$	$2^{-\frac{1}{2}}(n_3-n_1)(n_1^2+n_3^2)^{-\frac{1}{2}}$
$\vec{q}^{26}$	0	$1-\frac{(n_1^2-n_3^2)^2}{32\,n_1^4}$	$\frac{n_1^2-n_3^2}{4\,n_1^2}$

TABLE V. Quadratic T_1 and hexagonal H_1 groups. $\theta = 180°$.

Not.	γ	$\vec{Q}$	$\vec{u}$	Type of mode	$\vec{e}_1 = \vec{e}_2'$	$\vec{e}_2 = \vec{e}_2'$	β_{11}
1	C_{11}	(1,0,0)	(1,0,0)	L	(0,1,0)	(0,0,1)	p_{12}^2/C_{11}
7	C_{33}	(0,0,1)	(0,0,1)	L	(1,0,0)	(0,1,0)	p_{13}^2/C_{33}
10	$\frac{1}{2}(C_{11} + C_{12}) + C_{66}$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	L	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	p_{31}^2/γ
13	γ^{13}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{13},u_3^{13})$	L	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$
14	γ^{14}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{14},u_3^{14})$	M	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$
15	$\frac{1}{2}(C_{44} + C_{66})$	$2^{-\frac{1}{2}}(0,1,1)$	(1,0,0)	T	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	0

TABLE VI. Hexagonal group H_2. $\theta = 180°$. The

Not.	γ	$\vec{Q}$	$\vec{u}$	Type of mode	$\vec{e}_1 = \vec{e}_1'$	$\vec{e}_2 = \vec{e}_2'$	β_{11}
1	C_{11}	(1,0,0)	(1,0,0)	L	(0,1,0)	(0,0,1)	p_{12}^2/C_{11}
2	C_{66}	(1,0,0)	(0,1,0)	T	(0,1,0)	(0,0,1)	p_{16}^2/C_{66}
3	C_{44}	(1,0,0)	(0,0,1)	T	(0,1,0)	(0,0,1)	0
7	C_{33}	(0,0,1)	(0,0,1)	L	(1,0,0)	(0,1,0)	p_{13}^2/C_{33}
10	C_{11}	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	L	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	p_{31}^2/C_{11}
11	C_{66}	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$	T	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	0
12	C_{44}	$2^{-\frac{1}{2}}(1,1,0)$	(0,0,1)	T	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	0
13	γ^{13}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{13},u_3^{13})$	L	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$
14	γ^{14}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{14},u_3^{14})$	M	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$
15	$\frac{1}{2}(C_{44} + C_{66})$	$2^{-\frac{1}{2}}(0,1,1)$	(1,0,0)	T	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$p_{16}^2/(C_{44} + C_{66})$

The values of $\vec{u}$ and γ are made explicit in Table IX.

$\beta_{12} = \beta_{21}$	β_{22}
0	p_{31}^2/C_{11}
0	p_{13}^2/C_{33}
0	$(p_{11} + p_{12} - 2p_{66})^2/4\gamma$
0	$\frac{(n_1^2 + n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}(n_1^4(p_{11}u_2 + p_{13}u_3) + n_3^4(p_{31}u_2 + p_{33}u_3) - 2\, n_1^2 n_3^2(p'_{44}u_2 + p'_{4\bar{4}}u_3))^2$
0	$\frac{(n_1^2 + n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}(n_1^4(p_{11}u_2 + p_{13}u_3) + n_3^4(p_{31}u_2 + p_{33}u_3) - 2\, n_1^2 n_3^2(p'_{44}u_2 + p'_{4\bar{4}}u_3))^2$
$\frac{(n_1^2 + n_3^2)^2}{8\, n_1^4 n_3^4\, \gamma}(n_1^2 p_{66} - n_3^2 p'_{44})^2$	0

values of $\vec{u}$ and γ are made explicit in Table IX.

$\beta_{12} = \beta_{21}$	β_{22}
0	p_{31}^2/C_{11}
0	0
p_{45}^2/C_{44}	0
0	p_{13}^2/C_{33}
0	p_{12}^2/C_{11}
0	p_{16}^2/C_{66}
p_{45}^2/C_{44}	0
$\frac{(n_1^2 + n_3^2)^2}{16\, n_1^4 n_3^4\, \gamma}(n_1^2 p_{16}u_2 + n_3^2 p_{45}(u_2 + u_3))^2$	$\frac{(n_1^2 + n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}(n_1^4(p_{11}u_2 + p_{13}u_3) + n_3^4(p_{31}u_2 + p_{33}u_3) - 2\, n_1^2 n_3^2(p_{44}u_2 + p_{4\bar{4}}u_3))^2$
$\frac{(n_1^2 + n_3^2)^2}{16\, n_1^4 n_3^4\, \gamma}(n_1^2 p_{16}u_2 + n_3^2 p_{45}(u_2 + u_3))^2$	$\frac{(n_1^2 + n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}(n_1^4(p_{11}u_2 + p_{13}u_3) + n_3^4(p_{31}u_2 + p_{33}u_3) - 2\, n_1^2 n_3^2(p_{44}u_2 + p_{4\bar{4}}u_3))^2$
$\frac{(n_1^2 + n_3^2)^2}{8\, n_1^4 n_3^4 (C_{44} + C_{66})}(n_1^2 p_{66} - n_3^2 p_{44})^2$	$\frac{(n_1^2 + n_3^2)^2}{64\, n_1^4 n_3^4 (C_{44} + C_{66})}(n_1^2 p_{16} + n_3^2 p_{45})^2$

TABLE VII. Quadratic T_1 and hexagonal H_1 groups. $\theta = 90°$. The values

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}_{\mu'}$	Longitudinal mode Not.	γ	$\beta_{\mu\mu'}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	(0,0,1)	1a	C_{11}	p_{31}^2/C_{11}
(1,0,0)	$(q_1^1,q_2^1,0)$	(0,0,1)	$(-q_1^1,-q_2^1,0)$			0
(1,0,0)	$(q_1^2,q_2^2,0)$	$(q_2^2,-q_1^2,0)$	(0,0,1)			0
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$			$(p_{11}-p_{12})^2/4C_{11}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	(0,-1,0)	(0,-1,0)	1b	C_{11}	p_{12}^2/C_{11}
(1,0,0)	$(q_1^3,0,q_3^3)$	(0,-1,0)	$(-q_1^3,0,-q_3^3)$			0
(1,0,0)	$(q_1^4,0,q_3^4)$	$(q_3^4,0,-q_1^4)$	(0,-1,0)			0
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(1,0,-1)$			$\dfrac{(n_1^2+n_3^2)^4}{64\,n_1^8 n_3^8\,C_{11}}(n_1^4 p_{11}-n_3^4 p_{31})^2$
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	(0,1,0)	(0,1,0)	7	C_{33}	p_{13}^2/C_{33}
(0,0,1)	$(q_1^9,0,q_3^9)$	(0,1,0)	$(-q_1^9,0,-q_3^9)$			0
(0,0,1)	$(q_1^{10},0,q_3^{10})$	$(-q_3^{10},0,q_1^{10})$	(0,1,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(-1,0,1)$			$\dfrac{(n_1^2+n_3^2)^4}{64\,n_1^8 n_3^8\,C_{33}}(n_1^4 p_{13}+n_3^4 p_{33})^2$
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,0,1)	(0,0,1)	10	γ^{10}	p_{31}^2/γ
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{13},q_2^{13},0)$	(0,0,1)	$(-q_1^{13},-q_2^{13},0)$			0
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{14},q_2^{14},0)$	$(q_2^{14},-q_1^{14},0)$	(0,0,1)			0
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,1,0)	(1,0,0)			p_{66}^2/γ
$2^{-\frac{1}{2}}(0,1,1)$	(0,-1,0)	(1,0,0)	(1,0,0)	13	γ^{13}	$(p_{12}u_2+p_{13}u_3)^2/2\gamma$
$2^{-\frac{1}{2}}(0,1,1)$	(0,-1,0)	(1,0,0)	(0,1,0)			0
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{17},q_3^{17})$	$(0,q_3^{17},-q_2^{17})$	(1,0,0)			0
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{18},q_3^{18})$	$(0,q_3^{18},-q_2^{18})$	$(0,-q_2^{18},-q_3^{18})$			$\dfrac{((n_1q_2^{18})^2+(n_3q_3^{18})^2)^2((n_3q_2^{18})^2+(n_1q_3^{18})^2)^2}{2\,n_1^8 n_3^8\,\gamma}\Big\{q_2^{18}q_3^{18}(n_3^4(p_{31}u_2+p_{33}u_3)-n_1^4(p_{11}u_2+p_{13}u_3)) + ((q_2^{18})^2-(q_3^{18})^2)n_1^2n_3^2(p'_{44}u_2+p'_{44}u_3)\Big\}^2$

of γ and $\vec{u}$ are given in Table IX; the values of $\vec{q}$ are given in Table IV.

