in Eq. (6) was first suggested in Ref. 1, where it was also shown that $\Delta\gamma(T)$ obeys a simple power law $(T-T_N)^{-1.17\pm0.03}$, a result semiquantitatively explained by Huber's theory.

[7] M. S. Seehra, J. Appl. Phys. 42, 1290 (1971).

[8] Other approximations made in Ref. 5 which led to Eqs. (6) and (7) are the decoupling of the four-spin correlation function into product of two pairs of correlation functions and the assumption of the isotropy of the pair correlation function. The validity of these assumptions is discussed briefly in the text.

[9] M. P. Schulhof, R. Nathans, P. Heller, and A. Linz, Phys. Rev. B 4, 2254 (1971).

PHYSICAL REVIEW B VOLUME 6, NUMBER 9 1 NOVEMBER 1972

Theory of Spin Resonance in Dilute Magnetic Alloys

David C. Langreth*
Department of Physics, Rutgers University, New Brunswick, New Jersey 08903

and

John W. Wilkins†
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850

(Received 15 June 1971)

The scheme of Kadanoff and Baym is shown to be useful for the rigorous derivation of Bloch or Boltzmann equations for spin systems. The scheme is applied to a system of conduction electrons in a metal interacting via an exchange interaction J with a low density of local spins. The coupled Bloch equations appropriate to conduction-electron spin resonance are derived rigorously to second order in J. The prominent features of the derived equations are that (i) the disturbed magnetizations are shown to relax to the instantaneous local equilibrium *magnetization* with the result that correct static susceptibilities are obtained, (ii) the instantaneous magnetizations are *not* proportional to the effective magnetic fields causing the magnetizations to precess, and (iii) the equations are consistent with conservation of the total magnetization.

I. INTRODUCTION

The principal purpose of this paper is to show how the Bloch-like equations for paramagnetic resonance in a dilute magnetic alloy may be derived and to give that derivation to second order in the interaction J between the spin of the conduction electron and the electronic spin of a magnetic impurity.[1] This enterprise turns out to be extraordinarily complicated and so we want at the first to set out the principal results and some arguments for them.

There are, roughly speaking, three sorts of questions that concern us in deriving (or writing down) Bloch equations: (i) what forces drive the spins; (ii) what mechanisms cause the spin to relax; and (iii) toward what "equilibrium" value do the spins relax?

A. Conduction Electrons Alone

Let us illustrate these concerns by writing a characteristic Bloch equation for just the conduction-electron magnetization density $\vec{M}_s$ [2,3]:

$$\frac{\partial\vec{M}_s}{\partial t}+\gamma\vec{H}\times\vec{M}_s+\vec{\nabla}\cdot\overleftrightarrow{J}_s=-\frac{\vec{M}_s-\chi_s\vec{H}}{T_{sl}}\ . \qquad (1.1)$$

For the moment suppose that the gradient term $\vec{\nabla}\cdot\overleftrightarrow{J}$ is zero. Traditionally the driving field for the magnetization is written

$$\vec{H}=H_0\hat{z}+H_1e^{-i\Omega t}(\hat{x}-i\hat{y})\ , \qquad (1.2)$$

where H_0 is the dc applied field (which causes the spins to precess) and H_1 is the amplitude of the ac field (which can cause spin-flip transitions and in any case change the instantaneous axis and frequency of precession of the magnetization). Here $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in the three Cartesian directions. The instantaneous or local equilibrium value of the magnetization is $\chi_s\vec{H}$, where χ_s is the Pauli susceptibility and it is clearly toward this value that the nonequilibrium magnetization $\vec{M}_s$ will relax. In this case the relaxation mechanism is one in which spin angular momentum is carried off to the lattice—hence the order of the subscripts $sl\,(s\rightarrow l)$.

While the form of the right-hand side of (1.1) seems obvious it is a common occurrence[4] for the $\chi_s\vec{H}$ term to be dropped from the x and y components of (1.1). We can readily see the effect of that error on the susceptibility. First we observe that for $H_1 \ll H_0$, the z component of (1.1) (with $\vec{\nabla}\cdot\overleftrightarrow{J}_s$ still zero) is essentially independent of time and

$$(M_s)_z\simeq\chi_sH_0\ . \qquad (1.3)$$

Then defining

$$M_+ = (\vec{M}_s)_x + i(\vec{M}_s)_y \ , \tag{1.4}$$

we have the equation

$$\left(i(\Omega_0 - \Omega) + \frac{1}{T_{sl}}\right) M_+ = \chi_s\left(i\Omega_0 + \frac{H}{T_{sl}}\right) H_+ \ , \tag{1.5}$$

where $\Omega_0 = \gamma H_0$. The first term on the right-hand side of (1.5) results from the identification (1.3), whereas the second would be missing if we made the (erroneous) assumption that M_+ relaxes toward zero (instead of $\chi_s H_+$). The ac value of the susceptibility

$$\chi_+(\Omega) = M_+/H_+ \tag{1.6}$$

is readily calculated:

$$\chi_+(\Omega) = \chi_s\left(\frac{i\Omega_0 + 1/T_{sl}}{i(\Omega_0 - \Omega) + 1/T_{sl}}\right) \ . \tag{1.7}$$

Note that $\chi_+(\Omega = 0) = \chi_s$ as it should. But for the incorrect relaxation assumption the factor of $1/T_{sl}$ would be missing in the numerator and a very inappropriate dc value would result. The necessity for $\chi_s\vec{H}$ in (1.1) has been recognized by theorists[5] who have been unable to derive it and has been ignored by experimentalists who should have known better.

Let us turn finally to the divergence term in (1.1). Since the electrons (of spin $\frac{1}{2}$) can move, the magnetization density can change by the divergence of the magnetization current (actually a tensor) $\overleftrightarrow{J}_s$. This is analogous to a similar term in the continuity equation: $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$. The analogy can be extended further since the ordinary current is proportional to the gradient of the density. In a similar way one might expect that

$$\overleftrightarrow{J}_s = -D_s\vec{\nabla}(\vec{M}_s - \chi_s\vec{H}) \ , \tag{1.8}$$

where D_s is the spin diffusion constant. The $\chi_s\vec{H}$ might be surprising at first but on reflection two different considerations lead to it. First a magnetic moment $\vec{\mu}$ (which any spin possesses) is acted on by the force $-\vec{\mu}\cdot\vec{\nabla}H$ (in addition to any gradient in the magnetization density). Second in a static nonuniform field $\vec{H}(x)$, the equilibrium situation requires

$$\vec{\nabla}\cdot\overleftrightarrow{J}_s = -D_s\nabla^2(\vec{M}_s - \chi\vec{H}) \tag{1.9}$$

to be zero which is clearly the case only if the disturbance from local equilibrium $\vec{M}_s - \chi_s\vec{H}$ drives the diffusion term.

The presence of the $\chi_s\vec{H}$ term in (1.9) has been recognized for some time[6] but only recently have some of its experimental implications been discussed.[7] The presence of the diffusionlike term (1.9) in (1.1) permits a new kind of experiment conduction-electron spin resonance (CESR). For a wave propagating into the sample

$$M_+(z,\ t) = M_+ e^{i(kz - \Omega t)} \ , \tag{1.10}$$

the susceptibility has an additional factor $D_s k^2$ in the numerator and denominator. At resonance ($\Omega = \Omega_0$) the damping of the wave (Imk) is zero, and hence the transmission is enhanced.[8]

So far in this simple example we have supposed that the conduction electrons were noninteracting. Of course, that is not the case. The Fermi-liquid theory permits a phenomenological way of including the contribution of other electrons to the effective magnetic field that any electron (or more properly quasiparticle) sees. Since the magnetization $\vec{M}_s$ commutes with the exchange interaction in the phenomenological theory (as well as with the Coulomb interaction in the Hamiltonian), then $\vec{M}_s$ precesses about only the external field. However, the magnetization current $\overleftrightarrow{J}$ does not so commute and its precession is entirely due to the exchange interaction. This effect gives rise to spin waves which have been experimentally observed in metals where the exchange interaction is small.[9–11] In some transition metals where the interaction is large there appear to be a number of complications including an anisotropic gyromagnetic ratio—so that resonance effects have not yet been clearly observed.

B. Conduction Electrons and Localized Spins

When a number N of localized spins, whose magnetic moment density is $\vec{M}_d$, are added to a metal containing n conduction electrons, a number of effects arise from the interaction

$$-(J/n\gamma^2)\vec{M}_s\cdot\vec{M}_d \ . \tag{1.11}$$

1. Driving Terms

First, although the total magnetization $\vec{M} = \vec{M}_s + \vec{M}_d$ commutes with (1.11) the individual magnetizations do not. Accordingly (1.11) supplies an additional effective magnetic field which causes $\vec{M}_s$ and $\vec{M}_d$ to precess with different frequencies. The effective magnetic field for the conduction electrons $H_{\mathrm{eff},s}$ and for local spins $H_{\mathrm{eff},d}$ to second order in J is

$$\vec{H}_{\mathrm{eff},s(d)} = \vec{H} + (\alpha_1 + \alpha_2)\vec{M}_{d(s)} \ , \tag{1.12}$$

where

$$\alpha_1 = J/\gamma^2 n \tag{1.13}$$

and

$$\alpha_2 = (J/n)^2(\rho/\gamma^2)\ln k_B T/D \ . \tag{1.14}$$

Here ρ is the density of states at the Fermi surface for a band of width D. Note that $\alpha_2/\alpha_1 \sim (J/D)\ln k_B T/D$, and that this is the leading Kondo logarithmic term. Throughout this work we assume that the gyromagnetic ratio γ is the same for the conduction electrons as for the localized spins. It will be clear, as we go on, however, that to first order in J, our equations are valid even if the gyromagnetic ratios are different, if one replaces γ by γ_s or γ_d as appropriate.

2. Relaxation Mechanisms

Second, (1.11) provides a new relaxation mechanism—namely, the conduction electrons and local spins can exchange spin angular momentum. To second order in the interaction the rate of loss of magnetization from the conduction electron to the local spins ($s \rightarrow d$) is

$$\frac{1}{T_{sd}} = \frac{2\pi}{\hbar}\left(\frac{J}{n}\right)^2 \rho N \tfrac{1}{3} S(S+1) , \tag{1.15}$$

while the reverse rate, from the local spins to the conduction electrons ($d \rightarrow s$), is

$$\frac{1}{T_{ds}} = \frac{\pi}{\hbar}\left(\frac{J}{n}\right)^2 \rho^2 kT . \tag{1.16}$$

These rates can be *understood* in terms of the "Golden Rule"—namely, that a rate is proportional to the product of the square of the matrix element $(J/n)^2$ and the number of final states. For (1.15) the number of final states is clearly proportional to the number (N) of local spins. On the other hand, for (1.16) the number of final states for the scattered conduction electron is proportional to the thermal spread ($k_B T$) of those states (clearly this is the case only when $k_B T$ is large compared to the magnetic splitting $\vec{\mu} \cdot \vec{H}$).

The Golden Rule is not, however, sufficient to handle the spin weightings correctly. To get these rates exactly right (even to lowest order in J) it is necessary to subtract from the rate of electrons scattered out of the initial spin state, the rate of electrons scattered back in. When this is done one obtains rates in exact agreement[12] with (1.15) and (1.16). As we shall see in Sec. IV, this "scattering-back" term occurs naturally in our treatment, and the neglect of such a term is one reason for the difficulty that some other formal treatments have had in obtaining (1.15) and (1.16).

One more observation, that initially appears more profound, is that

$$\frac{\chi_{s0}}{T_{sd}} = \frac{\chi_{d0}}{T_{ds}} , \tag{1.17}$$

where the Pauli susceptibility is

$$\chi_{s0} = \tfrac{1}{2} \rho (\hbar\gamma)^2 \tag{1.18}$$

and the Curie susceptibility is

$$\chi_{d0} = \frac{N \tfrac{1}{3} S(S+1)}{k_B T} (\hbar\gamma)^2 . \tag{1.19}$$

If one supposed that additional relaxation terms in (1.1) due to the interaction (1.11) were of the form

$$\frac{-\vec{M}_s}{T_{sd}} + \frac{\vec{M}_d}{T_{ds}} , \tag{1.20}$$

then (1.17) could be a sort of detailed-balance "law" since in equilibrium $\vec{M}_s = \chi_{s0}\vec{H}$ and $\vec{M}_d = \chi_{d0}\vec{H}$. But, of course, as we have already discussed the magnetizations do not relax to zero but to their local equilibrium values. There is as far as we know no *a priori* reason for (1.17). The final observation to be made about relaxation mechanisms arising from (1.11) is that since this interaction term commutes with the total magnetization $\vec{M}$ it can not lead, by itself, to any relaxation of $\vec{M}$. Accordingly any relaxation terms due to (1.11) that appear in the Bloch equation for $\vec{M}_s$ must appear in exactly the same form, but with the opposite sign, in the equation for $\vec{M}_d$.

So far we have discussed relaxation mechanisms due only to (1.11). As in our first example a spin-orbit interaction that couples the conduction electron to the lattice provides for a relaxation of the conduction-electron magnetization directly to the lattice.[13] It might appear that the local spins would not relax to the lattice but the experimental evidence[14] to the contrary suggests a $1/T_{dl}$ on the order of a few hundred Gauss. Its temperature and concentration dependence are not well documented. No published mechanism has yet appeared, let alone any calculations of its magnitude. We will offer no such calculation here, and commend the problem to our colleagues.

3. Local Equilibrium Magnetization

In the absence of the interaction (1.11) the local equilibrium of $\vec{M}_s$ and $\vec{M}_d$ that should appear in the relaxation and diffusion terms are clearly $\chi_{s0}\vec{H}$ and $\chi_{d0}\vec{H}$. But the presence of (1.11) produces a nettling complexity that does not become apparent until second order in J. The obvious (but *wrong*) guess could be that $\vec{M}_{s(d)}$ relaxes to $\chi_{s(d)0}\vec{H}_{\text{eff},s(d)}$ and that is correct to first order in J.[15] But to second order in J the appropriate quantity in the relaxation of conduction-electron magnetization is

$$\delta\vec{M}_s = \vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d) , \tag{1.21}$$

while for local spins the quantity is

$$\delta\vec{M}_d = \vec{M}_d - \chi_{d0}[\vec{H} + (\alpha_1 + 2\alpha_2)\vec{M}_s] . \tag{1.22}$$

Note the absence of α_2 in (1.21) and its doubled presence in (1.22). One test of (1.21) and (1.22) is that they give the correct static susceptibility. For a dc field, $\delta\vec{M}_s = \delta\vec{M}_d = 0$. Solving the resulting equation we get

$$\begin{aligned} \chi_s &= \chi_{s0}(1 + \alpha_1 \chi_{d0}) , \\ \chi_d &= \chi_{d0}[1 + (\alpha_1 + 2\alpha_2)\chi_{s0}] , \end{aligned} \tag{1.23}$$

and hence

$$\chi - \chi_{s0} = \chi_{d0}[1 + 2(\alpha_1 + \alpha_2)\chi_{s0}] , \qquad (1.24)$$

which is correct[15] to that order in perturbation theory. Unfortunately (1.24) would also have resulted if we had guessed that the same factor $(\alpha_1 + \alpha_2)$ should occur in (1.21) and (1.22). The strongest argument for the asymmetry in the two equations we know is that (1.21) is known[16] to be correct to fourth order in perturbation theory and we argue in Appendix D that it is true to all orders. Furthermore our formalism also gives $-D_s\nabla^2\delta\vec{M}_s$ in the diffusion term.

The results of our discussion can be summarized by writing down the Bloch-like equations for $\vec{M}_s$ and $\vec{M}_d$:

$$\frac{\partial\vec{M}_s}{\partial t} + \gamma\vec{H}_{\mathrm{eff},s}\times\vec{M}_s - D_s\nabla^2\delta\vec{M}_s = -\left(\frac{1}{T_{sl}} + \frac{1}{T_{sd}}\right)\delta\vec{M}_s + \left(\frac{1}{T_{ds}} + \frac{\alpha_2\chi_{s0}}{T_{dl}}\right)\delta\vec{M}_d \qquad (1.25)$$

and

$$\frac{\partial\vec{M}_d}{\partial t} + \gamma\vec{H}_{\mathrm{eff},d}\times\vec{M}_d = -\left(\frac{1+\alpha_2\chi_{s0}}{T_{dl}} + \frac{1}{T_{ds}}\right)\delta\vec{M}_d + \frac{1}{T_{sd}}\delta\vec{M}_s . \qquad (1.26)$$

These are essentially the equations of Hasegawa[17] with the following exceptions: (i) His $H_{\mathrm{eff},s(d)}$ were only given to first order in J; (ii) his equations relaxed toward zero $\vec{M}_s$ and $\vec{M}_d$; (iii) the diffusion term and $1/T_{dl}$ were missing; and (iv) he allowed for different gyromagnetic ratios for $\vec{M}_s$ and $\vec{M}_d$. We could do (iv) but it would complicate our analysis greatly. To linear order in J, however, the extension is trivial.

