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Statistical Mechanics of One-Dimensional Ginzburg-Landau Fields

D. J. Scalapino* and M. Sears*†
University of California, Santa Barbara, California 93106

and

R. A. Ferrell‡
University of Maryland, College Park, Maryland 20742
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We have been studying the statistical mechanics of one-dimensional Ginzburg-Landau fields for real, complex, and phase-only fields. Here, results for the specific heat, the order-parameter–order-parameter, and energy-density–energy-density correlation functions, will be presented. Formally, these solutions are of interest because they describe the behavior of systems which are nearly ordered but do not undergo sharp phase transitions. Physically, the real-field results may have application in some organic chain systems, while the complex field and phase-only fields are associated with superconducting strips and linear arrays of coupled weak links, respectively.

I. INTRODUCTION

In bulk three-dimensional systems, the onset of a second-order phase transition is characterized by the continuous growth of an order parameter when the temperature decreases below a critical temperature T_c. The nature of the order parameter varies with the type of phase transition. Some examples are the relative density between a gas-liquid phase, the superconducting pair amplitude, magnetization, and molecular orientation. In general the order parameter is a field which, as the examples illustrate, can be real, complex, vector, or tensor in character.

In addition to the order parameter, its spatial correlations as well as correlations of its intensity are often of interest. Landau introduced a simple phenomenological approach to determine the thermodynamics and the static spatial correlations in the vicinity of the critical temperature.[1] A free-energy functional is constructed by expanding the free-energy density in powers of the order parameter ψ and its spatial gradient. Then equations for ψ as well as its various correlations are obtained by requiring the functional to be stationary. This leads to the so-called mean-field solutions. Perhaps the best-known example of this is the Ginzburg–Landau theory of superconductivity.[2]

As is known, these results are only approximate, leading, for example, to an incorrect temperature dependence for ψ and the range of its correlations near T_c as well as to the prediction of long-range order in reduced dimensions. Now, it can be argued that this simply represents a breakdown of the analytic form assumed for the free-energy density. However, it appears more likely that the difficulty is associated with the importance of fluctuations near T_c, and that the Landau free-energy functional should be used as an energy functional in a canonical average over-all ψ fields.

In general, such functional integrals are extremely difficult to evaluate. While various perturbation and truncation procedures have been developed to deal with this type of problem, none appear sufficiently powerful to give a detailed treatment of the critical region. However, for one-dimensional systems the transfer matrix technique can be used to reduce the functional integration to a one-particle quantum-mechanical problem.[3] In this form, straightforward numerical procedures can be used to obtain an essentially exact solution.

Here we present in detail results[4,5] obtained from an analysis of one-dimensional systems characterized by real, complex, and phase-only order parameters. Results for the temperature dependence of the order-parameter, the specific heat, the order-parameter–order-parameter, and the intensity-intensity correlation functions will be given. Formally, these solutions are of interest because they describe the behavior of systems which are nearly ordered but do not undergo sharp phase transitions at any finite temperature. Physically, the real-field results may have application for linear chain molecules, while the complex and phase-only fields are associated with superconducting whiskers and linear arrays of weak links, respectively. It has also been suggested that near T_c, the application of a magnetic field to a bulk superconductor can separate the Landau levels of the order parameter field sufficiently that only the lowest one is important, resulting in an effectively one-dimensional system.[6] Finally, we hope that

these exact results for one-dimensional systems will stimulate further experimental work with the goal of making detailed comparisons to test the validity of the functional generalization of the Landau theory.

II. FORMULATION

Here we take a simple form for the energy functional $F[\psi]$;

$$F[\psi]=\int_0^L \frac{dx}{\xi_0}\left(a|\psi|^2+b|\psi|^4+c\left|\frac{d\psi}{dx}\right|^2\right). \tag{2.1}$$

The coefficient a vanishes linearly as the reduced temperature $t=T/T_c$ approaches unity;

$$a=(t-1)a' . \tag{2.2}$$

a', b, and c are positive constants and ξ_0 is a temperature-independent length set by

$$\xi_0=(c/a')^{1/2} . \tag{2.3}$$

The form of F in Eq. (1) is essentially that of the Ginzburg–Landau theory of superconductivity[4] in the absence of a vector potential. In this work, we will use this form for three cases: (i) ψ real, (ii) ψ complex, and (iii) ψ phase only. In this latter form, the magnitude of ψ is fixed, but its phase φ is allowed to vary so that only the derivative in Eq. (1) is relevant and

$$F[\varphi]=\int_0^L dx\, c'|\nabla\varphi|^2 . \tag{2.4}$$

Because of the quadratic nature of Eq. (2.4), the continuous phase-only problem is trivially soluble. A more interesting case is the discrete phase-only problem associated with a linear array of weak links. In this case,