Not.	γ	$\beta_{\mu\mu'}$ (Pure shear or mixed mode)	Not.	γ	$\beta_{\mu\mu'}$ (Pure shear mode)
2a	C_{66}	0 0 0 0	3a	C_{44}	0 $(q_1^1 p'_{4\bar{4}})^2/C_{44}$ $(q_2^2 p'_{4\bar{4}})^2/C_{44}$ 0
2b	C_{66}	0 $\frac{((n_3 q_1^3)^2 + (n_1 q_3^3)^2)^2}{n_3^4 C_{66}}(q_1^3 p_{66})^2$ $\frac{((n_1 q_1^4)^2 + (n_3 q_3^4)^2)^2}{n_3^4 C_{66}}(q_3^4 p_{66})^2$ 0	3b	C_{44}	0 0 0 0
8/9	C_{44}	Pure shear degenerated mode: 0 $\frac{((n_3 q_1^9)^2 + (n_1 q_3^9)^2)^2}{n_1^4 C_{44}}(q_3^9 p'_{44})^2$ $\frac{((n_1 q_1^9)^2 + (n_3 q_3^9)^2)^2}{n_1^4 C_{44}}(q_1^{10} p'_{44})^2$ 0			
11	$\frac{1}{2}(C_{11} - C_{12})$	0 0 0 0	12	C_{44}	0 $((q_1^{13} + q_2^{13}) p'_{4\bar{4}})^2/2C_{44}$ $((q_1^{14} - q_2^{14}) p'_{4\bar{4}})^2/2C_{44}$ 0
14	γ^{14}	See mode 13	15	$\frac{1}{2}(C_{44} + C_{66})$	0 $p_{66}^2/(C_{44} + C_{66})$ $\frac{((n_1 q_2^{17})^2 + (n_3 q_3^{17})^2)^2}{2\, n_1^4 n_3^4\, \gamma}(n_1^2 q_3^{17} p_{66} - n_3^2 q_2^{17} p'_{44})^2$ 0

TABLE VIII. Hexagonal H_2 group. $\theta = 90°$. The values of γ and

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}'_{\mu'}$	Longitudinal mode: Not.	γ	$\beta_{\mu\mu'}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	(0,0,1)	1a	C_{11}	p_{31}^2/C_{11}
(1,0,0)	$(q_1^1,q_2^1,0)$	(0,0,1)	$(-q_1^1,-q_2^1,0)$			0
(1,0,0)	$(q_1^2,q_2^2,0)$	$(q_2^2,-q_1^2,0)$	(0,0,1)			0
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$			$(p_{11} - p_{12})^2/4C_{11}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	(0,-1,0)	(0,-1,0)	1b	C_{11}	p_{12}^2/C_{11}
(1,0,0)	$(q_1^3,0,q_3^3)$	(0,-1,0)	$(-q_1^3,0,-q_3^3)$			$\frac{((n_3q_1^3)^2 + (n_1q_3^3)^2)^2}{n_3^4\ C_{11}}(q_1^3p_{16})^2$
(1,0,0)	$(q_1^4,0,q_3^4)$	$(q_3^4,0,-q_1^4)$	(0,-1,0)			$\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_3^4\ C_{11}}(q_3^4p_{16})^2$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$			$\frac{(n_1^2 + n_3^2)^2}{64\ n_1^4n_3^4\ C_{11}}(n_1^4p_{11} - n_3^4p_{31})^2$
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	(0,1,0)	(0,1,0)	7	C_{33}	p_{13}^2/C_{33}
(0,0,1)	$(q_1^9,0,q_3^9)$	(0,1,0)	$(-q_1^9,0,-q_3^9)$			0
(0,0,1)	$(q_1^{10},0,q_3^{10})$	$(-q_3^{10},0,q_1^{10})$	(0,1,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(-1,0,1)$			$\frac{(n_1^2 + n_3^2)^2}{64\ n_1^8n_3^8\ C_{33}}(n_1^4p_{13} - n_3^4p_{33})^2$
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,0,1)	(0,0,1)	10	C_{11}	p_{31}^2/γ
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{13},q_2^{13},0)$	(0,0,1)	$(-q_1^{13},-q_2^{13},0)$			0
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{14},q_2^{14},0)$	$(q_2^{14},-q_1^{14},0)$	(0,0,1)			0
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,1,0)	(1,0,0)			p_{66}^2/γ
$2^{-\frac{1}{2}}(0,1,1)$	(0,-1,0)	(1,0,0)	(1,0,0)	13	γ^{13}	$(p_{12}u_2 + p_{13}u_3)^2/2\gamma$
$2^{-\frac{1}{2}}(0,1,1)$	(0,-1,0)	(1,0,0)	(0,1,0)			$(p_{16}u_2)^2/2\gamma$
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{17},q_3^{17})$	$(0,q_3^{17},-q_2^{17})$	(1,0,0)			$\frac{((n_1q_2^{17})^2 + (n_3q_3^{17})^2)^2}{2\ n_1^4n_3^4\ \gamma}(n_1^2q_3^{17}p_{16}u_2 + n_3^2q_2^{17}p_{45}(u_2 + u_3))^2$
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{18},q_3^{18})$	$(0,q_3^{18},-q_2^{18})$	$(0,-q_2^{18},-q_3^{18})$			$\frac{((n_1q_2^{18})^2 + (n_3q_3^{18})^2)^2((n_3q_2^{18})^2 + (n_1q_3^{18})^2)^2}{2\ n_1^8n_3^8\ \gamma}\{q_2^{18}q_3^{18}(n_3^4(p_{31}u_2 + p_{33}u_3) - n_1^4(p_{11}u_2 + p_{13}u_3)) + n_1^2n_3^2(p'_{44}u_2 + p'_{44}u_3)((q_2^{18})^2 - (q_3^{18})^2)\}$

$\vec{u}$ are given in Table IX; the values of $\vec{q}$ are given in Table IV.

Pure shear or mixed mode			Pure shear mode		
Not.	γ	$\beta_{\mu\mu'}$	Not.	γ	$\beta_{\mu\mu'}$
2a	C_{66}	0 0 0 p_{16}^2/C_{66}	3a	C_{44}	0 $(q_1^1 p'_{4\bar{4}} + q_2^1 p_{45})^2/C_{44}$ $(q_2^2 p'_{4\bar{4}} - q_1^2 p_{45})^2/C_{44}$ 0
2b	C_{66}	p_{16}^2/C_{66} $\frac{((n_3q_1^3)^2 + (n_1q_3^3)^2)^2}{n_3^4\ C_{66}}(q_1^3 p_{66})^2$ $\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_3^4\ C_{66}}(q_3^4 p_{66})^2$ $\frac{(n_1^2 + n_3^2)^4}{64\ n_3^8\ C_{66}}p_{16}^2$	3b	C_{44}	0 $\frac{((n_3q_1^3)^2 + (n_1q_3^3)^2)^2}{n_1^4\ C_{44}}(q_3^3 p_{45})^2$ $\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_1^4\ C_{44}}(q_1^4 p_{45})^2$ 0
		Pure shear degenerated mode			
8/9	C_{44}	0 $\frac{((n_3q_1^9)^2 + (n_1q_3^9)^2)^2}{n_1^4\ C_{44}}(q_3^9)^2(p'^2_{44} + p_{45}^2)$ $\frac{((n_1q_1^{10})^2 + (n_3q_3^{10})^2)^2}{n_1^4\ C_{44}}(q_1^{10})^2(p'^2_{44} + p_{45}^2)$ 0			
11	C_{66}	0 0 0 p_{16}^2/γ	12	C_{44}	0 $((q_1^{13} + q_2^{13})p'_{4\bar{4}} + (q_2^{13} - q_1^{13})p_{45})^2/2C_{44}$ $((q_2^{14} - q_1^{14})p'_{4\bar{4}} + (q_1^{14} + q_2^{14})p_{45})^2/2C_{44}$ 0
14	γ^{14}	See mode 13	15	$\frac{1}{2}(C_{44} + C_{66})$	$p_{16}^2/(C_{44} + C_{66})$ $p_{66}^2/(C_{44} + C_{66})$ $\frac{((n_1q_2^{17})^2 + (n_3q_3^{17})^2)^2}{2\ n_1^4 n_3^4\ \gamma}(n_1^2 q_3^{17} p_{66} - n_3^2 q_2^{17} p'_{44})^2$ $\frac{((n_1q_2^{18})^2 + (n_3q_3^{18})^2)^2((n_3q_2^{18})^2 + (n_1q_3^{18})^2)^2}{2\ n_1^4 n_3^8\ \gamma}\left\{n_1^2 q_2^{18} q_3^{18} p_{16} + n_3^2((q_2^{18})^2 - (q_3^{18})^2)p_{45}\right\}^2$

TABLE IX. Quadratic T_1

Not.	γ
10	$\frac{1}{2}(C_{11}+C_{12})+C_{66}$
13	$\frac{1}{4}\left\{C_{11}+C_{33}+2C_{44}+\left((C_{11}-C_{33})^2+4(C_{13}+C_{44})^2\right)^{\frac{1}{2}}\right\}$
14	$\frac{1}{4}\left\{C_{11}+C_{33}+2C_{44}-\left((C_{11}-C_{33})^2+4(C_{13}+C_{44})^2\right)^{\frac{1}{2}}\right\}$

TABLE X. Quadratic group T_2. $\theta=180°$. The

Not.	γ	$\vec{Q}$	$\vec{u}$	Type of mode	$\vec{e}_1=\vec{e}_1'$	$\vec{e}_2=\vec{e}_2'$	β_{11}
1	γ^1	(1,0,0)	$(u_1^1,u_2^1,0)$	L	(0,1,0)	(0,0,1)	$(p_{12}u_1-p_{16}u_2)^2/\gamma$
2	γ^2	(1,0,0)	$(u_1^2,u_2^2,0)$	M	(0,1,0)	(0,0,1)	$(p_{12}u_1-p_{16}u_2)^2/\gamma$
3	C_{44}	(1,0,0)	(0,0,1)	T	(0,1,0)	(0,0,1)	0
7	C_{33}	(0,0,1)	(0,0,1)	L	(1,0,0)	(0,1,0)	p_{13}^2/C_{33}
10	γ^{10}	$2^{-\frac{1}{2}}(1,1,0)$	$(u_1^{10},u_2^{10},0)$	L	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	$\left(p_{31}(u_1+u_2)\right)^2/2\gamma$
11	γ^{11}	$2^{-\frac{1}{2}}(1,1,0)$	$(u_1^{11},u_2^{11},0)$	M	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	$\left(p_{31}(u_1+u_2)\right)^2/2\gamma$
12	C_{44}	$2^{-\frac{1}{2}}(1,1,0)$	(0,0,1)	T	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	0
22	γ^{22}	$(Q_1,Q_2,0)$	$(u_1^{22},u_2^{22},0)$	L	(0,0,1)	$(Q_2,-Q_1,0)$	$\left(p_{31}(u_1Q_1+u_2Q_2)\right)^2/\gamma$
23	γ^{23}	$(Q_1,Q_2,0)$	$(u_1^{23},u_2^{23},0)$	M	(0,0,1)	$(Q_2,-Q_1,0)$	$\left(p_{31}(u_1Q_1+u_2Q_2)\right)^2/\gamma$
24	C_{44}	$(Q_1,Q_2,0)$	(0,0,1)	T	(0,0,1)	$(Q_2,-Q_1,0)$	0

and hexagonal H_1 groups.

u_1	u_2	u_3
$2^{-\frac{1}{2}}$	$2^{-\frac{1}{2}}$	0
0	$(C_{13} + C_{44})((C_{13} + C_{44})^2 + (2\gamma^{13} - C_{11} - C_{44})^2)^{-\frac{1}{2}}$	$(2\gamma^{13} - C_{11} - C_{44})((C_{13} + C_{44})^2 + (2\gamma^{13} - C_{11} - C_{44})^2)^{-\frac{1}{2}}$
0	$(2\gamma^{13} - C_{11} - C_{44})((C_{13} + C_{44})^2 + (2\gamma^{13} - C_{11} - C_{44})^2)^{-\frac{1}{2}}$	$-(C_{13} + C_{44})((C_{13} + C_{44})^2 + (2\gamma^{13} - C_{11} - C_{44})^2)^{-\frac{1}{2}}$

values of γ and $\vec{u}$ are made explicit in Table XII.