Furthermore these are essentially the equations we published previously[1] with the exceptions of (i) an improved treatment of $\delta\vec{M}_s$ and $\delta\vec{M}_d$ and (ii) the absence of the hyperfine interaction. Again the hyperfine interaction effects could be included but at the cost of increasing complexity.

Note added in proof. The results of our calculation based on the s-d exchange model [our Eq. (2.1)] are identical, to an equivalent order in perturbation theory, with those for the Anderson model [P. W. Anderson, Phys. Rev. 124, 41 (1961)]. The detailed demonstration, based on an application of the Schrieffer-Wolff unitary transformation [Phys. Rev. 149, 491 (1966)], is given in a note added to Appendix D. For example, Eq. (1.21) is an example of the "compensation theorem"—an Anderson-model statement that the conduction-electron magnetization is unchanged by the introduction of an impurity state (under the assumption of constant matrix elements and the neglect of potential scattering). (Anderson's proof of this "theorem" is readily extended to all orders in perturbation theory.) The relevant point to realize is that the definitions of $\vec{M}_s$ and $\vec{M}_d$ in the s-d exchange model [our Eq. (2.1)] do not correspond one for one to the definition of the "conduction-electron spins" and "d-electron spins" in the Anderson model. Rather one has at least approximately

$$\vec{M}_s^{\mathrm{Anderson}} = \vec{M}_s - \alpha_1\chi_{s0}\vec{M}_d ,$$

$$\vec{M}_d^{\mathrm{Anderson}} = \vec{M}_d(1 + \alpha_1\chi_{s0}) ,$$

when potential scattering is neglected. These results explain the apparent inconsistency between our prediction of relaxation toward local instantaneous equilibrium and the work of T. Sasada and H. Hasegawa [Progr. Theoret. Phys. (Kyoto) 45, 1072 (1971)] which appeared after our manuscript was written and which makes use of the Anderson model. Indeed, the substitution of the above relations into Eq. (4.1) of Sasada and Hasegawa yields a set of equations with relaxation terms of the same form as our results, Eqs. (1.25) and (1.26) (with α_2 and $1/T_{dl}$ set equal to zero; this is shown in Appendix D). Thus, the results of Sasada and Hasegawa contradict neither our results, nor the recent results of M. B. Walker [Phys. Rev. (to be published)] nor of S. E. Barnes and J. Zitkova [Phys. Rev. (to be published)], both of which agree with ours in this respect.

C. Plan of Paper

The principal results of this paper are Eqs. (1.25) and (1.26) and the methods used to derive them. Since we believe that this paper constitutes the first derivation of these equations we are constrained to offer considerable detail of the method. The basic difficulty in deriving a Bloch or Boltzmann equation is that one must avoid assuming the form of the equation in its derivation for in so doing important terms may be thrown away. This is not too difficult when only a single unknown is involved. But the presence of two unknowns (such as $\vec{M}_s$ and $\vec{M}_d$) requires a formalism that can derive an equation for one unknown without assuming the solution for the others. That is usually too difficult for most Green's-function schemes and errors are often made with the relaxation and driving terms.[18] In this paper we show explicitly that the method of Kadanoff and Baym[19] which does not have this defect, can be cast in a form which is most useful in deriving rigorous results in spin systems.[20] The basic procedures are set out in Sec. II, but the major formal development is deferred to Appendix A. This organizational pattern of placing the computational details in Appendices is used throughout. The rest of the organization of the paper will be presented at end of Sec. II after the formal scheme has been described.

II. PRELIMINARIES

A. Hamiltonian and Approximations

Our aim is to derive a generalization of the Bloch equations to apply to n conduction electrons of energy ϵ_p and spin $s=\frac{1}{2}$ and to N localized moments of spin S. Essentially we wish to derive Hasegawa's phenomenological equations[17] directly from the Hamiltonian

$$\sum_{i=1}^{n}\epsilon_{p_i}-\sum_{i=1}^{n}\gamma\vec{s}_i\cdot\vec{H}+\sum_{j=1}^{N}E_{p_j}-\sum_{j=1}^{N}\gamma\vec{S}_j\cdot\vec{H}-\sum_{i,j}\frac{J}{n}\vec{s}_i\cdot\vec{S}_j\ . \tag{2.1}$$

Equation (2.1) already displays some of the basic features of our notation, in that upper-case letters generally refer to the localized spins and lower-case letters refer to the conduction electrons. The one prominent exception to this rule is that we shall use the symbol $\vec{M}_s$ to refer to the conduction-electron magnetization density, while $\vec{M}_d$ will denote the corresponding localized spin quantity. Equation (2.1) also hints at a calculational technique we shall use: By formally giving the localized spins a band energy E_p, we can straightforwardly apply all the standard Green's-function diagrammatic techniques, and then take the infinite-mass limit $E_p\rightarrow 0$ and really localize the localized electrons[21]; in practice, of course, this limit will occur very early in the calculation. Another feature evident from (2.1) is that we assume the gyromagnetic ratios γ for each spin species are identical. To take them unequal vastly complicates the details of the calculation, and will be deferred to a later work, where the effects of the hyperfine interaction will be included[22] from the beginning.

What we do here is to derive Hasegawa's equations directly from the Hamiltonian (2.1) with a number of approximations mentioned here. The first is that γH as well as the frequency of whatever disturbs the system from equilibrium is much smaller than $k_B T$.[23] Thus we keep only the lowest nonvanishing order of each of these quantities. Furthermore, we assume that the deviations from thermal equilibrium are sufficiently small that we can use linear-response theory to describe them, and that externally induced fields and magnetizations vary spatially on a scale large compared with the typical interparticle spacing of the local spins. Thus we assume that external perturbations are weak and slowly varying in space and time; how these assumptions are effected systematically will be shown in detail later.

Our most restrictive approximation, however, is that we shall assume that the coupling constant J is sufficiently weak, that it can be treated by perturbation theory. In particular, we will systematically and rigorously produce every term of order J^2 or lower on either side of Hasegawa's equations. Of special interest is, that in order J^2, Kondo-effect logarithmic terms appear for the first time.[1] These can be used to estimate whether, for example, the Kondo effect will alter line shapes in any important way, but should not be used quantitatively, even though they are exact to this order in J.

One further nonessential approximation, which we make and which is fully consistent with the neglect of the momentum dependence of J, is to take the conduction-electron density of states (for a single spin)

$$\rho(p_0)\equiv\sum_p\delta(p_0-\epsilon_p)=\rho \tag{2.2}$$

to be a constant independent of the energy p_0.

B. Method of Calculation

A number of years ago Kadanoff and Baym[19] derived a set of transport equations, which provide an ideal starting point for deriving Boltzmann-like equations. One trouble with the theory as originally formulated is that the exact equations [(8.27) and (8.28) in Ref. 19] are so cumbersome that they take 12 printed lines to write, and would therefore appear useless in a practical calculation. By making a gradient expansion, assuming quantities varied slowly in space and time, Kadanoff and Baym derived a rather elegant generalized Boltzmann equation which can and has been solved in practice. Here, however, while we would be willing to make the gradient expansion in the space and time variables by our earlier mentioned assumptions, the spin variables must remain as discrete quantum-mechanical matrices. Therefore we cannot use Kadanoff and Baym's generalized Boltzmann equation.

To obviate this difficulty, we use a notation[24] which renders Kadanoff and Baym's exact equations as simple in form as their approximate ones. Then we can do the gradient expansion in the space and time variables, while leaving the spin matrices intact. The derivation is carried out in Appendix A. We summarize the results here.

The formalism is based on knowing the two pieces of the self-energy $\Sigma^{\gtrless}$ as a functional of the two pieces of the Green's function $G^{\gtrless}$. This knowledge can be obtained to any desired degree of accuracy by standard perturbation theory, which works formally just as it would in equilibrium. Note that the quantities here are to be considered a function of two time variables, two spacial variables, and two spin variables. Exactly as in equilibrium $G^{>}$ and $G^{<}$ are defined as

$$G^{<}_{\alpha\beta}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)=\langle\langle\psi^{\dagger}_{\beta}(\vec{x}_2,\ t_2)\psi_{\alpha}(\vec{x}_1,\ t_1)\rangle\rangle\ ,$$

$$G^{>}_{\alpha\beta}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)=\langle\langle\psi_\alpha(\vec{x}_1,\ t_1)\psi^\dagger_\beta(\vec{x}_2,\ t_2)\rangle\rangle\ , \tag{2.3}$$

where α and β represent spin components and $\langle\langle\cdots\rangle\rangle$ represents the statistical average appropriate to a system that was in thermal equilibrium at time $t=-\infty$. Exactly as in equilibrium one defines the spectral weight function A and the lifetime function Γ as

$$A=G^{>}+G^{<}\ , \tag{2.4a}$$

$$\Gamma=\Sigma^{>}+\Sigma^{<}\ , \tag{2.4b}$$

and the real part of the Green's function G and the real part of the self-energy Σ as

$$G(t_1,\ t_2)=(1/2i)A(t_1,\ t_2)\,\mathrm{sgn}(t_1-t_2)\ , \tag{2.5a}$$

$$\Sigma(t_1,\ t_2)=\Sigma^{(1)}\delta(t_1-t_2)+(1/2i)\Gamma(t_1,\ t_2)\,\mathrm{sgn}(t_1-t_2)\ , \tag{2.5b}$$

where we have supressed the nonessential variables. All parts of the self-energy diagonal in time (the Hartree-like terms) are to be included in $\Sigma^{(1)}$. To make closer contact with the equilibrium case, we note that if one lets $\tau=t_1-t_2$; $t=\frac{1}{2}(t_1+t_2)$ and Fourier transforms with respect to the difference variable τ, then (2.5) implies that

$$G(p_0,\ t)=\int\frac{d\omega}{2\pi}\,\frac{A(\omega,\ t)}{p_0-\omega}\ , \tag{2.6a}$$

$$\Sigma(p_0,\ t)=\Sigma^{(1)}+\int\frac{d\omega}{2\pi}\,\frac{\Gamma(\omega,\ t)}{p_0-\omega}\ . \tag{2.6b}$$

In terms of this notation, the generalized Kadanoff-Baym (GKB) equation (see Ref. 24 and Appendix A) is

$$(1/i)[(G_0^{-1}+\gamma\vec{S}\cdot\vec{H}-\Sigma),\ G^{<}]-(1/i)[\Sigma^{<},\ G]=-\tfrac{1}{2}\{\Gamma,\ G^{<}\}+\tfrac{1}{2}\{\Sigma^{<},\ A\}\ , \tag{2.7}$$

where $[A,\ B]$ is the commutator and $\{A,\ B\}$ is the anticommutator. Matrix multiplication is implied not only in the space and spin variables, but also in the time variables which are integrated along the real axis from $-\infty$ to ∞. The final equation which makes (2.7) a closed system is just the ordinary Dyson equation

$$(G_0^{-1}+\gamma S\cdot H-\Sigma\mp i\tfrac{1}{2}\Gamma)(G\pm i\tfrac{1}{2}A)=\underline{1}\ , \tag{2.8}$$

which holds in equilibrium as well as nonequilibrium (see Ref. 24 and Appendix A). Of course, we have written the above equations with upper-case letters representing the local spins, but analogous equations exist for the conduction electrons as well. Note that in the above formulas G_0^{-1} is (as usual) the inverse of the bare Green's function

$$[G_0^{-1}(t_1,\ t_2;\ \vec{x}_1,\ \vec{x}_2)]_{\alpha\beta}=-i\frac{\partial}{\partial t_1}\,\delta(t_1-t_2)\delta(\vec{x}_1-\vec{x}_2)\delta_{\alpha\beta}$$

$$-\delta(t_1-t_2)\textstyle\sum_p E_p e^{ip\cdot(\vec{x}_1-\vec{x}_2)}\delta_{\alpha\beta}\ , \tag{2.9}$$

and $\underline{1}$ (as in $\gamma\vec{S}\cdot\vec{H}\,\underline{1}$) is the unit matrix in space and time, whose elements are $\delta(t_1-t_2)\,\delta(\vec{x}_1-\vec{x}_2)$. We thus have an exact set of equations which in principle predicts the exact behavior of a nonequilibrium system. We, however, wish to consider only slowly varying disturbances, and to make a gradual expansion of all quantities keeping all the largest terms.

The general procedure now is to write all the basic quantities in terms of sum and difference variables of the space and time arguments

$$\tau=t_1-t_2,\qquad t=\tfrac{1}{2}(t_1+t_2)\ ,$$
$$\vec{r}=\vec{x}_1-\vec{x}_2,\qquad \vec{x}=\tfrac{1}{2}(\vec{x}_1+\vec{x}_2)\ , \tag{2.10}$$

and Fourier transform with respect to the difference variables, and write all quantities as a function of the variables $(\vec{p},\ p_0;\ \vec{x},\ t)$, where $\vec{p}$ is the Fourier-transform variable corresponding to $\vec{r}$ and p_0 is the Fourier-transform variable corresponding to τ.

Of course, one cannot in general express the Fourier transform (Ft) with respect to the difference variables of a matrix commutator exactly in a useful form, but since we are expanding in slowly varying quantities, we use the gradient expansion

$$\frac{1}{i}[A,\ B]_{\mathrm{Ft}}=\frac{1}{i}[A,\ B]+\frac{1}{2}\left\{\frac{\partial A}{\partial p_0},\ \frac{\partial B}{\partial t}\right\}-\frac{1}{2}\left\{\frac{\partial A}{\partial t},\ \frac{\partial B}{\partial p_0}\right\}+\cdots-\tfrac{1}{2}\{\nabla_p A,\ \nabla_x B\}+\tfrac{1}{2}\{\nabla_x A,\ \nabla_p B\}+\cdots \tag{2.11a}$$

and

$$\frac{1}{i}\{A,\ B\}_{\mathrm{Ft}}=\{A,\ B\}+\frac{1}{2}\left[\frac{\partial A}{\partial p_0},\ \frac{\partial B}{\partial t}\right]-\frac{1}{2}\left[\frac{\partial A}{\partial t},\ \frac{\partial B}{\partial p_0}\right]+\cdots-\tfrac{1}{2}[\nabla_p A,\ \nabla_x B]+\tfrac{1}{2}[\nabla_x A,\ \nabla_p B]+\cdots\ , \tag{2.11b}$$

where on the right-hand side of (2.11) there is no longer any matrix multiplication in space and time variables, but only in spin variables; the space-time variables $\vec{p}$, p_0, $\vec{x}$, t are set equal in both members of a product. [The writing of (2.11) presented a notational problem; the reader is warned that the scalar product is implied between $\nabla_x A$ and $\nabla_p B$ and similar terms.] Henceforth we will generally suppress the variables x and t, although the dependence on them is implied. In the later section of this paper we will show how the GKB plus the gradient expansion above can be reduced to Hasegawa's equations.

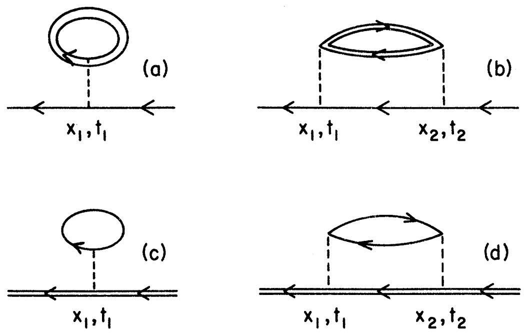

FIG. 1. First- and second-order (in J) self-energy diagrams for conduction electrons (a) and (b), and for localized spins (c) and (d). Single solid lines represent the fully interacting propagator for the conduction electrons, while double solid lines represent the localized spins. Vertical dashed line represents the instantaneous interaction $-(J/N)\vec{s}\cdot\vec{S}$. Care must be taken to preserve the matrix order, with lines entering a vertex multiplying the vertex on the right and those leaving multiplying it on the left. External lines are shown for clarity but are not included in the definitions of the self-energies. See also the discussion including Ref. 25 in Sec. II C.

C. Expansion of Self-Energies

In order to have a closed set of equations, one must know what the self-energy functions $\Sigma^{\gtrless}$ are in terms of $G^{\gtrless}$. To a given order of perturbation theory this is straightforward, because in the imaginary-time domain, the self-energy Σ has a well-defined diagrammatic expansion in powers of the Green's function G. One writes this down, and then uses the techniques described in Appendix A to continue the results to obtain the real-time physical functions $\Sigma^{\gtrless}$. In our particular approximation of neglecting terms of higher order than J^2, this analytic continuation is particularly easy to effect.