$$F[\varphi]=-\textstyle\sum_i E_i\cos(\varphi_i-\varphi_{i-1}) . \tag{2.5}$$

Using $F[\psi]$ as the energy associated with an order-parameter configuration ψ, the partition function for a canonical ensemble can be expressed as a functional integral

$$Z=\int \delta\psi\, e^{-\beta F[\psi]} . \tag{2.6}$$

A precise definition of this functional integration will be given below. Proceeding in the same spirit, the order-parameter–order-parameter and intensity-intensity correlation functions are formally given by

$$C_1(x)=\langle\psi(x)\psi^*(0)\rangle=\int \delta\psi(e^{-\beta F[\psi]}/Z)\psi(x)\psi^*(0) \tag{2.7}$$

and

$$C_2(x)=\langle\delta|\psi(x)|^2\delta|\psi(0)|^2\rangle = \int \delta\psi(e^{-\beta F[\psi]}/Z)\,\delta|\psi(x)|^2\delta|\psi(0)|^2 . \tag{2.8}$$

Here $\delta|\psi(x)|^2=|\psi(x)|^2-\langle|\psi|^2\rangle$ is the intensity fluctuation. The usual mean-field thermodynamic properties are obtained from a saddle-point approximation of Eq. (2.6) in which a mean field ψ is determined from

$$\delta F/\delta\psi^*=0 . \tag{2.9}$$

The lowest-order fluctuation contributions to Eqs. (2.7) and (2.8) involve a harmonic approximation in which $F[\psi]$ is expanded to second order in the fluctuations about the mean field. In the Sec. III these results will be reviewed.

As is well known, the transfer-matrix technique allows one to replace the functional integration by an eigenvalue problem. In this one-dimensional case, the eigenvalue problem can be reduced to a one-particle quantum-mechanical problem. This approach has been discussed by several authors[5] and is simply related to Feynman's path integral formulation of quantum mechanics.[7] Here we give a brief summary of the ideas as they apply to the real- and complex-field cases.

It is convenient to consider boundary conditions appropriate to a ring of length L. Dividing L into N segments of length ΔX, an explicit expression for the partition function Eq. (2.6) is

$$Z=\prod_{i=1}^{N}\int d\tilde{\psi}_i\,\exp[-\beta_c(\Delta x/\xi_0)f(\psi_{i+1},\psi_i)] , \tag{2.10}$$

with

$$f(\psi_{i+1},\psi_i)=a|\psi_{i+1}|^2+b|\psi_{i+1}|^4+c\left|\frac{\psi_{i+1}-\psi_i}{\Delta x}\right|^2 . \tag{2.11}$$

The ring boundary conditions imply ψ_{N+1} equals ψ_1. The integration elements $d\tilde{\psi}_i$ depend upon the nature of the field. For the real and complex fields we take

$$d\tilde{\psi}_i=\left(\frac{\beta_c c}{\pi\Delta x\xi_0}\right)^{1/2}d\psi_i \qquad (\psi \text{ real}) , \tag{2.12a}$$

$$d\tilde{\psi}_i=\left(\frac{\beta_c c}{\pi\Delta x\xi_0}\right)d(\mathrm{Re}\psi_i)\,d(\mathrm{Im}\psi_i) \quad (\psi \text{ complex}) . \tag{2.12b}$$

In all these expressions, Eqs. (2.9), (2.12a), and (2.12b), the parameter β has been set equal to its value $\beta_c=(kT_c)^{-1}$ at the transition temperature. The significant temperature variation is contained in the parameter a. The length Δx is of order ξ_0, so that the factors in Eqs. (2.12a) and (2.12b) are constants and play no role in the calculation of the thermodynamics or correlation functions.

In order to further reduce Eq. (2.10), we formally introduce an additional variable $\tilde{\psi}_1'$ and write

$$Z=\int d\tilde{\psi}_1' d\tilde{\psi}_1 d\tilde{\psi}_2\cdots d\tilde{\psi}_N\delta(\tilde{\psi}_1-\tilde{\psi}_1') .$$

$$\times \exp[-\beta_c(\Delta x/\xi_0) f(\tilde{\psi}_1', \psi_N)] \exp[\beta_c(\Delta x/\xi_0) f(\psi_N, \psi_{N-1})]$$

$$\cdots \exp[-\beta_c(\Delta x/\xi_0) f(\psi_2, \psi_1)] . \quad (2.13)$$

Now, the δ function is expanded in terms of a complete set of normalized eigenstates

$$\delta(\tilde{\psi}_1 - \tilde{\psi}_1') = \sum_n \Psi_n^*(\tilde{\psi}_1') \Psi_n(\tilde{\psi}_1) , \quad (2.14)$$

so that

$$Z = \sum_n \int d\tilde{\psi}_1' \cdots d\tilde{\psi}_N \Psi_n^*(\tilde{\psi}_1') \exp[-\beta_c(\Delta x/\xi_0) f(\psi_1', \psi_N)]$$

$$\cdots \exp[-\beta_c(\Delta x/\xi_0) f(\psi_2, \psi_1)] \Psi_n(\tilde{\psi}_1) . \quad (2.15)$$