$\beta_{12} = \beta_{21}$	β_{22}
0	$(p_{31}u_1)^2/\gamma$
0	$(p_{31}u_1)^2/\gamma$
p_{45}^2/C_{44}	0
0	p_{13}^2/C_{33}
0	$\frac{1}{8\gamma}((p_{11} + p_{12} - 2p_{66})(u_1 + u_2) - 2p_{61}(u_1 - u_2))^2$
0	$\frac{1}{8\gamma}((p_{11} + p_{12} - 2p_{66})(u_1 + u_2) - 2p_{61}(u_1 - u_2))^2$
p_{45}^2/C_{44}	0
0	$\frac{1}{\gamma}\left\{((p_{11} - 2p_{66})Q_1Q_2 + p_{16}(Q_2^2 - Q_1^2))(u_1Q_2 + u_2Q_1) + p_{12}(u_1Q_1^3 + u_2Q_2^3) - 2Q_1Q_2p_{61}(u_1Q_1 - u_2Q_2)\right\}^2$
0	$\frac{1}{\gamma}\left\{((p_{11} - 2p_{22})Q_1Q_2 + p_{16}(Q_2^2 - Q_1^2))(u_1Q_2 + u_2Q_1) + p_{12}(u_1Q_1^3 + u_2Q_2^3) - 2Q_1Q_2p_{61}(u_1Q_1 - u_2Q_2)\right\}^2$
$(2Q_1Q_2p_{45} - (Q_1^2 - Q_2^2)p'_{4\bar{4}})^2/C_{44}$	0

TABLE XI. Quadratic T_2 group. $\theta = 90°$. The values of γ and

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}_{\mu'}$	Longitudinal mode		
				Not.	γ	$\beta_{\mu\mu'}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	(0,0,1)	1a	γ^1	$(p_{31}u_1)^2/\gamma$
(1,0,0)	$(q_1^1,q_2^1,0)$	(0,0,1)	$(-q_1^1,-q_2^1,0)$			0
(1,0,0)	$(q_1^2,q_2^2,0)$	$(q_2^2,-q_1^2,0)$	(0,0,1)			0
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$			$((p_{11}-p_{12})u_1+2p_{16}u_2)^2/4\gamma$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	(0,-1,0)	(0,-1,0)	1b	γ^1	$(p_{12}u_1-p_{16}u_2)^2/\gamma$
(1,0,0)	$(q_1^3,0,q_3^3)$	(0,-1,0)	$(-q_1^3,0,-q_3^3)$			$\frac{((n_1q_3^3)^2+(n_3q_1^3)^2)^2}{n_3^4\gamma}(q_1^3(p_{66}u_2+p_{61}u_1))^2$
(1,0,0)	$(q_1^4,0,q_3^4)$	$(q_3^4,0,-q_1^4)$	(0,-1,0)			$\frac{((n_1q_1^4)^2+(n_3q_3^4)^2)^2}{n_3^4\gamma}(q_3^4(p_{66}u_2+p_{61}u_1))^2$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(1,0,-1)$			$\frac{(n_1^2+n_3^2)^4}{64\,n_1^8n_3^8\,\gamma}(n_1^4(p_{11}u_1+p_{16}u_2)-n_3^4p_{31}u_1)^2$
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	(0,1,0)	(0,1,0)	7	C_{33}	p_{13}^2/C_{33}
(0,0,1)	$(q_1^9,0,q_3^9)$	(0,1,0)	$(-q_1^9,0,-q_3^9)$			0
(0,0,1)	$q_1^{10},0,q_3^{10})$	$(-q_3^{10},0,q_1^{10})$	(0,1,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(-1,0,1)$			$\frac{(n_1^2+n_3^2)^4}{64\,n_1^4n_3^4\,C_{33}}(n_1^4p_{13}-n_3^4p_{33})^2$
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,0,1)	(0,0,1)	10	γ^{10}	$(p_{31}(u_1+u_2))^2/2\gamma$
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{13},q_2^{13},0)$	(0,0,1)	$(-q_1^{13},-q_2^{13},0)$			0
$2^{-\frac{1}{2}}(1,1,0)$	$(q_1^{14},q_2^{14},0)$	$(q_2^{14},-q_1^{14},0)$	(0,0,1)			0
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,1,0)	(1,0,0)			$(p_{66}(u_1+u_2)+p_{61}(u_1-u_2))^2/2\gamma$
$(Q_1,Q_2,0)$	$(q_1^{22},q_2^{22},0)$	(0,0,1)	(0,0,1)	22	γ^{22}	$(p_{31}(u_1Q_1+u_2Q_2))^2/\gamma$
$(Q_1,Q_2,0)$	$(q_1^{23},q_2^{23},0)$	(0,0,1)	$(-q_1^{23},-q_2^{23},0)$			0
$(Q_1,Q_2,0)$	$(q_1^{24},q_2^{24},0)$	$(q_2^{24},-q_1^{24},0)$	$(-q_1^{24},-q_2^{24},0)$			0
$(Q_1,Q_2,0)$	$(q_1^{22},q_2^{22},0)$	$(q_2^{22},-q_1^{22},0)$	$(-q_1^{22},-q_2^{22},0)$			$\{q_1^{22}q_2^{22}((p_{11}-p_{12})(u_2Q_2-u_1Q_1)-2p_{16}(u_1Q_2+u_2Q_1)) + ((q_1^{22})^2-(q_2^{22})^2)(p_{61}(u_1Q_1-u_2Q_2)+p_{66}(u_1Q_2+u_2Q_1))\}^2/\gamma$

$\vec{u}$ are given in Table XII, the values of $\vec{q}$ are given in Table IV.

Pure shear or mixed mode

Not.	γ	$\beta_{\mu\mu'}$
2a	γ^2	$(p_{31}u_1)^2/\gamma$
		0
		0
		$((p_{11} - p_{12})u_1 + 2p_{16}u_2)^2/4\gamma$
2b	γ^2	$(p_{12}u_1 - p_{16}u_2)^2/\gamma$
		$\dfrac{((n_1q_3^3)^2 + (n_3q_1^3)^2)^2}{n_3^4\ \gamma}(q_1^3(p_{66}u_2 + p_{61}u_1))^2$
		$\dfrac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_3^4\ \gamma}(q_3^4(p_{66}u_2 + p_{61}u_1))^2$
		0
11	γ^{11}	$(p_{31}(u_1 + u_2))^2/2\gamma$
		0
		0
		$(p_{66}(u_1 + u_2) + p_{61}(u_1 - u_2))^2/2\gamma$
23	γ^{23}	$(p_{31}(u_1Q_1 + u_2Q_2))^2/\gamma$
		0
		0
		$\{q_1^{22}q_2^{22}((p_{11} - p_{12})(u_2Q_2 - u_1Q_1) - 2p_{16}(u_1Q_2 + u_2Q_1)) + ((q_1^{22})^2 - (q_2^{22})^2)(p_{61}(u_1Q_1 - u_2Q_2) + p_{66}(u_1Q_2 + u_2Q_1))\}^2/\gamma$

Pure shear mode

Not.	γ	$\beta_{\mu\mu'}$
3a	C_{44}	0
		$(q_1^1p'_{4\bar{4}} + q_2^1p_{45})^2/C_{44}$
		$(q_2^2p'_{44} - q_1^2p_{45})^2/C_{44}$
		0
3b	C_{44}	0
		$\dfrac{((n_1q_3^3)^2 + (n_3q_1^3)^2)^2}{n_1^4\ C_{44}}(q_3^3p_{45})^2$
		$\dfrac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_1^4\ C_{44}}(q_1^3p_{45})^2$
		0
12	C_{44}	0
		$((q_1^{13} + q_2^{13})p'_{4\bar{4}} + (q_2^{13} - q_1^{13})p_{45})^2/2C_{44}$
		$((q_1^{14} - q_2^{14})p'_{44} + (q_1^{14} + q_2^{14})p_{45})^2/2C_{44}$
		0
24	γ^{24}	0
		$(p'_{4\bar{4}}(q_1Q_1 + q_2Q_2) + p_{45}(q_2Q_1 - q_1Q_2))^2/C_{44}$
		$(p'_{4\bar{4}}(q_1Q_2 - q_2Q_1) + p_{45}(q_1Q_1 + q_2Q_2))^2/C_{44}$
		0

Pure shear degenerated mode

Not.	γ	$\beta_{\mu\mu'}$
8/9	C_{44}	0
		$\dfrac{((n_3q_1^9)^2 + (n_1q_3^9)^2)^2}{n_1^4\ C_{44}}((q_3^9)^2(p'^2_{44} + p^2_{45}))$
		$\dfrac{((n_1q_1^{10})^2 + (n_3q_3^{10})^2)^2}{n_1^4\ C_{44}}((q_1^{10})^2(p'^2_{44} + p^2_{45}))$
		0

TABLE XII.