The diagrams for the self-energy of the conduction electrons to order J^2 is shown in Figs. 1(a) and 1(b); while those for the localized spins are shown in Figs. 1(c) and 1(d). The rules for evaluating such diagrams are standard and are derived, for example, in Ref. 19.[25] We must take special care in our case to preserve the order of the spin operators. Figure 1(a) makes only a contribution to $\sigma^{(1)}$ [see (2.5b)] and can be written immediately as

$$\sigma^{(1)}(\vec{x}_1,\ t_1,\ \vec{x}_2,\ t_2)=(-J/n)\,\mathrm{Tr}_S\vec{s}\cdot\vec{S}G^{<}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)\times\delta(t_1-t_2)\delta(\vec{x}_1-\vec{x}_2)\ . \quad (2.12)$$

Expressing this in terms of the sum and difference variables (2.10) and Fourier transforming with respect to the difference variables, yields

$$\sigma^{(1)}(\vec{p},\ p_0;\ \vec{x},\ t)=-(J/n)\,\mathrm{Tr}_S\vec{s}\cdot\vec{S}\sum_{p,p_0}G^{<}(\vec{p},\ p_0;\ \vec{x},\ t)\ . \quad (2.13)$$

Since by definition

$$\vec{M}_d(\vec{x},\ t)=\mathrm{Tr}_S\gamma\vec{S}\sum_{p,p_0}G^{<}(\vec{p},\ p_0;\ \vec{x},\ t)\ , \quad (2.14)$$

we have

$$\sigma^{(1)}(\vec{x},\ t)=-(J/n\gamma^2)\gamma\vec{s}\cdot\vec{M}_d(\vec{x},\ t)\ , \quad (2.15)$$

which is independent of p and p_0. In an exactly similar manner the diagram of Fig. 1(c) for the localized spins may be evaluated as

$$\Sigma^{(1)}(x,\ t)=-(J/n\gamma^2)\gamma\vec{S}\cdot\vec{M}_s(\vec{x},\ t)\ . \quad (2.16)$$

We turn now to the second-order terms such as Fig. 1(b), which is given by

$$(J/n)^2\,\mathrm{Tr}_S\vec{s}\cdot\vec{S}g(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)\times G(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)\vec{s}\cdot\vec{S}G(\vec{x}_2,\ t_2;\ \vec{x}_1,\ t_1)\ , \quad (2.17)$$

where we have used the fact that the assumed interaction (2.1) is a contact interaction to do the spacial integrals. Because there are no remaining time integrals, (2.17) is trivial to continue to real time;

$$\sigma^{\gtrless}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)=(J/n)^2\,\mathrm{Tr}_S\vec{s}\cdot\vec{S}g^{\gtrless}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)\times G^{\gtrless}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)\vec{s}\cdot\vec{S}G^{\lessgtr}(\vec{x}_2,\ t_2;\ \vec{x}_1,\ t_1)\ . \quad (2.18)$$

In terms of sum and difference variables (2.10) this becomes

$$\sigma^{\gtrless}(\vec{r},\ \tau;\ \vec{x},\ t)=(J/n)^2\,\mathrm{Tr}_S\vec{s}\cdot\vec{S}g^{\gtrless}(\vec{r},\ \tau;\ \vec{x},\ t)\times G^{\gtrless}(\vec{r},\ \tau;\ \vec{x},\ t)\vec{s}\cdot\vec{S}G^{\lessgtr}(-\vec{r},\ -\tau;\ \vec{x},\ t)\ . \quad (2.19)$$

Finally, upon Fourier transforming over the difference variables, we have

$$\sigma^{\gtrless}(\vec{p},\ p_0;\ \vec{x},\ t)=(J/n)^2\,\mathrm{Tr}_S\sum_{k,k_0}\sum_{q,q_0}\times\vec{s}\cdot\vec{S}g^{\gtrless}(\vec{p}+\vec{k},\ p_0+k_0;\ \vec{x},\ t)G^{\gtrless}(\vec{q},\ q_0;\ \vec{x},\ t)\vec{s}\cdot\vec{S}\times G^{\lessgtr}(\vec{q}+\vec{k};\ q_0+k_0;\ \vec{x},\ t)\ . \quad (2.20)$$

Note that because of the simplicity of this term coupled with our assumption that J is a contact interaction, there was no need to use the gradient expansion to accomplish the Fourier transformation; in general, of course, this would not be true.

The expression for the diagram of Fig. 1(d) is derived trivially from (2.20) by interchanging upper- and lower-case letters:

$$\Sigma^{\gtrless}(\vec{p},\ p_0;\ \vec{x},\ t)=(J/n)^2$$

$$\times \mathrm{Tr}_s \sum_{k,k_0} \sum_{q,q_0} \vec{s}\cdot\vec{S}G^{\gtrless}(\vec{p}+\vec{k},\ p_0+k_0;\ \vec{x},\ t)$$

$$\times g^{\gtrless}(\vec{q},\ q_0;\ \vec{x},\ t)\vec{s}\cdot\vec{S}g^{\lessgtr}(\vec{q}+\vec{k};\ q_0+k_0;\ \vec{x},\ t)\ . \tag{2.21}$$

D. Plan of Remaining Sections

We are now in a position to derive the Bloch equations for $\vec{M}_s$ and $\vec{M}_d$. A brief example may help to motivate the process. Consider, for example, the time-derivative part of G_0^{-1} in (2.9). Then as shown in Appendix A [Eq. (A35)] (but in any case it does not seem very surprising)

$$\left(\frac{\partial}{\partial t_1}\delta(t_1-t_2)\delta(\vec{x}_1-\vec{x}_2),\ G^{<}(\vec{x}_1,\ t_1;\ \vec{x}_2,\ t_2)\right)_{\mathrm{Ft}}$$

$$=\frac{\partial}{\partial t}G^{>}(\vec{p},\ p_0;\ \vec{x},\ t)\ . \tag{2.22}$$

If we now use the prescription (2.14) for calculating $\vec{M}_d(\vec{x},\ t)$ on (2.22) we get $\partial\vec{M}_d(\vec{x},\ t)/\partial t$ which is one term of the Bloch equation. Roughly speaking, the prescription (2.14) applied to (2.7)

$$\mathrm{Tr}_S\gamma\vec{S}\sum_{p,p_0}(\mathrm{GKB})_{\mathrm{Ft}}, \tag{2.23}$$

together with the gradient expansion (2.11) produces the Bloch equation with the left-hand side (LHS) of GKB giving rise to the driving terms and the right-hand side (RHS) to the relaxation terms. But, of course, some organization of the algebra is necessary. For example, consider the magnetic field term of (2.7); for it the first term of (2.11a) in (2.23) yields

$$\sum_{p,p_0}\mathrm{Tr}_S\gamma\vec{S}[\gamma\vec{S}\cdot\vec{H},\ G^{<}(p,\ p_0)]\ . \tag{2.24}$$

To readily obtain the expected result $\gamma\vec{H}\times\vec{M}_d$ requires an ability to decompose $G^{<}$ into that part which is proportional to $\vec{M}_d$ and that which is not. Just how important such a decomposition will be on LHS becomes clear when we consider a term such as $[\Sigma,\ G^{<}]$. The self-energy Σ must be decomposed into pieces proportional to $\vec{S}\cdot\vec{H}$, $\vec{S}\cdot\vec{M}_d$, and $\vec{S}\cdot\vec{M}_s$ in order for the commutator to be taken. In Sec. III, we display decompositions for the Green's functions and the self-energies appropriate to the LHS of the Bloch equation; see also Table I.

But a decomposition that is appropriate for the driving terms is certainly not for the relaxation terms or RHS of the Bloch equation. There we want to be able to separate out a piece of $G^{<}$ proportional to $\delta\vec{M}_d$ and further show that the remainder makes no contribution to the relaxation terms; in other words that the RHS contains only terms proportional to $\delta\vec{M}_s$ or to $\delta\vec{M}_d$. If that is the case, then, as we showed in Sec. I, the correct dc susceptibilities follow automatically. The decompositions appropriate to the RHS are discussed in Sec. IV along with a calculation of the susceptibilities. The actual evaluation of the relaxation terms (RHS) is deferred to Sec. V. Finally we return to work out the driving terms (or LHS of Bloch equation) in Sec. VI. In Sec. VII, all the results are brought together to display the complete Bloch equations for conduction electrons and the local spins. The more tedious details of the calculations are deferred to Appendices B–F.

III. DECOMPOSITION OF $g^{\gtrless}$ ($G^{\gtrless}$) FOR LHS OF GKB AND APPROPRIATE SELF-ENERGIES

Essential to an explicit evaluation of the GKB equation (2.7) in Sec. II are calculations of the relevant imaginary parts of the self-energy to second order in J. But that calculation requires a prescription for expanding $g^{\gtrless}$ to first order in $\vec{H}$ and $\vec{M}_s$ and $G^{\gtrless}$ to first order in $\vec{H}$ and $\vec{M}_d$ in a precise way. Consider the following choice for $g^{\gtrless}$:

$$g^{<}(\vec{p},\ p_0)=2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})$$

$$\times f(p_0+\gamma\vec{s}\cdot\vec{H})+\Delta g^{<}(\vec{p},\ p_0) \tag{3.1}$$

and

$$g^{>}(\vec{p},\ p_0)=2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})[1-f(p_0+\gamma\vec{s}\cdot\vec{H})]$$

$$-\Delta g^{<}(\vec{p},\ p_0)\ , \tag{3.2}$$

where the Fermi factor $f(p_0)=(e^{\beta p_0}+1)^{-1}$. Since we will work to only first order in H, there is no confusion about the relative order of the δ function and the Fermi factor (if we were working to higher order in the magnetic field, symmetrized products would be required).

First we observe that

$$g^{<}(\vec{p},\ p_0)+g^{>}(\vec{p},\ p_0)=2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})\ , \tag{3.3}$$

which is the correct form for the electron spectral weight function in a magnetic field to zero order in J. We should, of course, consider spectral weight function to finite order in J and in the presence of nonequilibrium local spin magnetization $\vec{M}_d$, and we discuss this point when the LHS of the GKB equation is derived in Sec. VI. It is clear, however, that (3.1) and (3.2) are adequate for the calculation of the self-energies to order J^2.

Next, we show that the conduction-electron magnetization is uniquely determined by $\Delta g^{<}(\vec{p},\ p_0)$. By definition

$$\vec{M}_s=\mathrm{Tr}_s\gamma\vec{s}\sum_{p,p_0}g^{<}(\vec{p},\ p_0)\ . \tag{3.4}$$

Hence by (3.1)

$$\vec{M}_s=\mathrm{Tr}_s\gamma\vec{s}\sum_{p,p_0}2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})f(p_0+\gamma\vec{s}\cdot\vec{H})$$

$$+\mathrm{Tr}_s \gamma\vec{s} \sum_{p,p_0} \Delta g^{<}(\vec{p}, p_0) . \quad (3.5)$$

The first term in (3.5) is identically zero and connection between $\vec{M}_s$ and $\Delta g^{<}$ established. This follows from variable transformation $p_0+\gamma\vec{s}\cdot\vec{H}$ to p_0 which leaves the integrand proportional to $\vec{s}$ and hence zero after the spin trace is taken. It is useful in performing the spin trace to exhibit the explicit spin dependence of $\Delta g^{<}$—namely,

$$\Delta g^{<}(\vec{p}, p_0) = \vec{m}(\vec{p}, p_0)\cdot\vec{s} , \quad (3.6)$$

where clearly $\sum_{p_0}\vec{m}(\vec{p}, p_0)$ is that spin projection of the distribution function appropriate to the magnetization. Using the spin traces in Table IV, we have

$$\vec{M}_s = \gamma(2s+1)\tfrac{1}{3}s(s+1) \sum_{p,p_0} \vec{m}(\vec{p}, p_0) . \quad (3.7)$$

We should remark that the choice (3.1) is not the obvious one, since in equilibrium the obvious factor to be multiplying the δ function would be just $f(p_0)$ [not $f(p_0+\gamma\vec{s}\cdot\vec{H})$]. We can rearrange (3.1) to discuss this possible decomposition:

$$g^{<}(\vec{p}, p_0) = 2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})f(p_0)$$
$$+\{2\pi\delta(p_0-\epsilon_p)[f(p_0+\gamma\vec{s}\cdot\vec{H}) - f(p_0)] + \Delta g^{<}\} , \quad (3.8)$$

where in the second δ function we have omitted the addend of $\gamma\vec{s}\cdot\vec{H}$ since the square bracket is already of first order in the magnetic field. Now the first term in the curly brackets when used in calculating $\vec{M}_s$ by (3.4) clearly gives

$$\mathrm{Tr}_s\gamma\vec{s}(\gamma\vec{s}\cdot\vec{H})\rho\int dp_0\frac{\partial f}{\partial p_0}$$
$$= -(2s+1)\tfrac{1}{3}s(s+1)\rho\gamma^2\vec{H} = -\chi_{s0}\vec{H} , \quad (3.9)$$

where

$$\chi_{s0} = \tfrac{1}{2}\rho\gamma^2 \quad (3.10)$$

is the Pauli susceptibility for noninteracting conduction electrons. Since $\Delta g^{<}$ corresponds to the total magnetization, clearly the term in the curly brackets corresponds to $\vec{M}_s - \chi_{s0}\vec{H}$, that is, to the nonequilibrium part of the magnetization. In a Bloch equation it is a quantity such as this which relaxes to zero with some characteristic relaxation time. Accordingly a decomposition such as (3.8) will be useful when we are discussing the RHS—or relaxation terms—of the GKB, as we do in Sec. IV. But here we are interested in the LHS which has terms such as $\partial\vec{M}_s/\partial t$ involving the total magnetization, hence the decomposition (3.1), (3.2), and (3.6) is appropriate for the conduction electrons.

TABLE I. Left-hand side decomposition for $G^{\gtrless}$ ($g^{\gtrless}$). The equivalent equations (3.1), (3.2), (3.6), and (3.7) for conduction electrons are gotten from the ones listed by replacing every captial G, E, S, F, and M with g, ϵ, s, f, and m, respectively, and M_d with M_s.

$$G^{<}(\vec{p}, p_0) = 2\pi\,\delta(p_0-E_p+\gamma\vec{S}\cdot\vec{H})\,F(p_0+\gamma\vec{S}\cdot\vec{H}) + \Delta G^{<}(\vec{p}, p_0) \quad (3.11\mathrm{a})$$

$$G^{>}(\vec{p}, p_0) = 2\pi\,\delta(p_0-E_p+\gamma\vec{S}\cdot\vec{H})\,[1-F(p_0+\gamma\vec{S}\cdot\vec{H})] - \Delta G^{<}(\vec{p}, p_0) \quad (3.11\mathrm{b})$$

$$\Delta G^{<}(\vec{p}, p_0) = \vec{M}(\vec{p}, p_0)\cdot\vec{S} \quad (3.12)$$

$$\vec{M}_d = \gamma(2S+1)\tfrac{1}{3}S(S+1)\sum_{p,p_0}\vec{M}(\vec{p}, p_0) \quad (3.13)$$

Finally there is a very important reason, if not readily apparent at this stage, for choosing the decomposition of $g^{\gtrless}$ we did. This choice as we shall show in Sec. VI will cause certain terms in the GKB equation to be automatically zero. As there are a great many terms anyway (15 when last we counted) this is no small benefit.

We turn now to the local spins. Clearly a discussion similar to that for conduction electrons will go through with very few changes. In particular, we write

$$G^{<}(\vec{p}, p_0) = 2\pi\delta(p_0-E_p+\gamma\vec{S}\cdot\vec{H})$$
$$\times F(p_0+\gamma\vec{S}\cdot H) + \Delta G^{<}(\vec{p}, p_0) , \quad (3.11\mathrm{a})$$

$$G^{>}(\vec{p}, p_0) = 2\pi\delta(p_0-E_p+\gamma\vec{S}\cdot H)$$
$$\times[1-F(p_0+\gamma\vec{S}\cdot\vec{H})] - \Delta G^{<}(\vec{p}, p_0) , \quad (3.11\mathrm{b})$$

and

$$\Delta G^{<}(\vec{p}, p_0) = \vec{M}(\vec{p}, p_0)\cdot\vec{S} . \quad (3.12)$$

We have also recorded these equations in Table I for future reference. The Boltzmann factor $F(p_0)$ may be written in a number of ways. The most general is

$$F(p_0) = e^{-\beta(p_0-\mu)} . \quad (3.13)$$

To zero order in J,

$$F(p_0) = N e^{-\beta p_0}/(2S+1)\textstyle\sum_p e^{-\beta E_p} ,$$

where N is the number of local spins. Hence

$$\mathrm{Tr}_S \sum_{p,p_0} 2\pi\delta(p_0-E_p)F(p_0) = \mathrm{Tr}_S\sum_p F(E_p) = N . \quad (3.14)$$

Note that in the infinite-mass limit, the last equality becomes

$$(2S+1)\textstyle\sum_p F(0) = N , \quad (3.15)$$

which is often used in calculations in Appendix B. We observe that the factor of $F(p_0)$ in (3.11b) is negligible compared to unity.