This expression for Z can be directly evaluated if the Ψ_n are eigenfunctions of the transfer operator

$$\int d\tilde{\psi}_i \exp[-\beta_c(\Delta x/\xi_0) f(\psi_{i+1}, \psi_i)] \Psi_n(\tilde{\psi}_i)$$

$$= \exp[-\beta_c(\Delta x/\xi_0)\epsilon_n] \Psi_n(\tilde{\psi}_{i+1}), \quad (2.16)$$

so that

$$Z = \sum_n e^{-\beta_c (L/\xi_0) \epsilon_n} . \quad (2.17)$$

For a thermodynamic system in which $L/\xi_0 \to \infty$, only the ground state contributes and the free energy per unit length is

$$f = (-kT_c/L) \ln Z = \epsilon_0/\xi_0 . \quad (2.18)$$

Following this same procedure, the correlation functions can be expressed in terms of the Ψ_n eigenstates,

$$C_1(x) = \sum_n |\langle \Psi_n | \psi | \Psi_0 \rangle|^2 \exp[-\beta_c(x/\xi_0)(\epsilon_n - \epsilon_0)] \quad (2.19)$$

and

$$C_2(x) = \sum_n |\langle \Psi_n | \delta |\psi|^2 | \Psi_0 \rangle|^2 \exp[-\beta_c(x/\xi_0)(\epsilon_n - \epsilon_0)] . \quad (2.20)$$

For distance $x \gg \xi_0$, the lowest excited state coupled by the matrix element determines the behavior of the correlation functions.

The transfer-matrix-eigenvalue equation [Eq. (2.16)] can be reduced to a one-particle quantum-mechanical problem by expanding $\Psi_n(\tilde{\psi}_i)$ about $\Psi_n \times (\tilde{\psi}_{i+1})$ and carrying out the $d\tilde{\psi}_i$ integration. Only the even terms contribute and to leading order in Δx one finds

$$\int d\tilde{\psi}_i \exp[-\beta_c(\Delta x/\xi_0) f(\psi_{i+1}, \psi_i)] [\Psi_n(\tilde{\psi}_{i+1})$$

$$+ (\tilde{\psi}_i - \tilde{\psi}_{i+1}) \Psi'(\tilde{\psi}_{i+1})$$

$$+ \tfrac{1}{2}(\tilde{\psi}_i - \tilde{\psi}_{i+1})^2 \Psi''(\tilde{\psi}_{i+1}) + \cdots]$$

$$= \exp[-\beta_c(\Delta x/\xi_0)(a|\psi_{i+1}|^2 + b|\psi_{i+1}|^4)]$$

$$\times \left(1 + \frac{1}{4} \frac{\Delta x \xi_0}{\beta_c c} \frac{\partial^2}{\partial \psi_{i+1}^2}\right) \Psi_n(\tilde{\psi}_{i+1}) . \quad (2.21)$$

Formally, the derivative term can be exponentiated and combined with the "potential" to order Δx so that the transfer eigenvalue Eq. (2.16) becomes

$$\exp[-\beta_c(\Delta x/\xi_0) H] \Psi_n = \exp[-\beta_c(\Delta x/\xi_0)\epsilon_n] \Psi_n . \quad (2.22)$$

The effective Hamiltonian is just that of a particle moving in an anharmonic potential;

$$H = -\frac{1}{4} \frac{\xi_0^2}{\beta_c^2 c} \frac{\partial^2}{\partial \psi^2} + a|\psi|^2 + b|\psi|^4 . \quad (2.23)$$

For a real field, this is one-dimensional in ψ while for a complex field, the real and imaginary parts of ψ form a two-dimensional space with

$$\frac{\partial^2}{\partial \psi^2} = \frac{\partial^2}{\partial |\psi|^2} + \frac{1}{|\psi|} \frac{\partial}{\partial |\psi|} + \frac{1}{|\psi|^2} \frac{\partial^2}{\partial \varphi^2} . \quad (2.24)$$

The anharmonic potential $a|\psi|^2 + b|\psi|^4$ is shown in Fig. 1 for three different temperatures. For $T > T_c$, the minimum in the potential occurs at $|\psi| = 0$, but when $T < T_c$ the minimum is displaced to a finite value of $|\psi|$. For the real field, the potential is symmetric under the discrete parity operation $\psi \to -\psi$. For the complex field, the potential depends only upon the radial $|\psi|$ coordinate and exhibits a continuous guage symmetry reflected in the φ independence of the potential.