Not.	γ
1	$\frac{1}{2}\left\{C_{11} + C_{66} + ((C_{11} - C_{66})^2 + 4C_{16}^2)^{\frac{1}{2}}\right\}$
2	$\frac{1}{2}\left\{C_{11} + C_{66} - ((C_{11} - C_{66})^2 + 4C_{16}^2)^{\frac{1}{2}}\right\}$
10	$\frac{1}{2}\left\{C_{11} + C_{66} + ((C_{12} + C_{66})^2 + 4C_{16}^2)^{\frac{1}{2}}\right\}$

Not.	u_1
1	$C_{16}((\gamma^1 - C_{11})^2 + C_{16}^2)^{-\frac{1}{2}}$
2	$(\gamma^1 - C_{11})((\gamma^1 - C_{11})^2 + C_{16}^2)^{-\frac{1}{2}}$
10	$(C_{12} + C_{66})((2\gamma^{10} - C_{11} - C_{66} - 2C_{16})^2 + (C_{12} + C_{66})^2)^{-\frac{1}{2}}$
11	$(2\gamma^{10} - C_{11} - C_{66} - 2C_{16})((2\gamma^{10} - C_{11} - C_{66} - 2C_{16})^2 + (C_{12} + C_{66})^2)^{-\frac{1}{2}}$
22	$(C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66})Q_1Q_2)\left\{(\gamma^{22} - C_{11}Q_1^2 - C_{66}Q_2^2 - 2C_{16}Q_1Q_2)^2 + (C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66})Q_1Q_2)^2\right\}^{-\frac{1}{2}}$
23	$(\gamma^{22} - C_{11}Q_1^2 - C_{66}Q_2^2 - 2C_{16}Q_1Q_2)\left\{(\gamma^{22} - C_{11}Q_1^2 - C_{66}Q_2^2 - 2C_{16}Q_1Q_2)^2 + (C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66})Q_1Q_2)^2\right\}^{-\frac{1}{2}}$

TABLE XIII. Cubic groups

Not.	γ	$\vec{Q}$	$\vec{u}$	Type of mode	$\vec{e}_1 = \vec{e}_1'$	$\vec{e}_2 = \vec{e}_2'$	β_{11}
1	C_{11}	(1,0,0)	(1,0,0)	L	(0,1,0)	(0,0,1)	p_{13}^2/C_{11}
10	$\frac{1}{2}(C_{11} + C_{12}) + C_{44}$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	L	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	$(p_{12} + p_{13})^2/4\gamma$
11	$\frac{1}{2}(C_{11} - C_{12})$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$	T	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$	$(p_{12} - p_{13})^2/2(C_{11} - C_{12})$
25	$\frac{1}{3}(C_{11} + 2C_{12} + 4C_{44})$	$3^{-\frac{1}{2}}(1,1,1)$	$3^{-\frac{1}{2}}(1,1,1)$	L	$6^{-\frac{1}{2}}(1,1,-2)$	$2^{-\frac{1}{2}}(-1,1,0)$	$\frac{1}{9\gamma}(p_{11} + p_{12} + p_{13} - 2p_{44})^2$
26	$\frac{1}{3}(C_{11} - C_{12} + C_{44})$	$3^{-\frac{1}{2}}(1,1,1)$	D(111)*	T	$6^{-\frac{1}{2}}(1,1,-2)$	$2^{-\frac{1}{2}}(-1,1,0)$	$\frac{1}{216\gamma}(3(p_{12} + p_{13} + 4p_{44} - 2p_{11})^2 + (p_{13} - p_{12})^2)$

*/ D(111) : degenerated in the (111) plane.

Quadratic group T_2.

Not.	γ
11	$\frac{1}{2}\left\{C_{11} + C_{66} - ((C_{12} + C_{66})^2 + 4C_{16}^2)^{\frac{1}{2}}\right\}$
22	$\frac{1}{2}\left\{C_{11} + C_{66} + ((C_{11} - C_{66})^2(Q_1^2 - Q_2^2) + 4(C_{12} + C_{66})^2Q_1^2Q_2^2 + 4C_{16}^2 + 8C_{16}(C_{11} + C_{12})Q_1Q_2(Q_1^2 - Q_2^2))^{\frac{1}{2}}\right\}$
23	$\frac{1}{2}\left\{C_{11} + C_{66} - ((C_{11} - C_{66})^2(Q_1^2 - Q_2^2) + 4(C_{12} + C_{66})^2Q_1^2Q_2^2 + 4C_{16}^2 + 8C_{16}(C_{11} + C_{12})Q_1Q_2(Q_1^2 - Q_2^2))^{\frac{1}{2}}\right\}$

u_2	u_3
$(\gamma^1 - C_{11})((\gamma^1 - C_{11})^2 + C_{16}^2)^{-\frac{1}{2}}$	0
$- C_{16}((\gamma^1 - C_{11})^2 + C_{16}^2)^{-\frac{1}{2}}$	0
$(2\gamma^{10} - C_{11} - C_{66} - 2C_{16})((2\gamma^{10} - C_{11} - C_{66} - 2C_{16})^2 + (C_{12} + C_{66})^2)^{-\frac{1}{2}}$	0
$- (C_{12} + C_{66})((2\gamma^{10} - C_{11} - C_{66} - 2C_{16})^2 + (C_{12} + C_{66})^2)^{-\frac{1}{2}}$	0
$(\gamma^{22} - C_{11}Q_1^2 - C_{66}Q_2^2 - 2C_{16}Q_1Q_2)\left\{(\gamma^{22} - C_{11}Q_1^2 - C_{66}Q_2^2 - 2C_{16}Q_1Q_2)^2 + (C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66})Q_1Q_2)^2\right\}^{-\frac{1}{2}}$	0
$-(C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66})Q_1Q_2)\left\}(\gamma^{22} - C_{11}Q_1^2 - C_{66}Q_2^2 - 2C_{16}Q_1Q_2)^2 + (C_{16}(Q_1^2 - Q_2^2) + (C_{12} + C_{66})Q_1Q_2)^2\right\}^{-\frac{1}{2}}$	0

C_2 and C_1. $\theta = 180°$.

$\beta_{12} = \beta_{21}$	β_{22}
0	p_{12}^2/C_{11}
0	$(2p_{11} - 4p_{44} + p_{12} + p_{13})^2/16\gamma$
0	$(p_{12} - p_{13})^2/8(C_{11} - C_{12})$
0	$\frac{1}{9\gamma}(p_{11} + p_{12} + p_{13} - 2p_{44})^2$
$\frac{1}{72\gamma}(3(p_{12} - p_{13})^2 + (p_{12} + p_{13} + 4p_{44} - 2p_{11})^2)$	$\frac{1}{72\gamma}(3(p_{12} - p_{13})^2 + (p_{12} + p_{13} + 4p_{44} - 2p_{11})^2)$

TABLE XIV. Cubic group C_2 and C_1. $\theta = 90°$.

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{s}'_{\mu'}$	Longitudinal mode Not.	γ	$\beta_{\mu\mu'}$	Pure shear mode Not.	γ	$\beta_{\mu\mu'}$ (Pure shear degenerated mode for the first eight rows)	Pure shear mode Not.	γ	$\beta_{\mu\mu'}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	(0,0,1)	1a	C_{11}	p_{12}^2/C_{11}	2a	C_{44}	0			
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	$2^{-\frac{1}{2}}(1,-1,0)$			0			$p_{44}^2/2C_{44}$			
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	(0,0,1)			0			$p_{44}^2/2C_{44}$			
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$			$(p_{11} - p_{13})^2/C_{11}$			0			
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,-1)$	(0,1,0)	(0,1,0)	1b	C_{11}	p_{13}^2/C_{11}	2b	C_{44}	0			
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,-1)$	(0,1,0)	$2^{-\frac{1}{2}}(1,0,1)$			0			$p_{44}^2/2C_{44}$			
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,-1)$	$2^{-\frac{1}{2}}(-1,0,1)$	(0,1,0)			0			$p_{44}^2/2C_{44}$			
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,-1)$	$2^{-\frac{1}{2}}(-1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$			$(p_{11} - p_{12})^2/C_{11}$			0			
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,0,1)	(0,0,1)	10	$\frac{1}{2}(C_{11} + C_{12}) + C_{44}$	$(p_{12} + p_{13})^2/4\gamma$	11	$\frac{1}{2}(C_{11} - C_{12})$	$(p_{12} - p_{13})^2/4\gamma$	12	C_{44}	0
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,0,1)	(1,0,0)			0			0			$p_{44}^2/2C_{44}$
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,1,0)	(0,0,1)			0			0			$p_{44}^2/2C_{44}$
$2^{-\frac{1}{2}}(1,1,0)$	(-1,0,0)	(0,1,0)	(1,0,0)			p_{44}^2/γ			0			0

TABLE XV. Rhombohedric group R_1. $\theta = 180°$. The values of γ and $\vec{u}$ are made explicit in Table XVII. The values of $\vec{q}$ are made explicit in Table IV.