From the definition of the local spin magnetization

$$\vec{M}_d = \mathrm{Tr}_S\gamma\vec{S}\sum_{p,p_0} G^{<}(\vec{p}, p_0), \quad (3.16)$$

it follows from (3.10) and (3.12) that

TABLE II. Lifetimes and self-energies for conduction electrons.

$$\gamma_0(p_0)=2\pi(J/n)^2N\tfrac{1}{3}S(S+1)\ s(s+1)\ \textstyle\sum_k\delta(p_0-\epsilon_k) \qquad \text{(B4)}$$

$$\gamma_H(p_0)=\gamma\vec{s}\cdot\vec{H}\frac{\partial}{\partial p_0}\gamma_0(p_0) \qquad \text{(B10)}$$

$$\gamma_{M_d}(p_0)=2\pi(J/n)^2\{(2S+1)\tfrac{1}{3}S(S+1)\sum_{q,q_0}\vec{M}(\vec{q},\ q_0)\cdot\vec{s}\}$$

$$\times\textstyle\sum_k\{-\tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)[1-f(\epsilon_k)]+\tfrac{1}{2}\delta(p_0-q_0-\epsilon_k)f(\epsilon_k)\} \qquad \text{(B14)}$$

$$\gamma_{M_s}=0 \qquad \text{(B3)}$$

$$\sigma(p_0)=\sigma^{(1)}_{M_d}+\sum_\omega\frac{\gamma(\omega)}{p_0-\omega}$$

$$\sigma^{(1)}_{M_d}=-\alpha_1\gamma\vec{s}\cdot\vec{M}_d,\quad \alpha_1=(J/n\gamma^2)$$

$$\sigma^<_{M_d}(p_0)=2\pi(J/n)^2\{(2S+1)\tfrac{1}{3}S(S+1)\sum_{q,q_0}\vec{M}(\vec{q},\ q_0)\cdot\vec{s}\}$$

$$\times\textstyle\sum_k\tfrac{1}{2}\delta(p_0-q_0-\epsilon_k)f(\epsilon_k) \qquad \text{(B24)}$$

$$\sigma^<_{M_s}(p_0)=(J/n)^2N\tfrac{1}{3}S(S+1)\ [s(s+1)-1]\textstyle\sum_k\vec{m}(\vec{k},\ p_0)\cdot\vec{s} \qquad \text{(B26)}$$

$$\vec{M}_d=\mathrm{Tr}_S\gamma\vec{S}\sum_{p,p_0}\Delta G^<(\vec{p},\ p_0), \tag{3.17}$$

$$\vec{M}_d=\gamma(2S+1)\tfrac{1}{3}S(S+1)\sum_{p,p_0}\vec{M}(\vec{p},\ p_0)\ . \tag{3.18}$$

Finally we calculate the magnetization of local spins in a static magnetic field. Another way of viewing (3.8) is to say that in equilibrium the curly bracket is identical zero. Applying this to local spins, we would say that

$$(\Delta G^<)_{\text{equil}}=\gamma\vec{S}\cdot\vec{H}\,2\pi\delta(p_0)\frac{-\partial F}{\partial p_0}\ , \tag{3.19}$$

where $-\partial F/\partial p_0=\beta F(p_0)$ and we have already gone to the infinite-mass limit. Clearly the equilibrium magnetization of the local spins in a static magnetic field is

$$\mathrm{Tr}_S\gamma\vec{S}(\gamma\vec{S}\cdot\vec{H})\textstyle\sum_p\beta F(0)=\chi_{d0}H\ , \tag{3.20}$$

where

$$\chi_{d0}=N\gamma^2S(S+1)/3k_BT \tag{3.21}$$

is the Curie susceptibility for N noninteracting spins.

It may strike the reader that this formalism is not very efficient if 21 equations are required to derive the Pauli and Curie susceptibilities. However, our aim is considerably more ambitious. In this section (together with Appendices B and C) we will derive the equilibrium and nonequilibrium self-energies to order J^2 and that task requires only another 50 equations or so.

The procedure for the calculating γ and $\sigma^<$ for the conduction electrons and Γ and $\Sigma^<$ for the local spins is straightforward. The previously discussed decompositions for $g^{\gtrless}$ and $G^{\gtrless}$, in Table I, are inserted into the equations for $\sigma^{\gtrless}$ and $\Sigma^{\gtrless}$ to second order in J given in Sec. II. We systematically expand in powers of H, $\Delta g^<$, and $\Delta g^<$ and keep all terms of zero or first order in any of these quantities. The tedious details are relegated to Appendix B for the conduction electrons and Appendix C for the local spins. The results are exhibited in Tables II and III, respectively.

We will not repeat the results here but comment briefly on them. First, $\gamma_{M_s}(p_0)=0$, that is, it is possible to define a self-energy for the conduction electrons which is independent of the total magnetization of the conduction electrons (including their magnetization in a static uniform field). This is, of course, entirely consistent with the decomposition. In an analogous way $\Gamma_{M_d}(p_0)=0$.

The second interesting result is that

$$\gamma_0(p_0)+\gamma_H(p_0)=\gamma_0(p_0+\gamma\vec{s}\cdot\vec{H}) \tag{3.22}$$

and

$$\Gamma_0(p_0)+\Gamma_H(p_0)=\Gamma_0(p_0+\gamma\vec{S}\cdot\vec{H}) \tag{3.23}$$

to *first order* in the magnetic field. That this is the case will allow considerable simplification in the LHS. At first this result seems obvious as some sort of Ward identity, but on closer inspection involves expanding each internal Green's-function line to first order in H (with complicated spin traces resulting) and not just those for conduction electrons (say). Furthermore, the decomposition of $g^{\gtrless}$ (and $G^{\gtrless}$) clearly plays a crucial role. So it is not clear how to prove the results to all orders in H or J for that matter. Another peculiarity of the results (3.22) and (3.23) is that they depend on the g value for the conduction electrons and the local spin being the same. This is in general not the case and hence extra terms will appear on the LHS of the Bloch equations. In this paper such terms will not appear

TABLE III. Lifetimes and self-energies for local spins.

$$\Gamma_0(p_0)=2\pi(J/n)^2\ (2s+1)\ \tfrac{1}{3}s(s+1)\ S(S+1)$$

$$\times\sum_{k,q}f(\epsilon_k)[1-f(\epsilon_q)]\,\delta(p_0+\epsilon_k-\epsilon_q) \qquad \text{(C4)}$$

$$\Gamma_H(p_0)=\gamma\vec{S}\cdot\vec{H}\frac{\partial}{\partial p_0}\Gamma_0(p_0) \qquad \text{(C8)}$$

$$\Gamma_{M_s}(p_0)=-2\pi(J/n)^2\{(2s+1)\ \tfrac{1}{3}s(s+1)\sum_{q,q_0}\vec{m}(\vec{q},\ q_0)\cdot\vec{S}\}$$

$$\times\textstyle\sum_k\{\tfrac{1}{2}\delta(p_0-q_0+\epsilon_k)f(\epsilon_k)+\tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)\ [1-f(\epsilon_k)]\} \qquad \text{(C11)}$$

$$\Gamma_{M_d}(p_0)=0$$

$$\Sigma(p_0)=\Sigma^{(1)}_{M_s}+\sum_\omega\frac{\Gamma(\omega)}{p_0-\omega}$$

$$\Sigma^{(1)}_{M_s}=-\alpha_1\gamma\vec{S}\cdot\vec{M}_s,\quad \alpha_1=(J/n\gamma^2),\quad \alpha_2=(J/n)^2(\rho/\gamma^2)\ln k_BT/D$$

$$\Sigma^<_{M_s}(p_0)=2\pi N/(2S+1)\ (J/n)^2\{(2s+1)\ \tfrac{1}{3}s(s+1)\sum_{q,q_0}\vec{m}(\vec{q},\ q_0)\cdot\vec{S}\}$$

$$\times\textstyle\sum_k\{\tfrac{1}{2}\delta(p_0-q_0+\epsilon_k)[1-f(\epsilon_k)]+\tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)f(\epsilon_k)\} \qquad \text{(C17)}$$

$$\Sigma^<_{M_d}(p_0)=2\pi(J/n)^2\ (2s+1)\ \tfrac{1}{3}s(s+1)\ [S(S+1)-1]$$

$$\times\sum_{k,k_0}\vec{M}(\vec{p}+\vec{k},\ p_0+k_0)\cdot\vec{S}\sum_q f(\epsilon_q)[1-f(\epsilon_{\vec{k}+\vec{q}})]\,\delta(k_0+\epsilon_q-\epsilon_{\vec{k}+\vec{q}}) \qquad \text{(C21)}$$

since we assume always that $\gamma_s = \gamma_d = \gamma$.

In the Appendices B and C we calculate only the second-order contribution to σ and Σ. There are, of course, first-order terms which are calculated in Sec. VI, and given in Tables II and III. For the record, we exhibit the sum of first- and second-order terms at $p_0 = 0$,

$$\sigma_{M_d}(0) = -(\alpha_1 + \alpha_2)\gamma\vec{s}\cdot\vec{M}_d, \tag{3.24}$$

$$\sigma_{M_d} = -\alpha_1[1 + (\rho J/n)\ln(k_B T/D)]\gamma\vec{s}\cdot\vec{M}_d \tag{3.25}$$

for the conduction electrons. The second term in the square bracket is the so-called Kondo logarithmic term. For the local spins $\Sigma_{M_s}(0)$ would have a similar form, except $\vec{s}\cdot\vec{M}_d$ is replaced by $\vec{S}\cdot\vec{M}_s$.

IV. DECOMPOSITION OF $g^{\gtrless}$ ($G^{\gtrless}$) FOR RHS OF GKB AND SUSCEPTIBILITIES TO ORDER J^2

We turn now to a discussion of a different decomposition of $g^{\gtrless}$ (and $G^{\gtrless}$) which was alluded to in Sec. III [(3.8)] and will be appropriate for treating the RHS, i.e., the relaxation terms, of the GKB equation. Consider the following choice for $g^{\gtrless}$ for the conduction electrons:

$$g^{<}(\vec{p}, p_0) = a(\vec{p}, p_0)f(p_0) + \delta g^{<}(\vec{p}, p_0)\ , \tag{4.1}$$

$$g^{>}(\vec{p}, p_0) = a(\vec{p}, p_0)[1 - f(p_0)] - \delta g^{<}(\vec{p}, p_0)\ . \tag{4.2}$$

Here $a(\vec{p}, p_0)$ is the complete nonequilibrium spectral weight function including all contributions to $\gamma(p_0)$ and $\sigma(p_0)$ to second order in J and any linear field dependence. Since $g^{>} + g^{<} = a$, and $\gamma(p_0)$ is independent of $\vec{M}_s$, we observe that a is also.

Further $a(\vec{p}, p_0)$ must satisfy the following sum rules:

$$\sum_{p_0} a(\vec{p}, p_0) = 1\ , \tag{4.3}$$

$$\mathrm{Tr}_s \sum_{p, p_0} a(\vec{p}, p_0)f(p_0) = n\ , \tag{4.4}$$

where n is the total number of conduction electrons. Both of the above equations are, of course, essentially definitions but it is reassuring to know that when the calculated $\gamma(p_0)$'s and $\sigma(p_0)$'s are included in $a(\vec{p}, p_0)$, (4.3) and (4.4) are satisfied. The details of this calculation are in Appendix D.

Even if $a(\vec{p}, p_0)$ were just $2\pi\delta(p_0 - \epsilon_p + \gamma\vec{s}\cdot\vec{H})$ it would not be the case that δg was proportional to $\vec{M}_s$ alone, but instead to $\vec{M}_s - \chi_{s0}\vec{H}$ as we showed in (3.8). Including the effect of interactions we should expect that the quasiequilibrium magnetization to which $\vec{M}_s$ relaxes includes some contribution from the local spins. The calculation of the contribution proceeds by the expansion of $a(\vec{p}, p_0)$ to first order in $\vec{M}_d$ (and second order in J^2):

$$a(\vec{p}, p_0) = a_0(p, p_0 + \gamma\vec{s}\cdot\vec{H})$$

$$+ 2\,\mathrm{Im}\{(p_0 - \epsilon_p - iO^+)^{-2}[\sigma_{M_d}(p_0) + \tfrac{1}{2}i\gamma_{M_d}(p_0)]\}\ , \tag{4.5}$$

where

$$a_0(p, p_0 + \gamma\vec{s}\cdot\vec{H}) = 2\,\mathrm{Im}[p_0 + \gamma\vec{s}\cdot\vec{H} - \epsilon_p - \sigma_0(p + \gamma\vec{s}\cdot\vec{H}) - \tfrac{1}{2}i\gamma_0(p_0 + \gamma\vec{s}\cdot\vec{H})]^{-1} \tag{4.6}$$

to the calculation

$$\mathrm{Tr}_s\gamma\vec{s}\sum_{p, p_0} a(\vec{p}, p_0)f(p_0)\ . \tag{4.7}$$

Here $a_0(p, p_0)$ is the equilibrium spectral weight function in zero magnetic field (hence $\vec{M}_d$ is zero) calculated in this case to second order in J. The term from $a_0(p, p_0 + \gamma\vec{s}\cdot\vec{H})$ gives [see (D19)] reassuringly $\chi_{s0}\vec{H}$, that is, the static Pauli susceptibility is unaffected to order J^2. The term from the $\vec{M}_d$ part of $a(\vec{p}, p_0)$ is more surprising: The J^2 contributions all cancel leaving only the first-order contributions to σ_{M_d}. The result (D13) is consistent with work of Yosida and Okiji[15] who confirmed that to fourth order in J the magnetization of the conduction electrons due to the local spins equals $-\alpha_1 \times \gamma\vec{s}\cdot\vec{M}_d$. We think this result is true to all orders in J and give a proof in Appendix D. In summary, then

$$\mathrm{Tr}_s\gamma\vec{s}\sum_{p, p_0}\delta g^{<}(\vec{p}, p_0) = \vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1\vec{M}_d)\ . \tag{4.8}$$

When this decomposition is used in the RHS of the GKB, we will find that the instantaneous or local equilibrium magnetization to which $\vec{M}_s$ relaxes is indeed $\chi_{s0}(\vec{H} + \alpha_1\vec{M}_d)$. While this is correct, it is not the result we reported earlier[17] where the fact that $\vec{H} + (\alpha_1 + \alpha_2)\vec{M}_d$ is the field in which $\vec{M}_s$ processes lead us to believe it to be appropriate for calculating the local equilibrium magnetization. It is not. However, the physics (in our minds if not so clearly in our earlier letter) remains the same: (i) On the LHS $\vec{M}_s$ precesses in the effective *field* given by $\vec{H} + (\alpha_1 + \alpha_2)\vec{M}_d$; (ii) on the RHS $\vec{M}_s$ relaxes to local equilibrium *magnetization* given by $\chi_{s0}(\vec{H} + \alpha_1\vec{M}_d)$. We have shown that both results are correct to order J^2 and in fact we believe (ii) to all orders.

There is finally one further thing about the decomposition choice (4.1) and (4.2) which will become clear in Sec. V and which plays a terribly important role in the derivation of the RHS terms. It will turn out that we will be able to show that the RHS is proportional to either $\delta g^{<}$ or $\delta G^{<}$ (the equivalent quantity for the local spins) and nothing else—a fact which considerably simplifies our analysis and understanding of the RHS. It should be noted that the different decomposition chosen for the LHS is also uniquely suited to the algebraic requirement of that side.

We turn now to a discussion of the appropriate decomposition of $G^{\gtrless}$ for use on the RHS of GKB. The obvious choice is

$$G^{<}(\vec{p}, p_0)=A(\vec{p}, p_0)F(p_0)+\delta G^{<}(\vec{p}, p_0) , \tag{4.9}$$

$$G^{>}(\vec{p}, p_0)=A(\vec{p}, p_0)[1-F(p_0)]-\delta G^{<}(\vec{p}, p_0) . \tag{4.10}$$

Here $A(\vec{p}, p_0)$ is the complete nonequilibrium spectral weight function for the local spins and includes all contribution to $\Gamma(p_0)$ and $\Sigma(p_0)$ to second order in J and any linear field dependence. But $\Gamma(p_0)$ is independent of $\vec{M}_d$, and hence so in $A(\vec{p}, p_0)$.

Of course, $A(\vec{p}, p_0)$ must satisfy the sum rules

$$\sum_{p_0} A(\vec{p}, p_0)=1 , \tag{4.11}$$

$$\mathrm{Tr}_S \sum_{p,p_0} A(\vec{p}, p_0)F(p_0)=N . \tag{4.12}$$

In Appendix D, it is shown just how the actual forms of $\Gamma(p_0)$ and $\Sigma(p_0)$ are consistent with these (defined) constraints. Much more interesting is the evaluation of $\delta G^{<}(\vec{p}, p_0)$ since it will tell us what magnetization the local spins relax to. The calculation is straightforward and detailed in Appendix D. In analogy with the case for the conduction electrons we find that

$$\mathrm{Tr}_S\gamma\vec{S} \sum_{p,p_0} A_0(p, p_0+\gamma\vec{S}\cdot\vec{H})F(p_0)=\chi_{d0}H , \tag{4.13}$$

i.e., that the Curie susceptibility is unaffected to order J^2. In this case the result follows trivially from (4.12) since by a change of variables and expansion in H, (4.13) becomes

$$\mathrm{Tr}_S\gamma\vec{S}(\gamma\vec{S}\cdot\vec{H}) \sum_{p,p_0} A_0(\vec{p}, p_0)\frac{-\partial F}{\partial p_0} \tag{4.14}$$

and $\partial F/\partial p_0=-\beta F$.