The formal relationship between this quantum-mechanical problem and the original statistical mechanics problem offers some insight into the nature of the solutions given in Sec. III. The role of time is exchanged for the distance times i and $\hbar$ is replaced by $kT_c\xi_0$. Thus the time correlations of the position of a particle in the potential are related to the space correlations of the order parameter. However, because t is replaced by ix, the oscillatory time behavior of the particle correlations is replaced by damped spatial correlations in the statistical mechanics problem. Thus the eigen energies set inverse correlation lengths rather than characteristic frequencies. Continuing with the analogy for T above a narrow critical region about T_c, the particle follows an essentially classical motion, while when T approaches the critical region the quantum mechanical motion of the particle becomes increasinlgy important. At temperatures sufficiently above T_c, the harmonic part of the potential dominates. However, in a critical region near T_c, the potential is very anharmonic, and the particle can tunnel between the potential minima. This leads to a near degeneracy of the ground state and the low-lying excited states as T decreases below T_c. This near degeneracy is reflected [Eq. (2.19)] in an increased range of the correlations. In summary, this formal analogy implies that the mean-field results of the original Landau approach [Eq. (2.9)] are to Hamilton's

equations in mechanics as the functional integral formalism [Eqs. (2.6)–(2.8)] is to quantum mechanics.

III. MEAN-FIELD AND PERTURBATION-THEORY RESULTS

The anharmonic potential $a|\psi|^2+b|\psi|^4$ was shown in Fig. 1 for different temperatures. At temperatures low or high compared with T_c, the ground-state energy is dominated by the potential minimum so that mean-field theory forms a useful starting point. For $T>T_c$, the minimum lies at the origin while for $T<T_c$ it occurs at $|\psi|^2$ equal to $-a/2b$. Within this mean-field approximation the free energy is just

$$F=\begin{cases}-(a^2/4b)\,L/\xi_0\,, & T<T_c\\ 0\,, & T>T_c\end{cases} \tag{3.1}$$

and the resulting mean-field specific heat has the well-known jump discontinuity as T decreases below T_c,

$$\Delta C = La'^2/\xi_0 2bT_c\,. \tag{3.2}$$

A perturbation theory for $T<T_c$ is obtained by expanding ψ about its mean-field value; for the real field we have

$$\psi=\pm(-a/2b)^{1/2}+\Delta\psi \tag{3.3}$$

and for the complex field

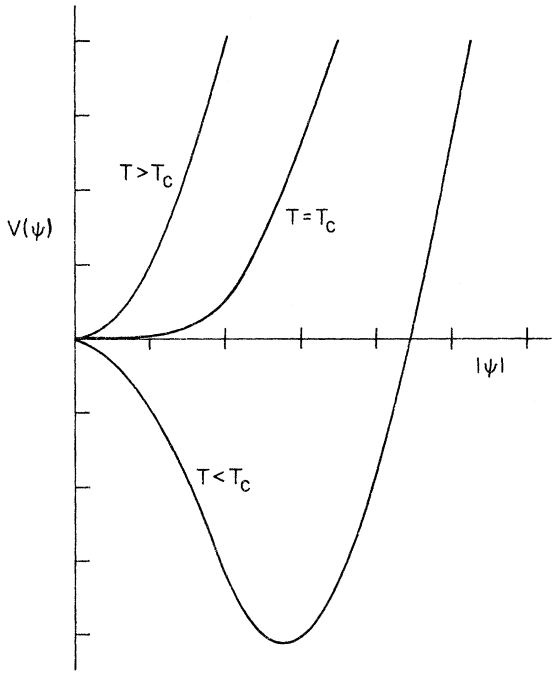

Fig. 1. Anharmonic potential vs $|\psi|$ for different temperatures.

$$\psi=[(-a/2b)^{1/2}+\Delta\psi]\,e^{i\varphi}\,. \tag{3.4}$$

Inserting these transformations into the expression for H, Eq. (2.23), and keeping only terms quadratic in $\Delta\psi$ yields for both real and complex ψ,

$$H\cong\frac{a^2}{4b}-\frac{1}{4}\frac{\xi^2}{\beta_c^2 c}\frac{\partial^2}{\partial\Delta\psi^2}-2a(\Delta\psi)^2\,. \tag{3.5}$$

Within this harmonic approximation the ground-state energy is

$$\epsilon_0=-\frac{a^2}{4b}+\left(\frac{-a}{2}\frac{\xi^2}{\beta_c c}\right)^{1/2}$$

$$=-kT_c\left(\frac{\Delta t}{2}\right)^{1/2}\left[\left(\frac{1-t}{\Delta t}\right)^2-\left(\frac{1-t}{\Delta t}\right)^{1/2}\right]. \tag{3.6}$$

Here Δt is a fractional measure of the temperature width of the critical region[8,9] relative to T_c;

$$\Delta t=\frac{\Delta T}{T_c}=2\left(\frac{b}{\beta_c a'^2}\right)^{2/3}. \tag{3.7}$$

Physically, ΔT measures the size of temperature region below T_c in which the thermal energy kT_c is sufficient to drive the order parameter to zero over a mean-field coherence length. Above T_c, the mean field vanishes and the harmonic approximation gives for a real field

$$\epsilon_0\cong\tfrac{1}{2}kT_c(\Delta t)^{1/2}\,[(t-1)/\Delta t]^{1/2}\,, \tag{3.8}$$

and for a complex field there is an extra factor of 2 which arises from the two-dimensional nature of the Hamiltonian.