Not.	$\vec{\gamma}$	$\vec{Q}$	$\vec{u}$	Type of mode	$\vec{e}_1 = \vec{e}_1'$	$\vec{e}_2 = \vec{e}_2'$	β_{11}	$\beta_{12} = \beta_{21}$	β_{22}
1	C_{11}	(1,0,0)	(1,0,0)	L	(0,1,0)	(0,0,1)	p_{12}^2/C_{11}	p_{41}^2/C_{11}	p_{31}^2/C_{11}
4	γ^4	(0,1,0)	$(0,u_2^4,u_3^4)$	L	(0,1,0)	(1,0,0)	$(p_{31}u_2)^2/\gamma$	0	$(p_{12}u_2 + p_{14}u_3)^2/\gamma$
5	γ^5	(0,1,0)	$(0,u_2^5,u_3^5)$	M	(0,1,0)	(1,0,0)	$(p_{31}u_2)^2/\gamma$	0	$(p_{12}u_2 + p_{14}u_3)^2/\gamma$
6	C_{66}	(0,1,0)	(1,0,0)	T	(0,1,0)	(1,0,0)	0	p_{41}^2/C_{66}	0
7	C_{33}	(0,0,1)	(0,0,1)	L	(1,0,0)	(0,1,0)	p_{13}^2/C_{33}	0	p_{13}^2/C_{33}
8	C_{44}	(0,0,1)	D(001)*	T	(1,0,0)	(0,1,0)	p_{14}^2/C_{44}	p_{14}^2/C_{44}	p_{14}^2/C_{44}
13	γ^{13}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{13},u_3^{13})$	L	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3 + p_{14}(u_2 + u_3))^2/2\gamma$	0	$\frac{(n_1^2+n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}\{n_1^4(p_{11}u_2 + p_{13}u_3 - p_{14}(u_2+u_3)) + n_3^4(p_{31}u_2 + p_{33}u_3) - 2n_1^2n_3^2(p_{44}'u_2 + p_{44}'u_3 - p_{41}u_2)\}^2$
14	γ^{14}	$2^{-\frac{1}{2}}(0,1,1)$	$(0,u_2^{14},u_3^{14})$	M	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	$(p_{12}u_2 + p_{13}u_3 + p_{14}(u_2 + u_3))^2/2\gamma$	0	$\frac{(n_1^2+n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}\{n_1^4(p_{11}u_2 + p_{13}u_3 - p_{14}(u_2+u_3)) + n_3^4(p_{31}u_2 + p_{33}u_3) - 2n_1^2n_3^2(p_{44}'u_2 + p_{44}'u_3 - p_{41}u_2)\}^2$
15	$\frac{1}{2}(C_{44} + C_{66}) + C_{44}$	$2^{-\frac{1}{2}}(0,1,1)$	(1,0,0)	T	(1,0,0)	$2^{-\frac{1}{2}}(0,1,-1)$	0	$\frac{(n_1^2+n_3^2)^2}{16\, n_1^4 n_3^4\, \gamma}(n_1^2(p_{66} + p_{14}) - n_3^2(p_{44}' + p_{41}))^2$	0
19	γ^{19}	$2^{-\frac{1}{2}}(0,1,-1)$	$(0,u_2^{19},u_3^{19})$	L	(1,0,0)	$2^{-\frac{1}{2}}(0,-1,-1)$	$(p_{12}u_2 - p_{13}u_3 - p_{14}(u_2 - u_3))^2/2\gamma$	0	$\frac{(n_1^2+n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}\{n_1^4(p_{11}u_2 - p_{13}u_3 + p_{14}(u_2-u_3)) + n_3^4(p_{31}u_2 - p_{33}u_3) - 2n_1^2n_3^2(p_{44}'u_2 - p_{44}'u_3 + p_{41}u_2)\}^2$
20	γ^{20}	$2^{-\frac{1}{2}}(0,1,-1)$	$(0,u_2^{20},u_3^{20})$	M	(1,0,0)	$2^{-\frac{1}{2}}(0,-1,-1)$	$(p_{12}u_2 - p_{13}u_3 - p_{14}(u_2 - u_3))^2/2\gamma$	0	$\frac{(n_1^2+n_3^2)^4}{128\, n_1^8 n_3^8\, \gamma}\{n_1^4(p_{11}u_2 - p_{13}u_3 + p_{14}(u_2-u_3)) + n_3^4(p_{31}u_2 - p_{33}u_3) - 2n_1^2n_3^2(p_{44}'u_2 - p_{44}'u_3 + p_{41}u_2)\}^2$
21	$\frac{1}{2}(C_{44} + C_{66}) - C_{14}$	$2^{-\frac{1}{2}}(0,1,-1)$	(1,0,0)	T	(1,0,0)	$2^{-\frac{1}{2}}(0,-1,-1)$	0	$\frac{(n_1^2+n_3^2)^2}{16\, n_1^4 n_3^4\, \gamma}(n_1^2(p_{66} - p_{14}) - n_3^2(p_{41} - p_{44}'))^2$	0

*/ D(001) : degenerated in the (001) plane.

TABLE XVI. Rhombohedric group R_1. $\theta = 90°$. The values of γ

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}_{\mu'}$	Longitudinal mode		
				Not.	γ	$\beta_{\mu\mu'}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	(0,0,1)	(0,0,1)			p_{31}^2/C_{11}
(1,0,0)	$(q_1^1,q_2^1,0)$	(0,0,1)	$(-q_1^1,-q_2^1,0)$	1a	C_{11}	$(q_2^1 p_{41})^2/C_{11}$
(1,0,0)	$(q_1^2,q_2^2,0)$	$(q_2^2,-q_1^2,0)$	(0,0,1)			$(q_1^2 p_{41})^2/C_{11}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(1,-1,0)$			$(p_{11} - p_{12})^2/4C_{11}$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	(0,-1,0)	(0,-1,0)			p_{12}^2/C_{11}
(1,0,0)	$(q_1^3,0,q_3^3)$	(0,-1,0)	$(-q_1^3,0,-q_3^3)$	1b	C_{11}	$\frac{((n_1q_3^3)^2 + (n_3q_1^3)^2)^2}{n_1^4C_{11}}(q_3^3p_{41})^2$
(1,0,0)	$(q_1^4,0,q_3^4)$	$(q_3^4,0,-q_1^4)$	(0,-1,0)			$\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_1^4C_{11}}(q_1^4p_{41})^2$
(1,0,0)	$2^{-\frac{1}{2}}(-1,0,1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(1,0,-1)$			$\frac{(n_1^2 + n_3^2)^4}{64\, n_1^8n_3^8\, C_{11}}(n_1^4p_{11} - n_3^4p_{31})^2$
(0,1,0)	$2^{-\frac{1}{2}}(0,-1,1,)$	(1,0,0)	(1,0,0)			$(p_{12}u_2 + p_{14}u_3)^2/\gamma$
(0,1,0)	$(0,q_2^5,q_3^5)$	(1,0,0)	$(0,-q_2^5,-q_3^5)$	4a	γ^4	0
(0,1,0)	$(0,q_2^6,q_3^6)$	$(0,q_3^6,-q_2^6)$	(1,0,0)			0
(0,1,0)	$2^{-\frac{1}{2}}(0,-1,1)$	$2^{-\frac{1}{2}}(0,1,1)$	$2^{-\frac{1}{2}}(0,1,-1)$			$\frac{(n_1^2 + n_3^2)^4}{64\, n_1^8n_3^8\, \gamma}(n_1^4(p_{11}u_2 + p_{14}u_3) - n_3^4p_{31}u_2)^2$
(0,1,0)	$2^{-\frac{1}{2}}(1,-1,0)$	(0,0,-1)	(0,0,-1)			$(p_{31}u_2)^2/\gamma$
(0,1,0)	$(q_1^7,q_2^7,0)$	(0,0,-1)	$(-q_1^7,-q_2^7,0)$	4b	γ^4	$(q_2^7(p'_{44}u_3 - p_{41}u_2))^2/\gamma$
(0,1,0)	$(q_1^8,q_2^8,0)$	$(-q_2^8,q_1^8,0)$	(0,0,-1)			$(q_1^8(p'_{44}u_3 - p_{41}u_2))^2/\gamma$
(0,1,0)	$2^{-\frac{1}{2}}(1,-1,0)$	$2^{-\frac{1}{2}}(1,1,0)$	$2^{-\frac{1}{2}}(-1,1,0)$			$((p_{11} - p_{12})u_2 - 2p_{14}u_3)^2/4\gamma$
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	(0,1,0)	(0,1,0)			p_{13}^2/C_{33}
(0,0,1)	$(q_1^9,0,q_3^9)$	(0,1,0)	$(-q_1^9,0,-q_3^9)$	7a	C_{33}	0
(0,0,1)	$(q_1^{10},0,q_3^{10})$	$(-q_3^{10},0,q_1^{10})$	(0,1,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(1,0,-1)$	$2^{-\frac{1}{2}}(1,0,1)$	$2^{-\frac{1}{2}}(-1,0,1)$			$\frac{(n_1^2 + n_3^2)^4}{64\, n_1^8n_3^8\, C_{33}}(n_1^4p_{13} - n_3^4p_{33})^2$
(0,0,1)	$2^{-\frac{1}{2}}(0,1,-1)$	(-1,0,0)	(-1,0,0)			p_{13}^2/C_{33}
(0,0,1)	$(0,q_2^{11},q_3^{11})$	(-1,0,0)	$(0,-q_2^{11},-q_3^{11})$	7b	C_{33}	0
(0,0,1)	$(0,q_2^{12},q_3^{12})$	$(0,-q_3^{12},q_2^{12})$	(-1,0,0)			0
(0,0,1)	$2^{-\frac{1}{2}}(0,1,-1)$	$2^{-\frac{1}{2}}(0,1,1)$	$2^{-\frac{1}{2}}(0,-1,1)$			$\frac{(n_1^2 + n_3^2)^4}{64\, n_1^8n_3^8\, C_{33}}(n_1^4p_{13} - n_3^4p_{33})^2$

and $\vec{u}$ are given in Table XVII; the values of $\vec{q}$ are given in Table IV.

Pure shear or mixed mode

Not.	γ	$\beta_{\mu\mu'}$
2a	γ^2	0
		$(q_1^1(p_{41}u_2 + p'_{44}u_3))^2/\gamma$
		$(q_2^2(p_{41}u_2 + p'_{44}u_3))^2/\gamma$
		0
2b	γ^2	0
		$\frac{((n_3q_1^3)^2 + (n_1q_3^3)^2)^2}{n_3^4\,\gamma}(q_1^3(p_{66}u_2 + p_{14}u_3))^2$
		$\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_3^4\,\gamma}(q_3^4(p_{66}u_2 + p_{14}u_3))^2$
		0
5a	γ^5	$(p_{12}u_2 + p_{14}u_3)^2/\gamma$
		0
		0
		$\frac{(n_1^2 + n_3^2)^4}{64\,n_1^8n_3^8\,\gamma}(n_1^4(p_{11}u_2 - p_{14}u_3) - n_3^4p_{31}u_2)^2$
5b	γ^5	$(p_{31}u_2)^2/\gamma$
		$(q_2^7(p'_{44}u_3 - p_{41}u_2))^2/\gamma$
		$(q_1^8(p'_{44}u_3 - p_{41}u_2))^2/\gamma$
		$((p_{11} - p_{12})u_2 - 2p_{14}u_3)^2/4\gamma$

Pure shear mode

Not.	γ	$\beta_{\mu\mu'}$
3a	γ^3	0
		$(q_1^1(p_{41}u_2 + p'_{44}u_3))^2/\gamma$
		$(q_2^2(p_{41}u_2 + p'_{44}u_3))^2/\gamma$
		0
3b	γ^3	0
		$\frac{((n_3q_1^3)^2 + (n_1q_3^3)^2)^2}{n_3^4\,\gamma}(q_1^3(p_{66}u_2 + p_{14}u_3))^2$
		$\frac{((n_1q_1^4)^2 + (n_3q_3^4)^2)^2}{n_3^4\,\gamma}(q_3^4(p_{66}u_2 + p_{14}u_3))^2$
		0
6a	C_{66}	0
		$\frac{((n_3q_2^5)^2 + (n_1q_3^5)^2)^2}{n_1^4n_3^4\,C_{66}}(n_1^2q_2^5p_{66} + n_3^2q_3^5p_{41})^2$
		$\frac{((n_1q_1^6)^2 + (n_3q_3^6)^2)^2}{n_1^4n_3^4\,C_{66}}(n_1^2q_3^6p_{66} - n_3^2q_2^6p_{41})^2$
		0
6b	C_{66}	0
		$(q_1^7p_{41})^2/C_{66}$
		$(q_2^8p_{41})^2/C_{66}$
		0