The more interesting part of the calculation is the dependence of $\delta G^{<}$ on $\vec{M}_s$. In contradistinction to the case for the conduction electrons, the J^2 contributions do not vanish and one finds that

$$\mathrm{Tr}_S\gamma\vec{S} \sum_{p,p_0} \delta G^{<}(\vec{p}, p_0) =\vec{M}_d-\chi_{d0}[\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s] . \tag{4.15}$$

This result [second term in (4.15)] for the local equilibrium magnetization to which $\vec{M}_d$ relaxes, while different from our early letter, is the correct one. It is also consistent with the susceptibility calculation of Yosida and Okiji.[15]

We close this section by calculating the static susceptibility. In thermal equilibrium (4.8) and (4.15) will be zero. Therefore

$$\begin{aligned}\vec{M}_s&=\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d) ,\\ \vec{M}_d&=\chi_{d0}[\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s] .\end{aligned} \tag{4.16}$$

Solution of (4.16) for the total magnetization yields $\vec{M}=\vec{M}_s+\vec{M}_d=\chi\vec{H}$ with

$$\chi=\chi_{s0}+\chi_{d0}\frac{(1+\alpha_1\chi_{s0})[1+(\alpha_1+2\alpha_2)\chi_{s0}]}{1-\alpha_1(\alpha_1+2\alpha_2)\chi_{d0}\chi_{s0}} , \tag{4.17}$$

where (4.17) should not be construed to be accurate to higher order than $J^2\ln k_BT/D$. Note that in the single-impurity limit (4.17) reduces to that calculated by Yosida and Okiji, and that the Curie constant agrees with the normal mean-field result.

V. RIGHT-HAND SIDE OR RELAXATION TERMS OF GKB

A. Introduction

There are several types of terms which occur on the right-hand side of the GKB (2.7)

$$-\tfrac{1}{2}\{\gamma, g^{<}\}+\tfrac{1}{2}\{\sigma^{<}, a\} , \tag{5.1}$$

and these will be considered in this section. First there are the terms in the conduction-electron equation which arise from the direct spin-flip interaction with the localized spins, and which give rise to the angular-momentum-conserving scattering rates $1/T_{sd}$ and $1/T_{ds}$; these are considered in Sec. V B. In Sec. V C we derive the analogous terms for the local spins. Next we consider the effect of mechanisms which remove spin angular momentum from the system, and hence drive the system toward thermal equilibrium, which for the conduction electrons, say, gives rise to the so-called $1/T_{sl}$ relaxation term. We do this without specifying in detail what the specific mechanism for this relaxation is to be, by considering as a prototype a model where the conduction electrons interact with a number of randomly oriented and positioned spins, which could represent, for example, the spin-orbit interaction of the conduction electrons with localized impurities, or alternatively the spin-orbit interaction with phonons, since the structure of the perturbation theory would be the same in either case. A similar model is used for the direct relaxation of the localized spins where the details are less clear. The main assumption is that these spin-flip mechanisms are sufficiently weak so as not to have to be included in the various self-energies, but rather can be treated as a weak Born-approximation effect on only the right-hand side of the GKB. We will show that these mechanisms tend to cause the electrons to relax toward an instantaneous mean magnetization value.

The general scheme will be to use the decompositions of $g^{\gtrless}$ and $G^{\gtrless}$ discussed in Sec. IV. For the moment we suppress the arguments of the functions in order to present the logic of the procedure more clearly. We write following (4.1) and (4.2) and

(4.9) and (4.10):

$$g^{<}=af+\delta g^{<}\,,\qquad g^{>}=a(1-f)-\delta g^{<}$$

and (5.2)

$$G^{<}=AF+\delta G^{<}\,,\qquad G^{>}=A\;.$$

These will be used to linearize the right-hand side. We keep terms of only the lowest nonvanishing order in the field and magnetizations. Provided that all fields are weak in comparison with k_BT/γ, which is our assumption throughout, this is a perfectly consistent procedure. It amounts to neglecting the effect of the magnetic field on the scattering rates themselves. This means automatically, of course, that the transverse relaxation time T_2 is the same as the relaxation time T_1 for the component of magnetization in the dc field direction.

The reason for making the particular decomposition (5.2) is that the "unperturbed part" of it, that is, the part not proportional to $\delta g^{<}$ or $\delta G^{<}$, causes the right-hand side of the GKB to vanish identically. To see this, we introduce the notation

$$g^{<}_{\text{qe}}=af\,,\quad g^{>}_{\text{qe}}=a(1-f)\;,\quad G^{<}_{\text{qe}}=AF\,,\quad G^{>}_{\text{qe}}=A\;, \tag{5.3}$$

which should be compared with (5.2). The subscript qe stands for quasiequilibrium, and serves to emphasize that although (5.3) looks like the equilibrium form for the various Green's function, the spectral weight functions in (5.3) are meant to retain their full nonequilibrium values. Likewise, we define

$$\sigma^{<}=\sigma^{<}_{\text{qe}}+\delta\sigma^{<}\,,\quad \Sigma^{<}=\Sigma^{<}_{\text{qe}}+\delta\Sigma^{<}$$

and (5.4)

$$\gamma=\gamma_{\text{qe}}+\delta\gamma,\qquad \Gamma=\Gamma_{\text{qe}}+\delta\Gamma\;,$$

where here the quasiequilibrium quantities are to be calculated assuming that their internal Green's functions are replaced by the quasiequilibrium values. If these quasiequilibrium values are now substituted in (5.1), one obtains

$$-\tfrac{1}{2}\{\gamma_{\text{qe}}\,,\;af\}+\tfrac{1}{2}\{\sigma^{<}_{\text{qe}}\,,\;a\}\,. \tag{5.5}$$

This vanishes because, just as in full thermal equilibrium, $\sigma^{<}_{\text{qe}}$ satisfies

$$\sigma^{<}_{\text{qe}}=\gamma_{\text{qe}}f\;, \tag{5.6}$$

which may be seen by inserting (5.3) into (2.20). (See start of Appendix B.) Of course, similar relations hold for the local spin functions.

Since $\delta\gamma$, $\delta\sigma^{<}$, $\delta\Gamma$, and $\delta\Sigma^{<}$ are by construction proportional to $\delta g^{<}$ and $\delta G^{<}$, it is quite clear that the right-hand side (5.1) will be proportional to $\vec{\text{M}}_s$ and $\vec{\text{M}}_d$ only in combinations implied by the decompositions (5.2), that is, $\vec{\text{M}}_s$ will occur only in the combination (4.9); $\vec{\text{M}}_s-\chi_{s0}(\vec{\text{H}}+\alpha_1\vec{\text{M}}_d)$ and $\vec{\text{M}}_d$ only in the combination (4.16), $\vec{\text{M}}_d-\chi_{d0}[\vec{\text{H}}+(\alpha_1+2\alpha_2)\vec{\text{M}}_s]$. It is thus clear toward what local magnetizations the magnetizations will relax.

We emphasize that the proportionality of the right-hand side of the GKB to $\delta g^{<}$ and $\delta G^{<}$, and to *no other field*- or *magnetization*-dependent quantities is true in general, and not dependent on a particular order of perturbation theory. Thus it will be consistent to keep all the powers of J implied by (4.8) and (4.15) even when they appear in a term that is already of order J^2, and in fact this retention is necessary to get the static susceptibility and linewidth correct to order J^2. Said another way, the right-hand side of the GKB will always be of the form

$$(1/T_a)\delta G^{<}-(1/T_b)\delta g^{<}\;, \tag{5.7}$$

irrespective of the order in J to which the rates $1/T_a$ and $1/T_b$ are calculated.

B. Conduction-Electron Terms

Here we consider specifically the terms on the right-hand side of the conduction-electron equation which result from the $J\vec{\text{s}}\cdot\vec{\text{S}}$ interaction with the local spins. First we point out two facts that will be useful throughout Sec. V. The first of these is that to linear order in the field and magnetizations, one may assume that $[\gamma,\,g^{<}]=[\sigma^{<},\,a]=0$ so that (5.1) may be written

$$-\gamma g^{<}+\sigma^{<}a\;. \tag{5.8}$$

Second, once (5.8) is expanded in $\delta G^{<}$ and $\delta g^{<}$, one may set the field and both magnetizations equal to zero in all other quantities; for example, $\gamma_{\text{qe}}\delta g^{<}\to\gamma_0\delta g^{<}$; $\delta\sigma^{<}a\to\delta\sigma^{<}a_0$, etc. The subscript "0" refers, as always, to the zero-field quantities.

In the conduction electron, Eq. (5.8) becomes

$$\begin{aligned}-\gamma_0\delta g^{<}&+\delta\sigma^{<}a_0-\delta\gamma a_0 f\\ &=-\gamma_0\delta g^{<}+\sigma^{<}_{\delta g^{<}}a_0+\sigma^{<}_{\delta G^{<}}a_0-\gamma_{\delta G^{<}}a_0 f\;,\end{aligned} \tag{5.9}$$

where we have symbolically split $\delta\sigma^{<}$ and $\delta\gamma$ into the pieces proportional to $\delta g^{<}$ and $\delta G^{<}$, as indicated by the notation. (Note that $\delta\gamma$ has no piece proportional to $\delta g^{<}$.) For example, if we introduce the definitions

$$\delta g^{<}=\delta\vec{\text{m}}\cdot\vec{\text{s}}\;,\qquad \delta G^{<}=\delta\vec{\text{M}}\cdot\vec{\text{S}}\;, \tag{5.10}$$

then to order J^2, $\sigma^{<}_{\delta g^{<}}$ is equal to $\sigma^{<}_{M_s}$ with $\vec{\text{m}}$ replaced by $\delta\vec{\text{m}}$, $\sigma^{<}_{\delta G^{<}}$ is equal to $\sigma^{<}_{M_d}$ with $\vec{\text{M}}$ replaced by $\delta\vec{\text{M}}$, and $\gamma_{\delta G^{<}}$ is equal to γ_{M_d} with $\vec{\text{M}}$ replaced by $\delta\vec{\text{M}}$; these quantities can be read out of Tables II and III. Note that the last line of (5.9) is arranged so that the first two terms give the $(-\vec{\text{M}}_s/T_{sd})$-type term in the Bloch equation and the last two terms give the $(\vec{\text{M}}_d/T_{ds})$-type term. To obtain these terms in the Bloch equation one substitutes in (5.9) from Table II, sums over p and p_0, multiplies by $\gamma\vec{s}$, and takes the trace over $\vec{s}$. Actually, we will not sum completely over p immediately, and hence derive a Boltzmann-like equation for the momentum decomposition of $\vec{M}_s$. Specifically we will integrate over ϵ_p but not solid angle:

$$\sum_p \equiv \int \frac{d^3p}{(2\pi)^3} = \int \rho\, d\epsilon_p \int \frac{d\hat{p}}{4\pi} \ . \tag{5.11}$$

Specifically it is the last integral over momentum direction which will not be done. This decomposition is possible because $\delta\vec{m}$ varies from its equilibrium value only in a small region near the Fermi surface.

Writing out the first two terms in the last line of (5.9) using (B4) and (B26) we get

$$\begin{aligned} &N(J/n)^2\, \tfrac{1}{3}S(S+1)s(s+1)\textstyle\sum_k(-2\pi)\delta(p_0-\epsilon_k)\vec{s}\cdot\delta\vec{m}(\vec{p},\ p_0) \\ &\quad + N(J/n)^2\,\tfrac{1}{3}S(S+1)[s(s+1)-1] \\ &\qquad \times \textstyle\sum_{p'}\vec{s}\cdot\delta\vec{m}(\vec{p}\,',\ p_0)2\pi\delta(p_0-\epsilon_p) \ . \end{aligned} \tag{5.12}$$

Since the integral over ϵ_p is sufficient to prove (D9) and (D13), the summation over p_0, ϵ_p, and trace over $\vec{s}$ yields

$$\begin{aligned} &-\frac{1}{T_{sd}}\, s(s+1)\int\frac{d\hat{p}\,'}{4\pi}\,[\vec{m}(\hat{p})-\vec{m}(\hat{p}\,')] \\ &\qquad -\frac{1}{T_{sd}}\,[\vec{M}_s-\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)] \ , \end{aligned} \tag{5.13}$$

where

$$1/T_{sd} = 2\pi(J/n)^2\rho N\tfrac{1}{3}S(S+1) \ . \tag{5.14}$$

The first term in (5.13) vanishes when summed over $\hat{p}$. It therefore contributes to the relaxation of the magnetization current, but not to the relaxation of the magnetization itself. The remaining term is a spin-flip term which contributes to the Bloch equation.

Similarly, writing out the last two terms in (5.9) using (B24) and (B14), respectively, we get

$$\begin{aligned} &2\pi\left(\frac{J}{n}\right)^2[(2S+1)\tfrac{1}{3}S(S+1)\sum_{q,q_0}\delta\vec{M}(\vec{q},\ q_0)\cdot\vec{s}\,] \\ &\quad\times\sum_k\tfrac{1}{2}\delta(p_0-q_0-\epsilon_k)\,f(\epsilon_k)2\pi\delta(p_0-\epsilon_p) \\ &\quad -2\pi\left(\frac{J}{n}\right)^2[(2S+1)\tfrac{1}{3}S(S+1)\sum_{q,q_0}\delta\vec{M}(\vec{q},\ q_0)\cdot\vec{s}\,] \\ &\qquad\times\sum_k\{\tfrac{1}{2}\delta(p_0-q_0-\epsilon_{\vec{p}+\vec{k}})\,f(\epsilon_{\vec{p}+\vec{k}}) \\ &\qquad\qquad -\tfrac{1}{2}\delta(p_0+q_0-\epsilon_{\vec{p}+\vec{k}})[1-f(\epsilon_{\vec{p}+\vec{k}})]\} \\ &\qquad\qquad\times 2\pi\delta(p_0-\epsilon_p)\,f(p_0) \ . \end{aligned} \tag{5.15}$$

On integrating over ϵ_p, p_0, and taking the trace over $\gamma\vec{s}$, we find that (5.15) becomes

$$(1/T_{ds})\{\vec{M}_d-\chi_{d0}[\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s]\} \ , \tag{5.16}$$

where

$$\frac{1}{T_{ds}} = \frac{\chi_{s0}}{\chi_{d0}}\,\frac{1}{T_{sd}} = \pi\left(\frac{\gamma\rho J}{n}\right)^2 k_BT \ . \tag{5.17}$$

To obtain these results, we have had to assume that $\delta\vec{M}(q,\ q_0)$ is sharply peaked at $q_0=0$, a fact which is discussed in Sec. VII. Combining (5.13) and (5.16) gives

$$\begin{aligned} &-\frac{1}{T_{sd}}\, s(s+1)\int\frac{d\hat{p}\,'}{4\pi}\,[\vec{m}(\hat{p})-\vec{m}(\hat{p}\,')] \\ &\qquad -\frac{1}{T_{sd}}\,[\vec{M}_s-\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)] \\ &\qquad\qquad +\frac{1}{T_{ds}}\{\vec{M}_d-\chi_{d0}[\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s]\} \end{aligned} \tag{5.18}$$

for the total right-side contribution of the local spins to the Boltzmann equation for the conduction electrons. To obtain this contribution to the conduction-electron Bloch equation, one only has to average over the direction of the momentum and note that

$$\vec{M}_s = \int\frac{d\hat{p}}{4\pi}\,\vec{m}(\hat{p}) \ , \tag{5.19}$$

so that the first term in (5.18) disappears.

One might legitimately ask what sense it makes to keep terms up to fourth order in J, say, for example, the α_2 term in the curly brackets of (5.18), while on the other hand $1/T_{ds}$ is calculated only to order J^2. This question was discussed in the last paragraph prior to Sec. V B.