Now, as discussed in Sec. II, Eq. (2.18), the free energy per ξ_0 length is just equal to ϵ_0, and the specific heat is proportional to the second derivation of ϵ_0 with respect to temperature. Normalizing the harmonic perturbation results for the specific heat to ΔC [Eq. (3.3)], it follows for $T<T_c$ that

$$\frac{C}{\Delta C}\cong 1+\tfrac{1}{8}[\Delta t/(1-t)]^{3/2} \tag{3.9}$$

is valid for both real and complex ψ. Using the harmonic approximations for $T>T_c$ we have

$$\frac{C}{\Delta C}\cong(1/8\sqrt{2})\,[\Delta t/(1-t)]^{3/2} \tag{3.10}$$

for the real field and as discussed, the result for the complex field merely has an extra factor of 2. The singularities at T_c are, of course, an artifice of perturbation theory and in Sec. IV, numerical solutions will be given which show a smooth behavior throughout the entire temperature region.

From the form of the Hamiltonian it follows that the average value of the square of the order parameter can be obtained by differentiating ϵ_0 with respect to a:

$$\langle|\psi^2|\rangle=\frac{\partial \epsilon_0}{\partial a}. \tag{3.11}$$

Using the harmonic approximation, this gives for $\langle|\psi|^2\rangle$ in the region $T<T_c$,

$$\langle|\psi|^2\rangle\cong\frac{a'\Delta t}{4b}\left[2\left(\frac{1-t}{\Delta t}\right)-\frac{1}{2}\left(\frac{\Delta t}{1-t}\right)^{1/2}\right], \tag{3.12}$$

Above T_c, for the real field, one finds

$$\langle|\psi|^2\rangle\cong\frac{a'\Delta t}{8b}\frac{1}{\sqrt{2}}\left(\frac{\Delta t}{t-1}\right)^{1/2} \tag{3.13}$$

with, as before, an additional factor of 2 for the complex case. Finally, within this same set of approximations, the energy-level differences vary as $kT_c(1-t)^{1/2}$ so that

$$\frac{1}{\xi_1}\approx\frac{1}{2\xi_2}\approx\frac{1}{\xi_0}(\Delta t)^{1/2}\left(\frac{1-t}{\Delta t}\right)^{1/2} \tag{3.14}$$

and both coherence lengths diverge at T_c.

IV. EXACT RESULTS

There exists a large literature on the quantum-mechanical anharmonic oscillator.[10] Of the various approaches, we chose to generate a truncated matrix representation of the Hamiltonian using a basis of n harmonic oscillator states.[11] The scale of the basis state wave functions is set by minimizing the ground harmonic-oscillator-state expectation value of H for T equal to T_c. The matrix is then diagonalized numerically giving eigenvalues and eigenvectors. As discussed in Sec. II, the statistical mechanics is dominated by the ground state and first few excited states. The number n of basis states was varied to obtain numerical convergence; we found that taking $n=20$ gave accurate results over the temperature region of interest.

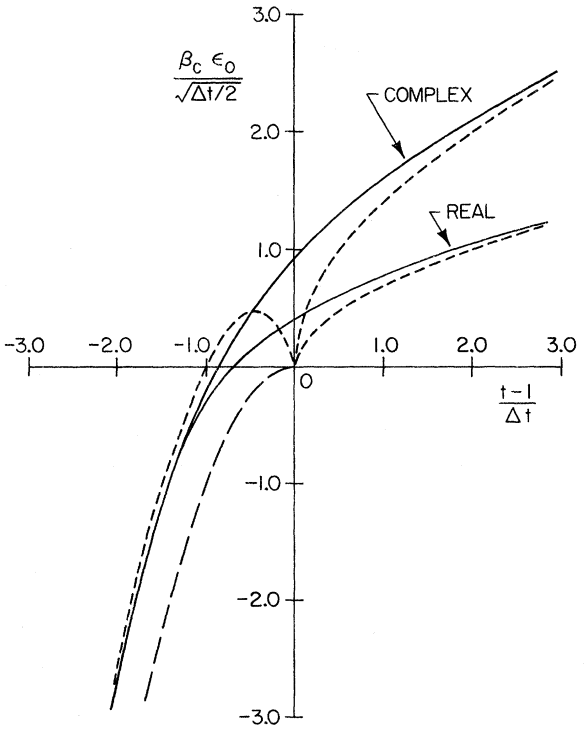

Fig. 2. Free-energy vs temperature. The mean field (— —) and harmonic approximations (-----) are shown for comparison while the exact solutions are plotted as the solid lines.