Pure shear degenerated mode

Not.	γ	$\beta_{\mu\mu'}$
(8,9)a	C_{44}	p_{14}^2/C_{44}
		$\frac{((n_3q_1^9)^2 + (n_1q_3^9)^2)^2}{n_1^4n_3^4\,C_{44}}(n_1^4(q_1^9p_{14})^2 + n_3^4(q_3^9p'_{44})^2)$
		$\frac{((n_1q_1^{10})^2 + (n_3q_3^{10})^2)^2}{n_1^4n_3^4\,C_{44}}(n_1^4(q_3^{10}p_{14})^2 + n_3^4(q_1^{10}p'_{44})^2)$
		$\frac{(n_1^2 + n_3^2)^4}{64\,n_3^8\,C_{44}}\,p_{14}^2$
(8,9)b	C_{44}	p_{14}^2/C_{44}
		$\frac{((n_3q_2^{11})^2 + (n_1q_3^{11})^2)^2}{n_1^4n_3^4\,C_{44}}(n_1^4q_2^{11}p_{14} + n_3^4q_3^{11}p'_{44})^2$
		$\frac{((n_1q_2^{12})^2 + (n_3q_3^{12})^2)^2}{n_1^4n_3^4\,C_{44}}(n_1^4q_3^{12}p_{14} - n_3^4q_2^{12}p'_{44})^2$
		$\frac{(n_1^2 + n_3^2)^4}{64\,n_3^8\,C_{44}}\,p_{14}^2$

TABLE XVI.

$\vec{Q}$	$\vec{q}$	$\vec{e}_\mu$	$\vec{e}'_{\mu'}$	Not. γ	Longitudinal mode $\beta_{\mu\mu'}$
$2^{-\frac{1}{2}}(0,1,1)$	$(0,-1,0)$	$(1,0,0)$	$(1,0,0)$	$13\ \gamma^{13}$	$\left(p_{12}u_2 + p_{13}u_3 + p_{14}(u_2 + u_3)\right)^2/2\gamma$
$2^{-\frac{1}{2}}(0,1,1)$	$(0,-1,0)$	$(1,0,0)$	$(0,1,0)$	$13\ \gamma^{13}$	0
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{17},q_3^{17})$	$(0,q_3^{17},-q_2^{17})$	$(1,0,0)$	$13\ \gamma^{13}$	0
$2^{-\frac{1}{2}}(0,1,1)$	$(0,q_2^{18},q_3^{18})$	$(0,q_3^{18},-q_2^{18})$	$(0,-q_2^{18},-q_3^{18})$	$13\ \gamma^{13}$	$\frac{((n_1q_2^{18})^2 + (n_3q_3^{18})^2)^2((n_3q_2^{18})^2 + (n_1q_3^{18})^2)^2}{2\ n_1^8n_3^8\ \gamma}\left\{n_1^4q_2^{18}q_3^{18}(p_{11}u_2 + p_{13}u_3 - p_{14}(u_2 + u_3)) - n_3^4q_2^{18}q_3^{18}(p_{31}u_2 + p_{33}u_3) + n_1^2n_3^2((q_3^{18})^2 - (q_2^{18})^2)(p'_{44}u_2 + p'_{4\bar{4}}u_3 - p_{41}u_2)\right\}^2$
$2^{-\frac{1}{2}}(0,-1,1)$	$(0,1,0)$	$(-1,0,0)$	$(-1,0,0)$	$19\ \gamma^{19}$	$\left(p_{12}u_2 - p_{13}u_3 + p_{14}(u_3 - u_2)\right)^2/2\gamma$
$2^{-\frac{1}{2}}(0,-1,1)$	$(0,1,0)$	$(-1,0,0)$	$(0,-1,0)$	$19\ \gamma^{19}$	0
$2^{-\frac{1}{2}}(0,-1,1)$	$(0,q_2^{25},q_3^{25})$	$(0,q_3^{25},-q_2^{25})$	$(-1,0,0)$	$19\ \gamma^{19}$	0
$2^{-\frac{1}{2}}(0,-1,1)$	$(0,q_2^{26},q_3^{26})$	$(0,q_3^{26},-q_2^{26})$	$(0,-q_2^{26},-q_3^{26})$	$19\ \gamma^{19}$	$\frac{((n_1q_2^{26})^2 + (n_3q_3^{26})^2)^2((n_3q_2^{26})^2 + (n_1q_3^{26})^2)^2}{2\ n_1^8n_3^8\ \gamma}\left\{n_1^4q_2^{26}q_3^{26}(p_{11}u_2 - p_{13}u_3 + p_{14}(u_2 - u_3)) - n_3^4q_2^{26}q_3^{26}(p_{31}u_2 - p_{33}u_3) + n_1^2n_3^2((q_3^{26})^2 - (q_2^{26})^2)(p'_{4\bar{4}}u_3 - p'_{44}u_2 - p_{41}u_2)\right\}^2$

TABLE XVII. Rhombohedric

Not.	γ
2	$\frac{1}{2}\left\{C_{44} + C_{66} + ((C_{44} - C_{66})^2 + 4C_{14}^2)^{\frac{1}{2}}\right\}$
3	$\frac{1}{2}\left\{C_{44} + C_{66} - ((C_{44} - C_{66})^2 + 4C_{14}^2)^{\frac{1}{2}}\right\}$
4	$\frac{1}{2}\left\{C_{11} + C_{44} + ((C_{11} - C_{44})^2 + 4C_{14}^2)^{\frac{1}{2}}\right\}$
5	$\frac{1}{2}\left\{C_{11} + C_{44} - ((C_{11} - C_{44})^2 + 4C_{14}^2)^{\frac{1}{2}}\right\}$
13	$\frac{1}{4}\left\{C_{11} + C_{33} + 2(C_{44} - C_{14}) + ((C_{11} - C_{33} - 2C_{14})^2 + 4(C_{13} + C_{44} - C_{14})^2)^{\frac{1}{2}}\right\}$
14	$\frac{1}{4}\left\{C_{11} + C_{33} + 2(C_{44} - C_{14}) - ((C_{11} - C_{33} - 2C_{14})^2 + 4(C_{13} + C_{44} - C_{14})^2)^{\frac{1}{2}}\right\}$
19	$\frac{1}{4}\left\{C_{11} + C_{33} + 2(C_{44} + C_{14}) + ((C_{11} - C_{33} + 2C_{14})^2 + 4(C_{13} + C_{44} - C_{14})^2)^{\frac{1}{2}}\right\}$
20	$\frac{1}{4}\left\{C_{11} + C_{33} + 2(C_{44} + C_{14}) - ((C_{11} - C_{33} + 2C_{14})^2 + 4(C_{13} + C_{44} + C_{14})^2)^{\frac{1}{2}}\right\}$

(*Continued*)

Pure shear or mixed mode			Pure shear mode		
Not.	γ	$\beta_{\mu\mu'}$	Not.	γ	$\beta_{\mu\mu'}$
14	γ^{14}	See mode 13	15	$\frac{1}{2}(C_{44} + C_{66}) + C_{14}$	0
					$(p_{66} + p_{14})^2/2\gamma$
					$\frac{((n_1q_2^{17})^2 + (n_3q_3^{17})^2)^2}{2\,n_1^4n_3^4\,\gamma}\left\{n_3^2q_2^{17}(p'_{44} + p_{41}) - n_1^2q_3^{17}(p_{66} + p_{14})\right\}^2$
					0
20	γ^{20}	See mode 19	21	$\frac{1}{2}(C_{44} + C_{66}) - C_{14}$	0
					$(p_{66} - p_{14})^2/2\gamma$
					$\frac{((n_1q_2^{25})^2 + (n_3q_3^{25})^2)^2}{2\,n_1^4n_3^4\,\gamma}\left\{n_3^2q_2^{25}(p_{41} - p'_{44}) - n_1^2q_3^{25}(p_{66} - p_{14})\right\}^2$
					0

group R_1.

u_1	u_2	u_3
0	$C_{14}((\gamma^2 - C_{66})^2 + C_{14}^2)^{-\frac{1}{2}}$	$(\gamma^2 - C_{66})((\gamma^2 - C_{66})^2 + C_{14}^2)^{-\frac{1}{2}}$
0	$(\gamma^2 - C_{66})((\gamma^2 - C_{66})^2 + C_{14}^2)^{-\frac{1}{2}}$	$-C_{14}((\gamma^2 - C_{66})^2 + C_{14}^2)^{-\frac{1}{2}}$
0	$C_{14}((\gamma^4 - C_{11})^2 + C_{14}^2)^{-\frac{1}{2}}$	$(\gamma^4 - C_{11})((\gamma^4 - C_{11})^2 + C_{14}^2)^{-\frac{1}{2}}$
0	$(\gamma^4 - C_{11})((\gamma^4 - C_{11})^2 + C_{14}^2)^{-\frac{1}{2}}$	$-C_{14}((\gamma^4 - C_{11})^2 + C_{14}^2)^{-\frac{1}{2}}$
0	$(C_{13} + C_{44} - C_{14})((2\gamma^{13} + 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} - C_{14})^2)^{-\frac{1}{2}}$	$(2\gamma^{13} + 2C_{14} - C_{11} - C_{44})((2\gamma^{13} + 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} + C_{14})^2)^{-\frac{1}{2}}$
0	$(2\gamma^{13} + 2C_{14} - C_{11} - C_{44})((2\gamma^{13} + 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} - C_{14})^2)^{-\frac{1}{2}}$	$-(C_{13} + C_{44} + C_{14})((2\gamma^{13} + 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} + C_{14})^2)^{-\frac{1}{2}}$
0	$(C_{13} + C_{44} + C_{14})((2\gamma^{19} - 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} + C_{14})^2)^{-\frac{1}{2}}$	$(2\gamma^{19} - 2C_{14} - C_{11} - C_{44})((2\gamma^{19} - 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} + C_{14})^2)^{-\frac{1}{2}}$
0	$(2\gamma^{19} - 2C_{14} - C_{11} - C_{44})((2\gamma^{19} - 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} + C_{14})^2)^{-\frac{1}{2}}$	$-(C_{13} + C_{44} + C_{14})((2\gamma^{19} - 2C_{14} - C_{11} - C_{44})^2 + (C_{13} + C_{44} + C_{14})^2)^{-\frac{1}{2}}$

of elastic waves. It is only the intensities that differ in some cases among the two groups. The calculation for the group C_2 can also be used for the group C_1 by taking into account the following remarks: $p_{12}=p_{13}$ for C_1, which produces, in particular, an intensity equal to zero in the case of 11, in polarization (1, 1); the distinction between the scattering cases called "a" and the scattering cases called "b" is useless because of the presence of quaternary axes in C_1. C_{11} and C_{44} are measured directly (modes 1a and b; 2a and b, 12), whereas C_{12} is calculated from measurements of γ^{10} and γ^{11}. For the isotropic group, the orientation $\vec{Q}$ is no longer important. For any direction the measurement of the elastic velocity of a longitudinal wave provides the value of C_{11} and the measurement of the transverse wave provides C_{44}.