C. Localized-Spin Terms

Here we consider specifically the terms on the right-hand side of the localized-spin equation which result from the $-J\vec{s}\cdot\vec{S}$ interaction with the conduction electrons. Generally the derivation is analogous to that in Sec. V B. Again we expand about the quasiequilibrium quantities (5.2)–(5.4), since as in (5.5) the quasiequilibrium piece of the right-hand side vanishes. The implication of (5.8) still holds and the analog of (5.9) (the right-hand side of the GKB) is

$$\begin{aligned} &-\Gamma_0\delta G^< + \delta\Sigma^< A_0 - \delta\Gamma A_0 F \\ &\qquad = -\Gamma_0\delta G^< + \Sigma^<_{\delta G^<}A_0 + \Sigma^<_{\delta g^<}A_0 - \Gamma_{\delta g^<}A_0F \ . \end{aligned} \tag{5.20}$$

The quantities in (5.20) can essentially be read out of Tables II and III, replacing $\vec{m}(\vec{p},\ p_0)$ by $\delta\vec{m}(\vec{p},\ p_0)$ and $\vec{M}(\vec{p},\ p_0)$ by $\delta\vec{M}(\vec{p},\ p_0)$. The first two terms in (5.20) give

$$-2\pi(J/n)^2(2s+1)\tfrac{1}{3}s(s+1)S(S+1)\sum_{k,q}f(\epsilon_k)[1-f(\epsilon_q)]$$

$$\times\delta(p_0+\epsilon_k-\epsilon_q)\vec{S}\cdot\delta\vec{M}(\vec{p},\ p_0)$$

$$+2\pi(J/n)^2(2s+1)\tfrac{1}{3}s(s+1)[S(S+1)-1]$$

$$\times\sum_{k,k_0}\vec{S}\cdot\delta\vec{M}(\vec{p}+\vec{k},\ p_0+k_0)$$

$$\times\sum_q f(\epsilon_q)[1-f(\epsilon_{\vec{k}+\vec{q}})]\delta(k_0-\epsilon_q-\epsilon_{\vec{k}+\vec{q}})2\pi\delta(p_0)\ . \tag{5.21}$$

If we sum over p and p_0 and take the trace over $\gamma\vec{S}$, using the fact that $\delta\vec{M}(\vec{p}, p_0)$ is proportional to $\delta(p_0)$, we see using (D27) that (5.21) becomes simply

$$-(1/T_{ds})[\vec{M}_d-\chi_{d0}(\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s)]\ . \tag{5.22}$$

Similarly, the last two terms in (5.20) may be written from Table III as

$$2\pi[N/(2S+1)](J/n)^2[(2s+1)\tfrac{1}{3}s(s+1)\sum_{q,q_0}\delta\vec{M}(\vec{q},q_0)\cdot\vec{S}]$$

$$\times\sum_k\{\tfrac{1}{2}\delta(p_0-q_0+\epsilon_k)[1-f(\epsilon_k)]$$

$$+\tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)f(\epsilon_k)\}2\pi\delta(p_0)$$

$$+2\pi(J/n)^2[(2s+1)\tfrac{1}{3}s(s+1)\sum_{q,q_0}\delta\vec{M}(\vec{q},\ q_0)\cdot\vec{S}]$$

$$\times\sum_k\{\tfrac{1}{2}\delta(p_0-q_0+\epsilon_k)f(\epsilon_k)$$

$$+\tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)[1-f(\epsilon_k)]\}2\pi\delta(p_0)F(p_0)\ . \tag{5.23}$$

Summing over p and p_0 and taking the trace over $\gamma\vec{S}$ gives, using (D27),

$$(1/T_{sd})[\vec{M}_s-\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)]\ . \tag{5.24}$$

Combining this with (5.22), we see that the complete right-hand side of the local-spin Bloch equation due to $(-J\vec{s}\cdot\vec{S})$-type scattering with the conduction electrons is

$$-(1/T_{ds})[\vec{M}_d-\chi_{d0}(\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s)]$$

$$+(1/T_{sd})[\vec{M}_s-\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)]\ . \tag{5.25}$$

Note that if we wished (5.25) could be written, using (5.14) and (5.17), in the form

$$-(1/T_{ds})_{\text{eff}}\vec{M}_d+(1/T_{sd})_{\text{eff}}\vec{M}_s\ , \tag{5.26}$$

where, referring to Appendix D, one sees that $(1/T_{ds})_{\text{eff}}$ and $(1/T_{sd})_{\text{eff}}$ are in the ratio of the true interacting s and d susceptibilities as required by detailed balance, rather than of the bare susceptibilities of (5.17).

D. Direct Relaxation Terms

As mentioned in Sec. V A, there exists in real materials a number of collision mechanisms by which the spin angular momentum of the conduction-electron–localized-spin system can be changed, and hence by which the system is driven toward thermal equilibrium. Without commiting ourselves to what the actual mechanism is in any given system, we take as a prototype a collection of ν randomly positioned and oriented angular momentum vectors $\vec{L}$, each interacting with the conduction electrons with the interaction

$$\lambda\vec{L}\cdot\vec{s}\ . \tag{5.27}$$

Our basic assumption is that λ is very small in comparison with other interaction energies in the problem, so that any self-energies arising from (5.27) may be neglected; its only effect then will be to act as an additional scattering term on the right-hand side of the conduction-electron equation. In particular, there will be a contribution to $\sigma^>$, $\sigma^<$, and γ due to their interaction, which we call $\tilde{\sigma}^>$, $\tilde{\sigma}^<$, and $\tilde{\gamma}$, respectively. We calculate $\tilde{\sigma}^{\gtrless}$ in Born approximation as

$$\tilde{\sigma}^{\gtrless}(\vec{p},\ p_0)=[\nu\lambda^2(2L+1)]\,\mathrm{Tr}_L\textstyle\sum_p\vec{L}\cdot\vec{s}g^{\gtrless}(\vec{p},\ p_0)\vec{L}\cdot\vec{s}\ . \tag{5.28}$$

The extra contribution to the conduction-electron equation [see (5.8)] is

$$-\tilde{\gamma}g^<+\tilde{\sigma}^<a\ . \tag{5.29}$$

Again it follows from inspection that the quasi-equilibrium forms (5.2)–(5.4) cause (5.29) to vanish, so that the linearized form of (5.29) is

$$-\tilde{\gamma}_0\delta g^<+\delta\tilde{\sigma}^<a_0=(\nu\lambda^2/2L+1)$$

$$\times\mathrm{Tr}_L\textstyle\sum_q[-\vec{L}\cdot\vec{s}a_0(q,\ p_0)\vec{L}\cdot\vec{s}\,\delta\vec{m}(\vec{p},\ p_0)\cdot\vec{s}$$

$$+\vec{L}\cdot\vec{s}\,\delta\vec{m}(\vec{q},\ p_0)\cdot\vec{s}\,\vec{L}\cdot\vec{s}a_0(p,\ p_0)]=\nu\lambda^2\tfrac{1}{3}L(L+1)$$

$$\times\textstyle\sum_q[-s(s+1)\,2\pi\delta(p_0-\epsilon_q)\delta\vec{m}(\vec{p},\ p_0)\cdot\vec{s}$$

$$+(s(s+1)-1)\delta\vec{m}(\vec{q},\ p_0)\cdot\vec{s}\,2\pi\delta(p_0-\epsilon_p)]\ . \tag{5.30}$$

Integrating over ϵ_p and p_0, multiplying by $\gamma\vec{s}$, and taking the trace over $\vec{s}$ gives

$$-\frac{s(s+1)}{T_{sl}}\int\frac{d\hat{p}'}{4\pi}[\vec{m}(\hat{p})-\vec{m}(\hat{p}')]$$

$$-(1/T_{sl})[\vec{M}_s-\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)]\ , \tag{5.31}$$

where

$$1/T_{sl}=2\pi\nu\tfrac{1}{3}L(L+1)\lambda^2\rho\ . \tag{5.32}$$

As before the first term goes away if we integrate over the direction of $\vec{p}$ to form the Bloch equation. The main thing to be noted about (5.31), however, is that it gives relaxation of $\vec{M}_s$ towards its instantaneous equilibrium value of $\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)$.

As a number of others have done, one might also imagine a mechanism by which the localized spins relax directly to the lattice. If such a mechanism can be described by a model of the same type that we have used for the conduction electrons, then it is clear from the arguments given above that an additional term of the type

$$-(1/T_{dl})\{\vec{M}_d-\chi_{d0}[\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s]\} \tag{5.33}$$

should be added to the Bloch equation for the localized spins. The important point is the relaxation towards local equilibrium, which, as we have seen, arises from rather more general considerations than a specific spin-flip mechanism.

VI. LEFT-HAND SIDE OF GKB

In this section we shall derive the "driving term" part of the equations which are essentially the Bloch-like equations of Hasagewa for conduction-electron and local-spin magnetization. However, we shall first derive a Boltzmann equation for the conduction electrons by which we mean an equation for

$$\vec{m}(\hat{p})=\mathrm{Tr}_s\gamma\vec{s}\int\rho\,d\epsilon_p\sum_{p_0}\Delta g^<(p,\,p_0)\ , \tag{6.1}$$

$$\vec{m}(\hat{p})=\gamma(2s+1)\tfrac{1}{3}s(s+1)\int\rho\,d\epsilon_p\sum_{p_0}\vec{m}(p,\,p_0)\ . \tag{B2}$$

The reason for this is that we shall discover terms of the form

$$\vec{v}_p\cdot\nabla[\vec{m}(\hat{p})\cdot\vec{s}\,]\ , \tag{6.2}$$

which correspond to a gradient of the magnetization current. If we had summed over the direction of the momentum then this term would have been an unknown. Instead, one can solve the Boltzmann equation, insert the solution for the magnetization current. When then the sum over momentum direction is taken a term of the form $D_s\nabla^2\vec{M}_s$ will be produced, where D_s is a diffusion constant. Such subtleties are unnecessary for the left-hand side for local spins where we derive an equation directly for

$$\vec{M}_d=\mathrm{Tr}_s\gamma\vec{S}\sum_{p,p_0}\Delta G^<(p,\,p_0)\ .$$

We can, of course, anticipate some of the results of our calculations. For the conduction electrons, the time derivative of magnetization is nearly balanced (on resonance) by its precession in an effective magnetic field which is the sum of the applied field and that due to the magnetization of the local spins. One expects these standard Bloch terms $(\partial\vec{M}_s/\partial t-\vec{M}_s\times\vec{H}_{\mathrm{eff},s})$, slightly modified by some very small terms of order J^2, plus the divergence of magnetization current. The surprising terms are those of order $J^2\ln T/D(\partial\vec{M}_d/\partial t+\gamma\vec{H}\times\vec{M}_d)$ which turn out to be essential to conserve angular momentum to order J^2 as is shown in Sec. VII.

For the local spins the effective precession field $\vec{H}_{\mathrm{eff},d}$ is the applied one plus that due to the conduction electrons. In this case the standard driving terms $(\partial\vec{M}_d/\partial t+\gamma\vec{H}_{\mathrm{eff},d}\times\vec{M}_d)$ are modified by a correction term of order $J^2\ln T/D$. There are also terms like $(\partial\vec{M}_s/\partial t+\gamma\vec{H}\times\vec{M}_s)$ but they are multiplied by a very small coefficient times J^2. Accordingly, when the conduction- and local-spin equations are added, all the correction terms vanish and

$$\frac{\partial(\vec{M}_s+\vec{M}_d)}{\partial t}+\gamma\vec{H}\times(\vec{M}_s+\vec{M}_d)+\vec{\nabla}\cdot\vec{J}_s=0$$

results which is required by the conservation of total angular momentum.

A. Conduction Electrons

The appropriate part of GKB equation for the conduction electrons is

$$(1/i)(g_0^{-1}+\gamma\vec{s}\cdot\vec{H}-\sigma,\ g^<)_{\mathrm{Ft}}-(1/i)[\sigma^<,\ g]_{\mathrm{Ft}}\ , \tag{6.3}$$

where only terms of order J^2 in σ or $\sigma^<$ are kept.

The obvious procedure at this stage is to use the gradient expansion of the Fourier transform of (6.3) given in Appendix A. There are two problems connected with a straightforward application of this procedure. First, a criterion is needed to decide which terms to keep in the gradient expansion. And that criterion arises from a consideration of the normal Bloch terms: $\partial\vec{M}_s/\partial t+\gamma\vec{H}\times\vec{M}_s$. $\vec{M}_s$, $\vec{M}_d$, $\vec{H}$, and the frequency of the applied field (all expressed in units of magnetic field) are of comparable size. Hence both parts of the Bloch terms are of second order in such small quantities. What we shall do is to keep all terms of second order in any of the small quantities: $\partial/\partial T$, $\vec{H}$, $\vec{M}_s$, $\vec{M}_d$, and $\vec{\nabla}$. Terms of higher order in these small quantities will be neglected.

The second problem associated with a simple-minded application of the gradient expansion to (6.3) is that even to second order a very large number of terms is generated. However, the use of the decomposition discussed in Sec. III causes many of these terms to be immediately eliminated. To understand the scheme let us write down an appropriate decomposition of the Fourier-transformed components:

$$g(p,\,p_0)=(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})^{-1}\ , \tag{6.4}$$

$$g^<(\vec{p},\,p_0)=2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})f(p_0+\gamma\vec{s}\cdot\vec{H})+\Delta g^<(\vec{p},\,p_0)\ , \tag{6.5a}$$

$$\Delta g^{<}(\vec{p},\ p_0)=\vec{m}(\vec{p},\ p_0)\cdot\vec{s}\ , \tag{6.5b}$$

$$\sigma(\vec{p},\ p_0)=\sigma_0(p_0+\gamma\vec{s}\cdot\vec{H})+\sigma_{M_d}(p_0)\ , \tag{6.6}$$

$$\sigma^{<}(\vec{p},\ p_0)=\gamma_0(p_0+\gamma\vec{s}\cdot\vec{H})\,f(p_0+\gamma\vec{s}\cdot\vec{H}) + \sigma^{<}_{M_d}(p_0)+\sigma^{<}_{M_s}(p_0)\ . \tag{6.7}$$

All the self-energies are calculated to order J^2 and are those given in Table II and calculated in Appendix B. It should be noted that $g(p,\ p_0)$ is only correct to zero order in J which is, however, sufficient since in (6.3) it appears with $\sigma^{<}$ which is of order J^2. A more subtle point concerns what we should write for $g^{>}(\vec{p},\ p_0)$ [see (3.2)] if it were needed in the calculation of the LHS. Since $g^{<}+g^{>}$ must equal the nonequilibrium spectral weight to order J^2 it is clear that the choice (3.1) and (3.2) is inconsistent with this requirement. We can treat (3.1)—i.e., (6.5a)—as exact and then must redefine $g^{>}$ [Eq. (3.2)] to satisfy the requirement to whatever order in J is desired. However, this latter step shall not be necessary since $g^{>}$ is not needed in the calculation of the LHS.

We observe that the Dyson equation (A24) guarantees that

$$(1/i)[g_0^{-1}+\gamma\vec{s}\cdot\vec{H}-\sigma_0(p_0+\gamma\vec{s}\cdot\vec{H}),\ 2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})] - (1/i)[\gamma_0(p_0+\gamma\vec{s}\cdot H),\ g(p,\ p_0+\gamma\vec{s}\cdot\vec{H})]=0\ . \tag{6.8}$$

Furthermore all the gradient expansions associated with this term are zero since the Dyson equation was satisfied before the Fourier transforming occured. A major simplification would occur if a Fermi factor $f(p_0+\gamma\vec{s}\cdot\vec{H})$ would be inserted next to the δ function in the first commutator of (6.8) and next to $g(p,\ p_0+\gamma\vec{s}\cdot\vec{H})$ in the second commutator so that the resulting expression were zero including the associated gradient expansions. Unfortunately the kinetic-energy term in g_0^{-1} plays a special role. The result is that

$$(1/i)[(g_0^{-1}+\gamma\vec{s}\cdot\vec{H}-\sigma_0),\ (g^{<}-\Delta g^{<})_{\mathrm{Ft}} - (1/i)[(\sigma^{<}-\sigma^{<}_{M_d}-\sigma^{<}_{M_s}),\ g]_{\mathrm{Ft}} = \tfrac{1}{2}\{\nabla_p\epsilon_p,\ \nabla_x f(p_0+\gamma\vec{s}\cdot\vec{H})\}\,2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})\ . \tag{6.9}$$

A proof of this, involving a discussion of terms arising from the gradient expansion of terms such as $g[g_0^{-1},f]_{\mathrm{Ft}}$, is deferred until near the end of this subsection. Then by combination of (6.9) and (6.3) the following terms remain to be calculated:

$$(1/i)[g_0^{-1},\ \Delta g^{<}]_{\mathrm{Ft}}+(1/i)[\gamma\vec{s}\cdot\vec{H},\ \Delta g^{<}]_{\mathrm{Ft}}\ , \tag{6.10a}$$

$$\vec{\nabla}_p\epsilon_p\cdot\nabla_x f(p_0+\gamma\vec{s}\cdot\vec{H})2\pi\delta(p_0-\epsilon_p+\gamma\vec{s}\cdot\vec{H})\ , \tag{6.10b}$$

$$-(1/i)[\sigma_0,\ \Delta g^{<}]_{\mathrm{Ft}}-(1/i)[\sigma^{<}_{M_s},\ g]_{\mathrm{Ft}}\ , \tag{6.11}$$

$$-(1/i)[\sigma_{M_d},\ \Delta g^{<}]_{\mathrm{Ft}}\ , \tag{6.12}$$

$$-(1/i)[\sigma_{M_d},\ g^{<}-\Delta g^{<}]_{\mathrm{Ft}}-(1/i)[\sigma^{<}_{M_d},\ g]_{\mathrm{Ft}}\ . \tag{6.13}$$