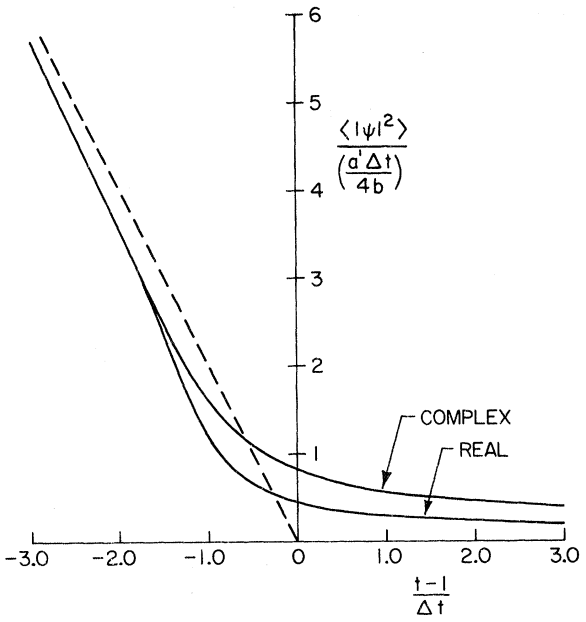

Fig. 3. Expectation value of field intensity vs temperature. The mean-field result is shown dashed.

The ground-state energy ϵ_0, which is the free energy for a section ξ_0 in length, is plotted in Fig. 2 for both the real and complex fields. The long-dashed line represents the mean-field result [Eq. (3.1)] and the short-dashed lines are the harmonic approximations [Eqs. (3.6) and (3.8)]. Asymptotically, both above and below T_c, the solutions approach the results of the harmonic approximations. Note that the exact free energies vary smoothly through the critical region.

The ground-state expectation value $\langle\Psi_0|\,|\psi|^2\,|\Psi_0\rangle$ can be obtained directly from the expression for $|\Psi_0\rangle$ as well as by differentiation of the ground-state energy as discussed in Eq. (3.11). Our numerical results for this expectation value are compared with the mean-field behavior in Fig. 3. Above T_c the mean field vanishes so that the exact result must lie above it. However, as the temperature is lowered below T_c, the exact result becomes smaller than the mean-field value and asymptotically approaches the mean-field behavior

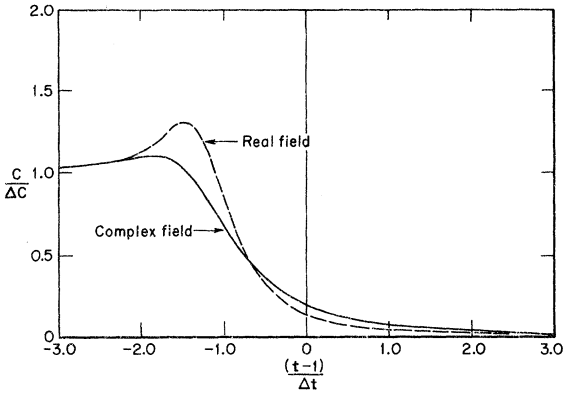

Fig. 4. Specific heat C vs temperature for real (dashed) and complex (solid) fields. ΔC is the molecular-field-step discontinuity.

from below. This contraction of the range of the ground-state wave function is associated with the quantum-mechanical motion of the particle in the anharmonic well. The ground state corresponds to an even-parity state with a finite amplitude at the origin.

The existence of the inflection point in $\langle\Psi_0|\,|\psi|^2|\Psi_0\rangle$ means that the specific heat of the one-dimensional system has a peak. The specific heat was calculated from the first temperature derivative of $\langle\Psi_0|\,|\psi|^2|\Psi_0\rangle$. The results obtained in this way for the real and complex fields are shown in Fig. 4. The mean-field result corresponds to a step discontinuity ΔC for $T < T_c$. Similar results for the specific heat of the complex field have been previously reported by Marcelja using another method.[12] The extra phase degree of freedom of the complex field spreads out the temperature region over which the field ordering occurs, giving rise to a smaller bump in the specific heat. The differences between the real and complex fields are closely associated with the topology of their potentials. For the real field, the particle must pass through the local potential maxima at $\psi = 0$ as it goes between the two potential minima. However, for the complex field, the two-dimensional nature of the potential allows the particle to orbit the origin. Furthermore, the two-dimensional phase space of the complex potential weights the larger $|\psi|$ regions, pushing the expectation value of $\langle|\psi|^2\rangle$ up and smoothing out the specific heat.