F. Rhombohedric Group R_1 (Classes $3m$, 32, $\bar{3}m$)

The calculations for this group are given in Tables XVI and XVII. The presence of elements C_{14}, C_{24}, and C_{56} which are different from zero, and which are called for by the small degree of symmetry of these classes, leads to complicated expressions for Eq. (5). This equation is completely factored in the direction (001) only. C_{11}, C_{33}, C_{44}, and C_{66} are measured directly, and so is C_{12} (modes 1a and b, 7a and b, 8a and b, and 6a and b). C_{14} is calculated from the measurements of γ^{15} and γ^{21}. Cross checks are given by the measurement of γ for the modes 2, 3, 4, and 5. Two values of C_{13} are obtained from the measurement of γ for one of the modes 13, 14, 19, and 20. The choice between two possible values of C_{13} can be made in the following manner: The values of γ^L and γ^T for the modes 13 and 14 are the roots of the equation

$$4\gamma^2 - 2\gamma(C_{11}+C_{33}+2C_{44}-2C_{14}) + (C_{33}+C_{44}) \times(C_{11}+C_{44}-2C_{14}) - (C_{13}+C_{44}-C_{14})^2 = 0 . \quad (42)$$

For the modes 19 and 20, Eq. (5) should read

$$4\gamma^2 - 2\gamma(C_{11}+C_{33}+2C_{44}+2C_{14}) + (C_{33}+C_{44}) \times(C_{11}+C_{44}+2C_{14}) - (C_{13}+C_{44}+C_{14})^2 = 0 . \quad (43)$$

If we write

$$(C_{13}+C_{44}+C_{14})^2 = E , \quad (C_{13}+C_{44}-C_{14})^2 = F ,$$

we find

$$(C_{13}+C_{44})\,C_{14} = \tfrac{1}{4}(F-E) . \quad (44)$$

The sign of C_{14} being determined by the modes 15 or 21, (44) lifts the indeterminacy of the sign of C_{13}.

G. R_2 Rhombohedric Group (Classes 3 and $\bar{3}$)

As in the case of the quadratic group T_2, the total factorization of Eq. (5) appears only in the direction (001). On the other hand, no partial factorization is possible in any other direction. For instance, in the direction (100) Eq. (5) becomes

$$\gamma^3 - \gamma^2(C_{11}+C_{44}+C_{66}) - \gamma(C_{14}^2+C_{25}^2-C_{11}C_{44} - C_{11}C_{66}-C_{44}C_{66}) - C_{11}C_{44}C_{66}+C_{14}^2C_{11}+C_{25}^2C_{66} = 0 . \quad (45)$$

If it is possible to measure the frequency shift between the three Brillouin lines in a sufficient number of orientations, the Parker and Meyer method should be applied. In any case, the lack of accuracy in the measurement of the constants is great.

VI. DETERMINATION OF ELASTIC CONSTANTS BY MEANS OF BACKSCATTERING MEASUREMENTS

A. Remark

In most cases the number of measurements possible in some specific directions is just equal to the number of constants to be determined. Should an intensity factor be too weak to permit a measurement or should complementary measurements be needed for verifications, it would then be necessary to perform measurements in further directions.

B. Orthorhombic Group

C_{11}, C_{22}, and C_{33} can be determined from γ^1, γ^4, and γ^7; C_{44}, C_{55}, and C_{66} from γ^{12}, γ^{15}, and γ^{18} (Table I). The absolute values of the sums $S_1=C_{12}+C_{66}$, $S_2=C_{13}+C_{55}$, and $S_3=C_{23}+C_{44}$ are obtained by measuring γ for the modes 10, 11, 16, 17, 13, and 14. The signs are determined by means of the method described in Sec. V A.

C. Quadratic Group T_1 and Hexagonal Groups H_1 and H_2

For the T_1 and H_1 groups (Table V and VI), C_{11} and C_{33} are determined by measuring γ^1 and γ^7. The value of the sum $\gamma^{13}+\gamma^{14}=\frac{1}{2}(C_{11}+C_{33}+2C_{44})$ is calculated from γ^{13} or γ^{14} by using the usual method for choosing between the two possible values.

The same method can be applied to H_2 and as a matter of fact, the measurement γ^{10} provides a verification of C_{11} only, whereas that of γ^{15} allows the determination of C_{12} by means of the relation $C_{66}=\frac{1}{2}(C_{11}-C_{12})$.

D. Quadratic Group T_2

The method is similar to the one described in Sec. V D, except for the measurement of C_{44}; as it is impossible to measure γ^8, C_{44} must be deduced from the value of γ for the modes 3, 12, or 24 (Table X).

E. Cubic Groups C_1, C_2 and Rhombohedric Group R_1

No special remark is to be made regarding these groups (see Secs. V E and V F) (Tables XIII and

XV).

VII. DETERMINATION OF PHOTOELASTIC CONSTANTS

A. Remarks

(i) In the systems under consideration, the only p'_{ij} constants which differ from the corresponding Pöckels p_{ij} term are the following: the orthorhombic system: p'_{44}, $p'_{4\bar{4}}$, p'_{55}, $p'_{5\bar{5}}$, p'_{66}, $p'_{6\bar{6}}$; and quadratic, rhombohedric, and hexagonal systesm: $p'_{44} = p'_{5\bar{5}}$, $p'_{4\bar{4}} = p'_{55}$.

(ii) As we shall mainly use β measurements for $\theta = 90°$, the index $\pi/2$ will be understood in the following text. The index π will be indicated only for backscattering measurement of β.

B. Orthorhombic Group

The absolute values of p_{12}, p_{13}, p_{21}, p_{23}, p_{31}, p_{32}, p'_{44}, p'_{55}, p'_{66}, $p'_{4\bar{4}}$, $p'_{5\bar{5}}$, and $p'_{6\bar{6}}$ are readily measured (Tables I and II). We assume that the sign of one of these constants is known (p_{12} for instance). The respective measurements of β^{13}_{11}, β^{16}_{11}, and β^{10}_{11} give the signs of $p_{12}p_{13}$, $p_{21}p_{23}$, and $p_{31}p_{32}$; they also provide a verification of the absolute values of these constants. Then we can write

$$p_{11} = \frac{n_2^4}{n_1^4} p_{21} \pm \frac{8n_2^4}{(n_1^2+n_2^2)^2} (C_{11}\beta^{1a}_{22})^{1/2} , \tag{46}$$

$$p_{11} = \pm \frac{n_3^4}{n_1^4} |p_{31}| \pm \frac{8n_2^4}{(n_1^2+n_3^2)^2} (C_{11}\beta^{1b}_{22})^{1/2} . \tag{47}$$

A comparison between the two possible values of p_{11} given by (46) and the four possible values given by (47) permits the determination of the algebraic value of p_{11}, of the sign of p_{31}, and then of the sign of p_{32}.

In the same manner, p_{22} is measured according to

$$p_{22} = \frac{n_3^4}{n_2^4} p_{32} \pm \frac{8n_3^4}{(n_2^2+n_3^2)^2} (C_{22}\beta^{4a}_{22})^{1/2} \tag{48}$$

and

$$p_{22} = \pm \frac{n_1^4}{n_2^4} |p_{12}| \pm \frac{8n_1^4}{(n_1^2+n_2^2)^2} (C_{22}\beta^{4b}_{22})^{1/2} . \tag{49}$$

Then p_{33} is determined from similar expressions as a function of β^{7a}_{22} and β^{7b}_{22}. The algebraic values of p_{12}, p_{13}, p_{21}, p_{31}, p_{32}, p_{11}, p_{22}, and p_{33} are thus found, depending however on the sign of p_{21}. There are two possibilities.

(a) The crystal possesses an important birefringence in one direction at least, so that $p_{i(j)}$ is not negligible with regard to p_{ij}. Suppose $n_2 \neq n_3$; starting from β^{9a}_{21} and β^{5b}_{21} we can write

$$p_{44}p_{4(4)} = \frac{n_1^4 C_{44}}{4} \left(\frac{\beta^{9a}_{21}}{[(n_1 q^{10}_1)^2 + (n_3 q^{10}_3)^2]^2 (q^{10}_1)^2} - \frac{\beta^{5b}_{21}}{[(n_1 q^{8}_1)^2 + (n_2 q^{8}_2)^2]^2 (q^{8}_1)^2} \right) , \tag{50}$$

which gives the sign of p_{44}, since

$$p_{4(4)} = \frac{1}{2}\left(\frac{1}{n_3^2} - \frac{1}{n_2^2}\right) . \tag{51}$$

No further information can be usefully inferred from 90°-scattering measurements. In expressions such as β^{12}_{12}, for instance, $p_{4\bar{4}}$ is included as a corrective term only and cannot be used for the purpose of experimental determination. The signs of the products $p'_{4\bar{4}}p'_{55}$, $p'_{5\bar{5}}p'_{66}$, and $p'_{6\bar{6}}p'_{44}$ can be fixed on the basis of $\beta^{12}_{12}(\pi)$, $\beta^{15}_{12}(\pi)$, and $\beta^{18}_{12}(\pi)$. We therefore know the algebraic values of p_{44}, p_{55}, and p_{66}. Proceeding from the measurement of $\beta^{s}_{22}(\pi)$ for one of the modes 10, 11, 13, 14, 16, and 17, the sign of p_{21} can be either confirmed or changed if necessary. It should be stressed that the new formulation of the photoelastic effect put forward by Nelson and Lax permits the determination of the signs of the photoelastic constants by means of Brillouin-scattering measurements: Within the present accuracy of intensity measurements (around 1%) the procedure can be applied when one of the main birefringences $[(n_1 - n_2)$ for instance$]$ is greater than 10^{-2}.