We shall show that Eqs. (6.10)–(6.13) have the following correspondence to parts of the Bloch equation:

$$(6.10\text{a})\rightarrow\frac{\partial\vec{M}_s}{\partial t}+\gamma(\vec{H}\times\vec{M}_s)+\vec{\nabla}\cdot\vec{J}_s\ , \tag{6.14a}$$

$$(6.11)\rightarrow \text{small (essentially zero) renormalization of (6.10)}\ , \tag{6.14b}$$

$$(6.12)\rightarrow(\alpha_1+\alpha_2)\gamma\vec{M}_d\times\vec{M}_s\ , \tag{6.14c}$$

$$(6.13)\rightarrow\alpha_2\chi_{s0}\left(\frac{\partial\vec{M}_d}{\partial t}+\gamma(\vec{H}\times\vec{M}_d)\right)\ . \tag{6.14d}$$

Some of these pieces are simple to calculate; others are deferred to Appendix E. In general the prescription is

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\,d\epsilon_p\sum_{p_0}[\text{any piece from (6.10)–(6.13)}]\ . \tag{6.15}$$

To obtain (6.14) a further integral over momentum direction $(\hat{p})$ is necessary. Consider, for example, the first term of (6.10). Then from (A33) we see that

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\,d\epsilon_p\sum_{p_0}(1/i)[g_0^{-1},\ \Delta g^{<}]_{\mathrm{Ft}} = \frac{\partial}{\partial t}\,\vec{m}(\hat{p})+\frac{\vec{p}}{m}\cdot\nabla_x\vec{m}(\hat{p})\ . \tag{6.16}$$

Equation (6.10b) will also contribute to the gradient term (6.16). In particular we see that

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\,d\epsilon_p\nabla_x(\gamma\vec{s}\cdot\vec{H})\cdot\nabla_p\epsilon_p\int dp_0\,\delta(p_0-\epsilon_p)\frac{\partial f}{\partial p_0} = -\vec{v}_p\cdot\nabla_x(\chi_{s0}\vec{H})\ . \tag{6.17}$$

The application of the gradient expansion (A31) to second term of (6.10) along with the substitution

$$\Delta g^{<}=\vec{m}\cdot\vec{s}\ , \tag{6.18}$$

and appropriate spin sums leads to

$$\gamma\vec{H}\times[\gamma(2s+1)\tfrac{1}{3}s(s+1)\int\rho\,d\epsilon_p\sum_{p_0}\vec{m}(\vec{p},\ p_0)] - \mathrm{Tr}_s\gamma\vec{s}\int\rho\,d\epsilon_p\sum_{p_0}\frac{1}{2}\left(\frac{\partial}{\partial t}\,\gamma\vec{s}\cdot\vec{H},\ \frac{\partial}{\partial p_0}\vec{m}(\vec{p},\ p_0)\cdot\vec{s}\right), \tag{6.19}$$

where the term involving $\partial H/\partial p_0$ has been obviously omitted. We see that the second term of (6.19) is of third order in small quantities and is hence neglected. In summary then the application of the prescription (6.15)–(6.10) produces the expected driving terms of the Boltzmann equation:

$$\frac{\partial \vec{m}(\hat{p})}{\partial t}+\gamma \vec{H}\times \vec{m}(\hat{p})+\vec{v}_p\cdot \vec{\nabla}_x[\vec{m}(\hat{p})-\chi_{s0}\vec{H}] , \qquad (6.20)$$

which clearly when summed over the momentum-direction variable is (6.14a).

What then are all the other terms? We expect that the effective field seen by the conduction electrons includes that due to the magnetization of the local spins. Such terms arise from (6.12). To first order in J,

$$\sigma_{M_d}^{(1)}(p_0)=\mathrm{Tr}_s\sum_{q,q_0}[-(J/n)\vec{s}\cdot\vec{S}\,G^{<}(q,\ q_0)]$$
$$=-(J/n\gamma^2)\gamma\vec{s}\cdot\vec{M}_d\ , \qquad (6.21)$$

so that the prescription (6.15),

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0} i[\sigma_{M_s}^{(1)}(p_0),\ \Delta g^{<}(\vec{p},\ p_0)]$$
$$=+\alpha_1\vec{M}_d\times\gamma\vec{m}(\hat{p})\ , \qquad (6.22)$$

produces a term which increases the effective magnetic field by $\alpha_1\vec{M}_d$. Likewise the second-order term in $\sigma_{M_d}(p_0)$, as shown in Appendix E, produces

$$\alpha_2\gamma\vec{M}_d\times\vec{m}(\hat{p})\ , \qquad (E12)$$

so that finally the precessional term is

$$\gamma[\vec{H}+(\alpha_1+\alpha_2)\vec{M}_d]\times\vec{m}(\hat{p})\ . \qquad (6.23)$$

It is clear that any additional terms in the gradient expansion for (6.12) would be of higher order in small quantities.

The terms arising from (6.11) can be disposed of quickly. Expanding σ_0 to first order in the field we see that the first term of (6.11) has a piece proportional to $\vec{H}\times\vec{m}(\hat{p})$. However, the constant of proportionality is shown in Appendix E [see the discussion near (E2)] to be zero. In a similar fashion all of (6.11) is found to be zero.

The terms in (6.13) are the most unusual part of our analysis for the LHS of GKB. They would, of course, be missed in any treatment which was not able to handle simultaneously nonequilibrium magnetizations of both the conduction electrons and the local spins as would the terms $(\alpha_1+\alpha_2)\vec{M}_d\times\gamma\vec{M}_s$. The first term in (6.13) produces two pieces: a $\vec{M}_d\times\vec{H}$ precessional term from the linear expansion in the field of $g^{<}-\Delta g^{<}$ and a $\partial\vec{M}_d/\partial t$ term from the gradient expansion. The evaluation, carried out in Appendix E, gives

$$\alpha_2\chi_{s0}\left(\frac{\partial\vec{M}_d}{\partial t}+\gamma\vec{H}\times\vec{M}_d\right). \qquad (6.24)$$

On the other hand, the second term of (6.13) is negligible [see (E4)].

Now we turn to a discussion of those terms in the gradient expansion involving momentum and spatial derivatives. It might appear that (6.11) would, through the gradient expansion, provide correction terms for the gradient of the magnetization current, $(\vec{v}_p\cdot\nabla)\vec{m}(\hat{p})$, but since σ_0 is independent of momentum (see Appendix B) the first term does not contribute. The second term is proportional to

$$\int\rho\, d\epsilon_p\sum_{k,p_0}\{\nabla\times[\vec{m}(\vec{k},p_0)\cdot\vec{s}],\ \nabla_p\epsilon_p/(p_0-\epsilon_p)^2\}, \qquad (6.25)$$

which is clearly the same kind of term as we found in (E4) modifying $\partial\vec{M}_s/\partial t$, and negligible for the same reason.

On the other hand (6.13) clearly appears to be the source of terms proportional to $(\vec{v}_p\cdot\nabla_x)\vec{M}_d$. It turns out that the only nonzero one arises from the first-order contribution to σ_{M_d},

$$\sigma_{M_d}^{(1)}=-\alpha_1\gamma\vec{s}\cdot\vec{M}_d,$$

in the first commutator of (6.13). Specifically we have

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0}(-\tfrac{1}{2})\{\nabla_x(-\alpha_1\gamma\vec{s}\cdot\vec{M}_d),\ \nabla_p\,\delta(p_0-\epsilon_p)\}$$
$$\times\ 2\pi f(p_0)=\ -\vec{v}_p\cdot\nabla_x(\chi_{s0}\alpha_1\vec{M}_d), \qquad (6.26)$$

which upon combination with (6.20) gives a gradient term of the form

$$\vec{v}_p\cdot\nabla_x[\vec{m}(\hat{p})-\chi_{s0}(H+\alpha_1\vec{M}_d)].$$

As we show in Appendix E, there is no term proportional to $\vec{v}_p\cdot\vec{\nabla}_x(\chi_{s0}\alpha_2\vec{M}_d)$ nor should there be. The reason for this, as discussed in Sec. IV, is that the conduction-electron magnetization $\vec{M}_s$ relaxes to $\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d)$ and hence it should be the difference between these quantities that drives the diffusion.

The divergence of the magnetization current, which is clearly second order in small quantities, becomes upon solution of the Boltzmann equation, as we shall see in Sec. VII, $D_s\nabla^2\vec{M}_s$ which at first appears to be of third order in small quantities. It is not, of course, but it is perhaps useful to examine some third-order small quantities and see how small they are compared to $D_s\nabla^2\vec{M}_s$. Since the field will be nonuniform inside the sample, the second term of (6.10), $(1/i)[\gamma\vec{s}\cdot\vec{H},\Delta g^{<}]$ will be a source of such a term. Application of the gradient expansion gives

$$\gamma \nabla_x \left(\vec{\mathrm{H}} \cdot \sum_{p,p_0} v_p \frac{\partial}{\partial \epsilon_p} \left[\gamma (2s+1) \tfrac{1}{3} s(s+1) \right] \vec{\mathrm{m}}(p, p_0) \right), \tag{6.27}$$

which is of order $(v_F \chi_{s0} H/\delta_e)\gamma H/D$, where δ_e is the characteristic diffusion length for the magnetization. Another term arising from (6.12) of the form

$$\sum_{q,q_0} \nabla_x \vec{\mathrm{M}}(q, q_0) \left(\frac{J}{n} \right)^2 \sum_{p,p_0} \sum_k \frac{f(\epsilon_k)}{p_0 - q_0 - \epsilon_k} \times v_p \frac{\partial}{\partial \epsilon_p} g^<(p, p_0) \tag{6.28}$$

is of order $(v_F \chi_{s0} H/\delta_e)\, \chi_{d0}\, \alpha_2 \gamma H/D$. Finally we note that

$$D_s \nabla^2 \vec{\mathrm{M}}_s \tag{6.29}$$

is of the order $(v_F \chi_{s0} H/\delta_e)(v_F \tau/\delta_e)$, since $D_s \sim v_F^2 \tau$. For typical experimental conditions[17] the last factor is of order unity. Thus the terms arising from (6.29), (6.27), and (6.28)

$$D_s \nabla^2 M_s \colon \nabla_x \vec{\mathrm{H}} \cdot \vec{\mathrm{M}}_s \colon \nabla_x \vec{\mathrm{M}}_d \colon \vec{\mathrm{M}}_s \tag{6.30}$$

are in the ratio

$$1 \colon \gamma H/D \colon \alpha_2 \chi_{s0} (\gamma H/D)\,. \tag{6.31}$$

Since $\gamma H/D \sim 10^{-6}$ and $\alpha_2 \chi_{d0} \sim (N/n)(\mu/k_B T)(\rho J/n)^2 \ln k_B T/D$ is less than unity, we can neglect such terms in comparison with $D_s \nabla^2 \vec{\mathrm{M}}_s$.

Finally we must turn to a proof of (6.9) which led to such simplifications in evaluating the LHS of the GKB equation. First, we observe that the kinetic-energy part of g_0^{-1} [see (2.9)] clearly by application of (2.11) gives the right-hand side of (6.9). Hence we must prove that the remainder of (6.9) is zero. Keeping in mind (6.8) we see that the time-frequency gradient expansion associated with the following three terms is the only possible nonzero entity:

$$\begin{aligned} &(1/i)\, 2\pi \delta(p_0 - \epsilon_p + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}}) \\ &\quad \times \{ [(p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}}), f(p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}})], \\ &\quad - [\sigma_0(p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}}), f(p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}})] \}, \\ &- (1/i)\, \gamma_0 (p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}}) \\ &\qquad \times [f(p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}}), g(p, p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}})]. \end{aligned} \tag{6.32}$$

An explicit gradient expansion would show that (6.32) is zero, or even easier (6.32) can be shown to be independent of H (and hence there is no $\partial H/\partial t$ term) when the sum over p_0 is performed by a variable transformation $p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}} \to p_0$. By either route (6.9) is proved. The observant reader might remark that the $\vec{\mathrm{p}}$ dependence of $g(p, p_0 + \gamma \vec{\mathrm{s}} \cdot \vec{\mathrm{H}})$ in the last term of (6.32) might give rise, via a gradient expansion, to a term of the form $(\vec{\mathrm{v}}_p \cdot \vec{\nabla}_x)\vec{\mathrm{H}}$. Fortunately the coefficient of such a term contains the integral $\int d\epsilon_p (p_0 - \epsilon_p)^{-2}$ which is zero by (D11).

In conclusion of this subsection, we collect all the terms, (6.20) and (6.23)–(6.25) on the LHS of the Boltzmann equation arising in connection with conduction electrons:

$$\begin{aligned} &\frac{\partial \vec{\mathrm{m}}(\hat{p})}{\partial t} + \gamma [\vec{\mathrm{H}} + (\alpha_1 + \alpha_2) \vec{\mathrm{M}}_d] \times \vec{\mathrm{m}}(\hat{p}) \\ &\quad + \vec{\mathrm{v}}_p \cdot \vec{\nabla}_x [\vec{\mathrm{m}}(\hat{p}) - \chi_{s0} (\vec{\mathrm{H}} + \alpha_1 \vec{\mathrm{M}}_d)] \\ &\quad + \alpha_2 \chi_{s0} \left(\frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d \right). \end{aligned} \tag{6.33}$$

B. Local Spins

Much of the treatment of the left-hand side of the equation for the local spins is very similar to that of Sec. VI A. We will proceed in a routine fashion pointing out the interesting differences along the way. The appropriate part of the GKB equation is

$$(1/i) [(G_0^{-1} + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}} - \Sigma), G^<]_{\mathrm{Ft}} - (1/i) [\Sigma^<, G]_{\mathrm{Ft}}, \tag{6.34}$$

where only terms of order J^2 in Σ or $\Sigma^<$ are kept. The appropriate decomposition is

$$G(p_0) = (p_0 + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}})^{-1}, \tag{6.35}$$

$$G^<(p_0) = 2\pi\, \delta(p_0 + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}})\, F(p_0 + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}}) + \Delta G^<(p_0), \tag{6.36a}$$

$$\Delta G^<(p_0) = \vec{\mathrm{M}}(\vec{\mathrm{p}}, p_0) \cdot \vec{\mathrm{S}}, \tag{6.36b}$$

$$\Sigma(p_0) = \Sigma_0 (p_0 + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}}) + \Sigma_{M_s}(p_0), \tag{6.37}$$

$$\Sigma^<(p_0) = \Gamma_0(p_0 + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}})\, F(p_0 + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}}) + \Sigma^<_{M_s}(p_0) + \Sigma^<_{M_d}(p_0), \tag{6.38}$$

where we have suppressed the useless momentum variable p. All the self-energies are those given in Table III and calculated in Appendix C. Note that the discussion following (6.7) is equally appropriate here.