Before leaving these thermodynamic properties, it is worthwhile comparing them with previously obtained results for small superconducting particles.[13] For particles with all dimensions small compared to the coherence length, the order parameter is uniform. Therefore, the functional integration simply reduces to one ordinary integration. The average of the square of the order parameter for the complex field of a small particle is shown in Fig. 5. This average lies above the mean-field result at all temperatures. For this zero-dimensional case, the statistical average simply weights each value of ψ with a Boltzmann factor determined by the energy $a|\psi|^2 + b|\psi|^4$. The two-dimensional phase space pushes the weight to larger $|\psi|$ so that $\langle|\psi|^2\rangle$ approaches the mean-field

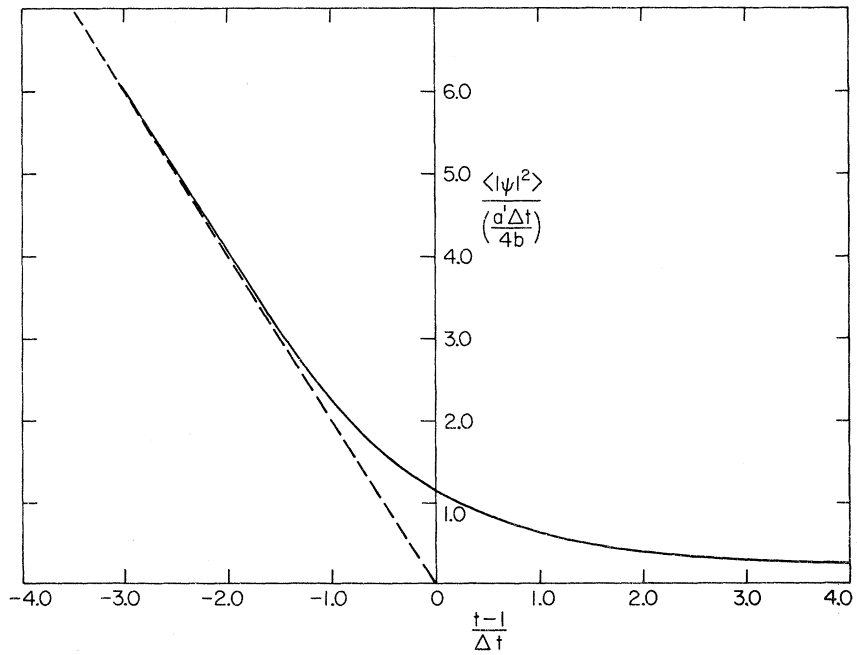

Fig. 5. Expectation value of the field intensity vs temperature for a superconducting small particle. Here $\Delta t = 2(b/2Va'^2\beta_c)^{1/2}$ where V is the volume of the small particle. The dashed curve is the mean-field result [after Mühlschlegel *et al.* (Ref. 13)].

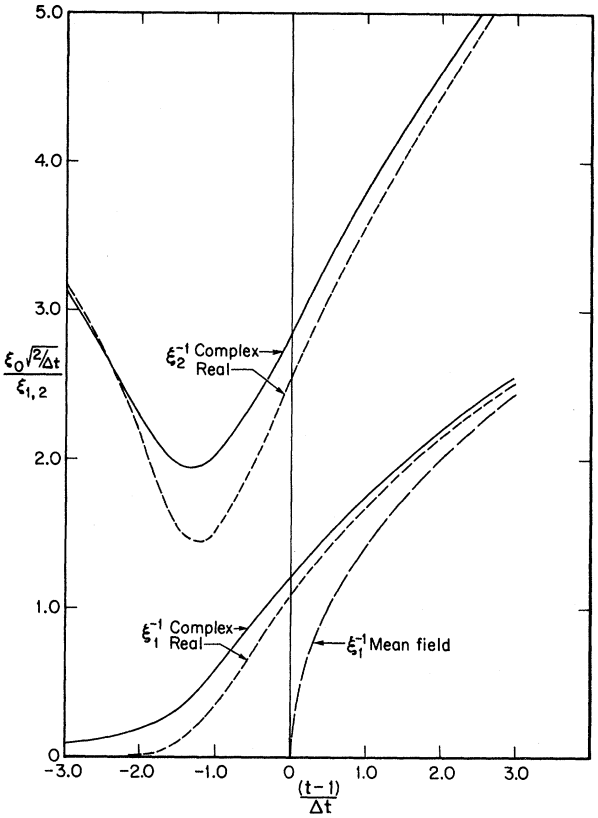

Fig. 6. Inverse correlation lengths ξ_1^{-1} and ξ_2^{-1} vs temperature for real (dashed) and complex (solid) fields.

result smoothly from above. Therefore, the specific heat of the small particles is monatomic in the critical region. Of course, at temperatures below the critical region the specific heat decreases from the mean-field jump. Measurements of the diamagnetic susceptibility of small particles and whiskers should provide information on $\langle|\psi|^2\rangle$ for these systems.