(b) The birefringence of the crystal is so weak that $p_{i(j)}$ can be considered as negligible within the accuracy of the experiments. In this case the calculation proceeds by comparing the signs of p_{44}, p_{55}, and p_{66} with those of the other constants, starting from the measurements of $\beta^{s}_{22}(\pi)$ for one of the modes 10, 11, 13, 14, 16, and 17. The sign of p_{21} will thus have to be determined by means of another method (static for instance).

C. Quadratic T_1 and Hexagonal H_1 Groups

The absolute values of p_{12}, p_{13}, p_{31}, p'_{44}, $p'_{4\bar{4}}$, and p_{66} are determined directly (Tables V and VII). If the sign of p_{12} is chosen arbitrarily, it is then possible to calculate the algebraic value of p_{11}, and to determine the signs of p_{31} and p_{13} (Sec. VII B). The measurement of β^{7}_{22} gives

$$p_{33} = \frac{n_1^4}{n_3^4} p_{13} \pm \frac{8n_1^4}{(n_1^2+n_3^2)^2} (C_{33}\beta^{7}_{22})^{1/2} . \tag{52}$$

There are two possibilities: (a) The indices n_1 and n_3 are different enough to permit the direct determination of the sign of p'_{44} by comparing $|p'_{44}|$ with $|p'_{4\bar{4}}|$. The sign of p_{66} is thus fixed by means of the measurement of $\beta^{15}_{21}(\pi)$. The experimental values of $\beta^{13}_{22}(\pi)$ and $\beta^{14}_{22}(\pi)$ are used in order to choose between the two possible values of p_{33} and to fix the signs of p_{12} and of the other constants. (b) $n_1 \simeq n_3$; the sign of the product $p_{44}p_{66}$ is determined by means of the $\beta^{15}_{21}(\pi)$ measurement. Proceeding from the $\beta^{13}_{22}(\pi)$ and $\beta^{14}_{22}(\pi)$ values, we can choose between the two possible values of p_{33} and fix the signs of p_{44} and p_{66} in relation to the sign of p_{12}, which is chosen arbitrarily.

D. Hexagonal H_2 Group

The absolute values of p_{12}, p_{13}, p_{31}, p_{66}, p_{45}, and p_{16} are determined readily (Tables VI and VIII). From the β^{9}_{12} measurement $|p'_{44}|$ is determined according to

$$|p'_{44}| = \left(\frac{n_1^4}{[(n_3 q^5_2)^2 + (n_1 q^5_3)^2]^2 (q^5_3)^2} C_{44} \beta^8_{12} - p^2_{45}\right)^{1/2} . \quad (53)$$

First, the sign of p_{12} is chosen arbitrarily; the algebraic values of p_{13}, p_{31}, and p_{11} are thus determined, as well as the two possible values of p_{33}. The sign of p_{66} is given by $p_{66} = \frac{1}{2}(p_{11} - p_{12})$.

Again we meet with two possibilities: (a) n_1 is different from n_3 so that $p_{4(4)}$ is not negligible as compared to p_{44} within the accuracy of the experiment. The two values of $p'_{4\bar{4}}$ are thus

$$p'_{4\bar{4}} = \pm |p'_{44}| - 2p_{4(4)} .$$

The signs of p_{44} and p_{45} are given by the values of β^{3a}_{12} and β^{3a}_{21}. The signs of p_{66} and p_{16} are determined by proceeding from the values of $\beta^{15}_{12}(\pi)$ and $\beta^{13}_{12}(\pi)$. The values of $\beta^{13}_{22}(\pi)$ and $\beta^{14}_{22}(\pi)$ are used to choose between the two possible values of p_{33} and to fix the sign of p_{12}.

(b) $n_1 \simeq n_3$. The signs of p_{44} and p_{45} are given by the respective values of $\beta^{15}_{12}(\pi)$ and $\beta^{3a}_{12}(\pi)$. The sign of p_{16} is given by the value of one of the constants $\beta^{15}_{22}(\pi)$, $\beta^{13}_{12}(\pi)$, or $\beta^{14}_{12}(\pi)$. The choice between the possible values of p_{33} is made on the basis of the value of $\beta^{13}_{22}(\pi)$ or of $\beta^{14}_{22}(\pi)$. Here again the sign of p_{12} will have to be determined by means of another method.

E. Quadratic Group T_2

The constants p_{13}, p_{31}, p'_{44}, and p_{45} are readily determined (Tables X and XI). The sign of p_{31} is chosen arbitrarily. The values of p_{11}, p_{12}, and p_{16} are found by the following procedure. Let us put

$$p_{11} u^1_1 + p_{16} u^1_2 = P_1 , \quad (54)$$

$$p_{12} u^1_1 - p_{16} u^1_2 = P_2 . \quad (55)$$

We may write

$$P_1 = \frac{n_3^4}{n_1^4} p_{31} u^1_1 \pm \frac{8 n_3^4}{(n_1^2 + n_3^2)^2} (\gamma^1 \beta^{1b}_{22})^{1/2} , \quad (56)$$

$$|P_2| = (\gamma^1 \beta^{1b}_{11})^{1/2} . \quad (57)$$

P_1 and P_2 can have two values. The choice between these values is made by considering

$$(P_1 - P_2)^2 = 4\gamma^1 \beta^{1a}_{22} . \quad (58)$$

In the same way, if we write

$$p_{11} u^2_1 + p_{16} u^2_2 = P_3 , \quad (59)$$

$$p_{12} u^2_1 - p_{16} u^2_2 = P_4 , \quad (60)$$

the algebraic values of P_3 and P_4 can be deduced from the values of β^{1b}_{11}, β^{1a}_{22}, and β^{1b}_{22}. Then we have

$$p_{11} = P_1 u^1_1 + P_3 u^1_2 , \quad (61)$$

$$p_{16} = P_1 u^1_2 + P_3 u^1_1 , \quad (62)$$

$$p_{12} = P_2 u^1_1 + P_4 u^1_2 . \quad (63)$$

On the other hand, the value of β^7_{22} gives four possible values for p_{33}, as opposed to two; the absolute values of p_{61} and p_{66} and the sign of their product are given by the measurements of β^s_{12} and β^s_{21} in the 1*b* and 2*b* modes and by that of β^s_{22} in the 10 and 11 modes. Proceeding from the measure of β^{22}_{22} and β^{23}_{22}, the sign of both constants can be determined as a function of that of p_{12}. If $p_{4(4)}$ is not negligible with respect to p_{44}, the signs of p_{44} and p_{45} can be specified by the measurement of β^{3a}_{22} and β^{3a}_{21}.

For the above-mentioned group, the calculations of intensities become very tedious when $\vec{Q}$ is not in a pure-mode direction. The expressions have been given explicitly only for $\vec{Q} = (Q_1, Q_2, 0)$ and $\vec{Q} = (0, 0, 1)$. As shown above, these expressions allow the determination of the absolute values of some of the constants; for others various values are possible. There exist relations between the signs of the constants. However, completion of the calculation requires measurements in the $(Q_1, 0, Q_3)$ or $(0, Q_2, Q_3)$ directions. A comparison between the experimental values and the possible values—to be obtained from computer—provides the necessary relations between signs.

F. Cubic Groups C_2 and C_1

For the C_2 group the absolute values of p_{12}, p_{13}, and p_{44} are readily measured (Tables XIII and XIV). The sign of p_{12} is chosen arbitrarily. The sign of p_{13} is determined by measuring β^{10}_{11} and β^{11}_{11}. The algebraic value of p_{11} is calculated from the values of β^{1a}_{22} and β^{1b}_{22}. The sign of p_{44} can be specified by measuring $\beta^{10}_{22}(\pi)$.

For the C_1 group the absolute values of p_{12} and p_{44} can be determined; an arbitrary choice of the sign of p_{12} allows the calculation of two possible values for p_{11}. The choice between these two possible values and the determination of the sign of p_{44} are made from the measurement of $\beta^{10}_{22}(\pi)$ and $\beta^{25}_{22}(\pi)$.

G. Rhombohedric Group R_1

The absolute values of p_{12}, p_{13}, p_{14}, p_{31}, and p_{41} are readily determined (Tables XV and XVI); those of p'_{44} and $p'_{4\bar{4}}$ can be deduced from

$$|p'_{44}| = \left(\frac{1}{(q^7_2)^2} (\gamma^4 \beta^{4b}_{12} + \gamma^5 \beta^{5b}_{12}) - p^2_{41}\right)^{1/2} \quad (64)$$

and

$$|p'_{4\bar{4}}| = \left(\frac{1}{(q^1_1)^2} (\gamma^2 \beta^{2a}_{12} + \gamma^3 \beta^{3a}_{12}) - p^2_{41}\right)^{1/2} . \quad (65)$$

The sign of p_{12} is chosen arbitrarily.

The algebraic values of p_{11} and p_{31} can be determined by a comparison between the two values resulting from the measurement of β_{22}^{1a} and the four values calculated from β_{22}^{1b}. Then the sign of p_{66} being known, the sign of p_{14} is determined from β_{12}^{15} or β_{12}^{21}; the sign of p_{41} from β_{12}^{6a} or β_{21}^{6a}, and that of p_{13} from a measurement of β_{11}^{s} in one of the modes 13, 14, 19, and 20.

Two possible values are given for p_{33} by measuring β_{22}^{7a}. Should the birefringence be important enough, the signs of p'_{44} and $p'_{4\bar{4}}$ will be given by a comparison between (64) and (65). The sign of p_{14} is given by β_{12}^{8b} or β_{21}^{8b}, which specifies at the same time the sign of all constants, p_{33} excepted. The choice between the two possible values of p_{33} is made from $\beta_{22}^{s}(\pi)$ in one of the modes 13, 14, 19, and 20.

Should p'_{44} and $p'_{4\bar{4}}$ be equal within the precision of the experiments, the sign of p_{44}—relative to the arbitrary sign of p_{12}—is given by the value of β_{12}^{8b}. The choice between the two possible values of p_{33} is effected as indicated above.

ACKNOWLEDGMENTS

The authors wish to thank Professor L. Cecchi for his interest and encouragement in carrying out this work. Valuable comments by Professor A. Zarembowitch are also gratefully acknowledged.

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