A direct calculation of the correlation functions $C_1(x)$ and $C_2(x)$ using five intermediate states showed that these functions rapidly approached the asymptotic forms determined by the lowest energy intermediate states. The resulting correlation lengths for C_1 and C_2 are

$$\xi_1^{-1} = \xi_0^{-1}\beta(\epsilon_1 - \epsilon_0)\,, \quad \xi_2^{-1} = \xi_0^{-1}\beta(\epsilon_2 - \epsilon_0)\,. \tag{4.1}$$

For the real field, ϵ_1 is the first excited state, an odd parity state, and ϵ_2 is the second excited state which has even parity. For the complex field, ϵ_1 is the lowest excited state having one unit of angular momentum while ϵ_2 is the second excited state of zero angular momentum. In Fig. 6 these inverse correlation lengths are plotted as functions of temperature. In contrast to the behavior of ξ_1, the energy-density fluctuation correlation length ξ_2 has a maximum slightly below T_c and remains short range throughout the transition. Physically, the loss of order-parameter–order-parameter correlations involves changes in the sign of ψ and for the real field this implies that ψ must vanish at some point. The finite energy associated with this provides a barrier for $T < T_c$ giving rise to the exponential growth of ξ_1 below T_c. However, $C_1(x)$ for the complex field can decay simply by phase fluctuations which by gradually changing φ involve arbitrarily small energies. Mathematically, these features reflect the fact that the Hamiltonian for the real field is invariant under the discrete parity transformation $\psi \to -\psi$ while the Hamiltonian for the complex field is invariant under the continuous gauge transformation $|\psi| \to |\psi| e^{i\varphi}$. The real field corresponds to a one-dimensional problem in which for $T < T_c$, the relative splitting of $\epsilon_1 - \epsilon_0$ depends upon "tunneling" through the region of zero ψ. This tunneling decreases exponentially with temperature. However, for the complex field, the potential becomes cylindrically symmetric, and the relevant splitting for $T < T_c$ depends only upon the additional energy associated with one unit of angular momentum. From the expression for the kinetic energy [Eq. (2.24)], it follows that the additional energy associated with one unit of angular momentum varies as $|\psi|^{-2}$ and hence one obtains the much slower $[\Delta t/(1-t)]$ variation of ξ_1^{-1} for the complex case.

The discrete phase-only case has been extensively studied as the classical-planar Heisenberg model of magnetism.[14,15] Here our major point is that arrays of Josephson junctions or weak links provide a physical realization of this model. The isolated superconducting regions can be well below their transition temperatures so that each has a well developed order parameter. However, because of the weak coupling between the superconductors, the critical region for the relative phase ordering of the different superconductors can be shifted to lower temperatures. Here we briefly summarize the expected behavior for such phase-only systems.

The eigenvalues of the transfer matrix are modified Bessel functions $I_n(\beta E_1)$. Therefore, the partition function for a linear array of N weak links is

$$Z = I_0^N(\beta E_1)\,. \tag{4.2}$$

If, contrary to the previous cases, E_1 is temperature independent, then the only temperature dependence is that due to β, and the specific heat per weak link is given by

$$\frac{C}{Nk} = (\beta E_1)^2\left[\frac{I_0''(\beta E_1)}{I_0(\beta E_1)} - \left(\frac{I_0'(\beta E_1)}{I_0(\beta E_1)}\right)^2\right]. \tag{4.3}$$

Alternatively, if the dominate temperature variation is associated with E_1, the specific heat is

$$\frac{C}{Nk}=\left(\frac{\partial E_1}{\partial kT}\right)^2\left[\frac{I''_0(\beta E_1)}{I_0(\beta E_1)}-\left(\frac{I'_0(\beta E_1)}{I_0(\beta E_1)}\right)^2\right]+\frac{\partial^2 E_1}{\partial (kT)^2}\frac{I'_0}{I_0}\,. \tag{4.4}$$

The correlation functions can also be expressed in terms of modified Bessel functions. The phase-phase correlation function between two regions separated by l weak links is

$$C_1(l)=\langle e^{i\varphi_l}e^{-i\varphi_0}\rangle=\left(\frac{I_1(\beta E_1)}{I_0(\beta E_1)}\right)^l\,. \tag{4.5}$$

The energy-density–energy-density correlation function for the discrete case is proportional to

$$\bar{C}_2(l)=\langle\cos(\varphi_{i+l}-\varphi_{i+l-1})\cos(\varphi_i-\varphi_{i-1})\rangle\,. \tag{4.6}$$

This can also be expressed in terms of Bessel functions, and one obtains

$$\bar{C}_2(l)=\frac{I''_0(\beta E_1)}{I_0(\beta E_1)}\delta_{l,0}\,. \tag{4.7}$$

As before, the energy density correlations are short range, in this case in fact, zero range.

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