Starting from the Dyson equation, we could show as in Sec. VI A (only here we are not bothered by momentum derivatives in the gradient expansion) that

$$(1/i)[(G_0^{-1} + \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}} - \Sigma_0), (G^< - \Delta G^<)]_{\mathrm{Ft}} - (1/i)[(\Sigma^< - \Sigma^<_{M_s} - \Sigma^<_{M_d}), G]_{\mathrm{Ft}} = 0. \tag{6.39}$$

Then by combination of (6.38) and (6.39) the following terms remain to be calculated:

$$(1/i)[G_0^{-1}, \Delta G^<]_{\mathrm{Ft}} + (1/i)[\gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{H}}, \Delta G^<]_{\mathrm{Ft}}, \tag{6.40}$$

$$-(1/i)[\Sigma_0, \Delta G^<]_{\mathrm{Ft}} - (1/i)[\Sigma^<_{M_d}, G]_{\mathrm{Ft}}, \tag{6.41}$$

$$-(1/i)[\Sigma_{M_s}, \Delta G^<]_{\mathrm{Ft}}, \tag{6.42}$$

$$-(1/i)[\Sigma_{M_s}, (G^< - \Delta G^<)]_{\mathrm{Ft}} - (1/i)[\Sigma^<_{M_s}, G]_{\mathrm{Ft}}. \tag{6.43}$$

Furthermore, as in Sec. VIA, there is a ready correspondence of the Eqs. (6.40)–(6.43) to pieces of the left-hand side of the Bloch equation:

$$(6.40) \rightarrow \frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d, \tag{6.44a}$$

$$(6.41) \rightarrow -\alpha_2 \chi_{s0} \left(\frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d \right), \tag{6.44b}$$

$$(6.42) \rightarrow (\alpha_1 + \alpha_2) \gamma \vec{\mathrm{M}}_s \times \vec{\mathrm{M}}_d, \tag{6.44c}$$

(6.43) → a negligible term involving

$$\frac{\partial \vec{\mathrm{M}}_s}{\partial t} \times (\gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_s). \tag{6.44d}$$

Each of the above terms follows straightforwardly from the prescription

$$\mathrm{Tr}_S \gamma \vec{\mathrm{S}} \sum_{p, p_0} [\text{ any piece from } (6.40)-(6.43)] , \tag{6.45}$$

although most of the tedious details are relegated to Appendix F. Remembering that

$$\vec{\mathrm{M}}_d = \mathrm{Tr}_S \gamma \vec{\mathrm{S}} \sum_{p, p_0} \Delta G^< (\vec{\mathrm{p}}, p_0)$$

$$= \gamma (2s+1) \tfrac{1}{3} s(s+1) \sum_{p, p_0} \vec{\mathrm{M}}(\vec{\mathrm{p}}, p_0) ,$$

we recover from (6.40)

$$\frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d. \tag{6.46}$$

The enhancement of the magnetic field in (6.46) by that due to the conduction electrons comes from (6.42). To first order in J

$$\Sigma^{(1)}_{M_s}(p_0) = \mathrm{Tr}_s \sum_{q, q_0} [-(J/n) \vec{\mathrm{s}} \cdot \vec{\mathrm{S}} g^< (\vec{\mathrm{q}}, q_0)]$$

$$= -\alpha_1 \gamma \vec{\mathrm{S}} \cdot \vec{\mathrm{M}}_s, \tag{6.47}$$

so that the prescription

$$\mathrm{Tr}_S \gamma \vec{\mathrm{S}} \sum_{p, p_0} -\frac{1}{i} [\Sigma^{(1)}_{M_s}(p_0), \Delta G^<(\vec{\mathrm{p}}, p_0)] = \alpha_1 \gamma \vec{\mathrm{M}}_s \times \vec{\mathrm{M}}_d. \tag{6.48}$$

Likewise the second-order term $\Sigma_{M_s(p_0)}$, as shown in Appendix F, produces

$$\alpha_2 \gamma \vec{\mathrm{M}}_s \times \vec{\mathrm{M}}_d, \tag{F14}$$

so that finally the precessional term is

$$\gamma [\vec{\mathrm{H}} + (\alpha_1 + \alpha_2) \vec{\mathrm{M}}_s] \times \vec{\mathrm{M}}_d. \tag{6.49}$$

There are correction terms to $\partial \vec{\mathrm{M}}_d / \partial t + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d$ with a coefficient proportional to $J^2 \ln(k_B T/D)$ which arise from (6.41). From each term of (6.41) such a correction arises, the first with a weighting of $-s(s+1)$ and of the second $[s(s+1) - 1]$, so that the terms nearly cancel, as shown in Appendix F, leaving

$$-\alpha_2 \chi_{s0} \left(\frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d \right). \tag{6.50}$$

Finally we observe that (6.43) leads, as shown in Appendix F, to a term proportional $(\partial \vec{\mathrm{M}}_s / \partial t + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_s)$ with a negligible coefficient.

In summary of this subsection, we collect all the terms on the left-hand side of the Bloch equation for the local spins:

$$\left(\frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d \right) (1 - \alpha_2 \chi_{s0}) + \gamma (\alpha_1 + \alpha_2) \vec{\mathrm{M}}_s \times \vec{\mathrm{M}}_d. \tag{6.51}$$

VII. SYNTHESIS

In this section we combine the results of Secs. V and VI. We have the following objectives in doing this: (i) to find the connection between the magnetization current and the magnetization—i.e., $\vec{\mathrm{J}}_s = -\vec{\nabla}[\vec{\mathrm{M}}_s - \chi_{s0}(\vec{\mathrm{H}} + \alpha_1 \vec{\mathrm{M}}_d)]$—by solving the Boltzmann equation for $\vec{\mathrm{m}}(\hat{p})$; (ii) to demonstrate that the Bloch equations are consistent with the concepts concerning angular momentum; and (iii) to evaluate the importance of various correction terms to the Hasegawa equations.

We reproduce the results (6.33), (5.18), and (5.31) of Secs. V and VI for the left- and right-hand sides of the Boltzmann equation for $\vec{\mathrm{m}}(\hat{p})$:

$$\frac{\partial \vec{\mathrm{m}}(\hat{p})}{\partial t} + \gamma [\vec{\mathrm{H}} + (\alpha_1 + \alpha_2) \vec{\mathrm{M}}_d] \times \vec{\mathrm{m}}(\hat{p})$$

$$+ \vec{\mathrm{v}}_p \cdot \vec{\nabla}_x [\vec{\mathrm{m}}(\hat{p}) - \chi_{s0}(\vec{\mathrm{H}} + \alpha_1 \vec{\mathrm{M}}_d)]$$

$$+ \alpha_2 \chi_{s0} \left(\frac{\partial \vec{\mathrm{M}}_d}{\partial t} + \gamma \vec{\mathrm{H}} \times \vec{\mathrm{M}}_d \right)$$

$$= -\left(\frac{1}{\tau} + \frac{s(s+1)}{T_{sd}} + \frac{s(s+1)}{T_{sl}} \right) \int \frac{dp'}{4\pi} [\vec{\mathrm{m}}(p) - \vec{\mathrm{m}}(p')]$$

$$-\left(\frac{1}{T_{sd}} + \frac{1}{T_{sl}} \right) [\vec{\mathrm{M}}_s - \chi_{s0}(\vec{\mathrm{H}} + \alpha_1 \vec{\mathrm{M}}_d)]$$

$$+ \frac{1}{T_{ds}} \{ \vec{\mathrm{M}}_d - \chi_{d0}[H + (\alpha_1 + 2\alpha_2) \vec{\mathrm{M}}_s] \}, \tag{7.1}$$

where in the first term of the right-hand side we have added a relaxation time τ due to ordinary potential scattering (from the spin-independent scattering of any impurities present) and electron-phonon scattering.

Before considering the divergence of the magnetization current, let us spend a moment with the second term on the left-hand side of (7.1). We will use it to show that $\vec{m}(\vec{p}, p_0)$ is proportional to $\delta(p_0 - \epsilon_p)\, \partial f(p_0)/\partial p_0$. To be specific suppose

$$\vec{H} = H_0 \hat{z} + \tfrac{1}{2}(\hat{x} - i\hat{y}) H_1 e^{-i\Omega t}, \tag{7.2}$$

where H_0 is the dc applied field and H_1 ($\ll H_0$) is the amplitude of the ac field; this is a conventional choice for spin-resonance experiments. Here $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in the three Cartesian directions. Then the second term of the LHS gives rise to

$$m_z(\hat{p}) H_1 - [m_x(\hat{p}) + i m_y(\hat{p})] H_0 . \tag{7.3}$$

A more convenient form for our proof would have been the equation of the form of (7.3) before the integrals over ϵ_p and p_0 were performed. Then we would have had

$$m_z(p, p_0) H_1 - [m_x(p, p_0) + i m_y(p, p)] H_0, \tag{7.4}$$

where clearly $m_z(p, p_0)$ arises essentially from dc magnetic field and hence as we showed in (3.8) is equal to

$$\mathrm{Tr}_s \gamma \vec{s}(\gamma \vec{s} \cdot \vec{H})\, 2\pi\, \delta(p_0 - \epsilon_p)\, \frac{\partial f}{\partial p_c} . \tag{7.5}$$

Since the first term of (7.4) is the inhomogeneous term in the "Boltzmann equation," it sets the scale for the nonequilibrium parts of the magnetization $m_x(\vec{p}, p_0)$ and $m_y(\vec{p}, p_0)$. Hence we see that they are proportional to $\delta(p_0 - \epsilon_p)\, \partial f/\partial p_0$. Thus our assumption

$$\vec{m}(\vec{p}, p_0) \propto \delta(p_0 - \epsilon_p)\, \frac{\partial f}{\partial p_0} \tag{7.6}$$

used in many parts of our calculation is proved. Clearly the same argument can be repeated for the local spins to show that $\vec{M}(p, p_0)$ is proportional to $\delta(p_0)$.

We return to (7.1) which upon multiplying by $\vec{v}_p$ and doing the integral over the direction of momentum $\vec{p}$, we find is

$$\begin{aligned}
&\frac{\partial \overleftrightarrow{J}_s}{\partial t} + \gamma[\vec{H} + (\alpha_1 + \alpha_2)\vec{M}_d] \times \overleftrightarrow{J}_s \\
&\qquad + \tfrac{1}{3} v_F^2\, \vec{\nabla}[\vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d)] \\
&\qquad = -\left(\frac{1}{\tau} + \frac{s(s+1)}{T_{sd}} + \frac{s(s+1)}{T_{sl}}\right) \overleftrightarrow{J}_s = -\frac{\overleftrightarrow{J}_s}{\tau_{\mathrm{tr}}} ,
\end{aligned} \tag{7.7}$$

where τ_{tr} is the transport relaxation time whose definition is obvious. Here v_F is the magnitude of the Fermi velocity and the factor of $\frac{1}{3}$ comes from the angular average. The first term of (7.7) is negligible by the following estimate. First since a drift velocity of even 1 cm/sec is enormous $\vec{J}_s$ is overestimated by $\vec{M}_s$. On the other hand $\partial/\partial t \sim \gamma H$ which is about 10^{10} sec^{-1} for a few kilogauss. This is to be compared with v_F^2/δ_e, where δ_e is the characteristic spatial variation of the magnetization and of the order of 10^{-3} cm for a typical experiment. Thus the ratio $\partial \overleftrightarrow{J}_s/\partial t : v_F^2 \vec{\nabla} \vec{M}_s$ is 10^{10}: 10^{19} and the first term is completely negligible as is the second. Accordingly, we have from (7.7)

$$\overleftrightarrow{J}_s = -D_s \nabla [M_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d)], \tag{7.8}$$

where the diffusion constant D_s is given by the familiar expression

$$D_s = \tfrac{1}{3} v_F^2 \tau_{\mathrm{tr}} . \tag{7.9}$$

Using (7.8) and summing (7.1) over the momentum direction $\hat{p}$, we have the Bloch-like equation:

$$\begin{aligned}
&\frac{\partial \vec{M}_s}{\partial t} + \gamma[\vec{H} + (\alpha_1 + \alpha_2)\vec{M}_d] \times \vec{M}_s \\
&\qquad - D_s \nabla^2 [\vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d)] \\
&\qquad + \alpha_2 \chi_{s0}\left(\frac{\partial \vec{M}_d}{\partial t} + \gamma \vec{H} \times \vec{M}_d\right) \\
&\qquad = -\left(\frac{1}{T_{sd}} + \frac{1}{T_{sl}}\right)[\vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d)] \\
&\qquad\quad + \frac{1}{T_{ds}} \{\vec{M}_d - \chi_{d0}[\vec{H} + (\alpha_1 + 2\alpha_2)\vec{M}_s]\} .
\end{aligned} \tag{7.10}$$

This equation is to solved in conjunction with the equation for $\vec{M}_d$ which results from combining (6.51) with (5.25) and (5.33)

$$\begin{aligned}
&\left(\frac{\partial \vec{M}_d}{\partial t} + \gamma \vec{H} \times \vec{M}_d\right)(1 - \alpha_2 \chi_{s0}) + \gamma(\alpha_1 + \alpha_2)\vec{M}_s \times \vec{M}_d \\
&\qquad = -\left(\frac{1}{T_{ds}} + \frac{1}{T_{dl}}\right)\{\vec{M}_d - \chi_{d0}[\vec{H} + (\alpha_1 + 2\alpha_2)\vec{M}_s]\} \\
&\qquad\quad + \frac{1}{T_{sd}} [\vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d)] .
\end{aligned} \tag{7.11}$$

The correction term on the left-hand side of both (7.10) and (7.11) involves

$$\alpha_2 \chi_{s0} = \tfrac{1}{2}(\rho J/n)^2 \ln(k_B T/D), \tag{7.12}$$

which for Mn in Cu is of the order of -0.02 at 1 °K. If this term were really effective, it might produce measurable shifts in the resonance frequency. As we shall show such a conclusion is not warrented to order J^2. For simplicity in writing we introduce the notation

$$\delta\vec{M}_s = \vec{M}_s - \chi_{s0}(\vec{H} + \alpha_1 \vec{M}_d), \tag{7.13a}$$

$$\delta\vec{M}_d = \vec{M}_d - \chi_{d0}[\vec{H} + (\alpha_1 + 2\alpha_2)\vec{M}_s], \tag{7.13b}$$

and

$$\frac{D\vec{M}_{s(d)}}{Dt}=\frac{\partial\vec{M}_{s(d)}}{\partial t}+\gamma\vec{H}\times\vec{M}_{s(d)}\,. \tag{7.14}$$

Then adding (7.10) and (7.11) we have the simple equation

$$\frac{D(\vec{M}_s+\vec{M}_d)}{Dt}-D_s\nabla^2\,\delta\vec{M}_s=-\frac{\delta\vec{M}_s}{T_{sl}}-\frac{\delta\vec{M}_d}{T_{dl}}\;. \tag{7.15}$$

This equation is an expression of the conservation of total (spin) angular momentum. For our original Hamiltonian (2.1) all terms in (7.15) are zero but the first which reflects the time development of the total magnetization in an external field. In that case $\vec{M}_s+\vec{M}_d$ would be a constant. In the real system, the mobility of the conduction electrons allows their magnetization to diffuse away. Further the presence of the "lattice" provides a sink for the spin angular momentum of the conduction electrons and the local spins.

If we solve (7.11) for $D\vec{M}_d/Dt$ and substitute into (7.10) and keep only corrections of order J^2 (remembering that $1/T_{sd}$ and $1/T_{ds}$ are proportional to J^2 but $1/T_{sl}$ and $1/T_{dl}$ are not) we discover that

$$\frac{D\vec{M}_s}{Dt}-D_s\nabla^2\,\delta\vec{M}_s+\gamma(\alpha_1+\alpha_2)\,\vec{M}_d\times\vec{M}_s$$
$$=-\left(\frac{1}{T_{sl}}+\frac{1}{T_{sd}}\right)\delta\vec{M}_s+\left(\frac{\alpha_2\chi_{s0}}{T_{dl}}+\frac{1}{T_{ds}}\right)\delta\vec{M}_d\,, \tag{7.16}$$

which is the equation we would have originally expected except for the small addition to the relaxation of the local spins. Since $1/T_{dl}$ of a few hundred Gauss is typical, this correction, based on a crude estimate of $\alpha_2\chi_{s0}$, could produce changes of a few Gauss—an effect hardly worth considering when compared with the theoretical (and experimental) uncertainties associated with T_{dl}. Finally we can combine (7.16) and (7.15) to get the Hasegawa equation for the local spins

$$\frac{D\vec{M}_d}{Dt}+\gamma\,(\alpha_1+\alpha_2)\,\vec{M}_s\times\vec{M}_d$$
$$=-\left(\frac{1+\alpha_2\,\chi_{s0}}{T_{dl}}+\frac{1}{T_{ds}}\right)\delta\vec{M}_d+\frac{1}{T_{sd}}\;\delta\vec{M}_s\,, \tag{7.17}$$

which again is the expected equation. The full form of (7.16) and (7.17) is given by making the substitutions (7.13a) and (7.13b) and (7.14).

ACKNOWLEDGMENTS

We wish to thank D. L. Cowan, J. H. Pifer, and R. Silsbee for helpful conversations. We are grateful for the hospitality of W. T. Fox and the staff of the Bronfmann Science Center, Williams College, Williamstown, Mass., where the bulk of the manuscript was written.

APPENDIX A: DERIVATION OF GENERALIZED KADONOFF-BAYM EQUATION

A number of years ago, Kadanoff and Baym[19] (KB) derived a set of Green's-function equations [Eqs. (8.27) and (8.28) of Ref. 19], which provide an exact description of nonequilibrium processes. Unfortunately, these equations were so complicated (taking nearly two printed pages to write) that they seemed hopeless. Neglecting spin, however, KB were able to simplify the form of these equations by assuming that all external disturbances varied slowly in space and time, and then making a gradient expansion in these slowly varying quantities. In this way they were able to derive much simpler equations which resembled the ordinary Boltzmann equation, and which were still exact to lowest order in these slowly varying quantities. It would seem that a generalization of their method would be ideally suited to our problem of spin transport. However, the spin matrices themselves must be treated exactly and quantum mechanically; it would clearly be inappropriate to attempt a gradient expansion in the distance away from the diagonal in an $S=\frac{1}{2}$ spin matrix as KB did for space and time. Furthermore, we eventually (although not in this paper) would like to treat the case $\Omega\sim\gamma H>k_BT$, where the gradient expansion in time breaks down. Thus it would seem as if one has to use the more complicated equations. However, there exists a notation[24] that renders the exact quantum-mechanical equations as simple in form as the Kadanoff-and Baym's Boltzmann equations; these equations we will call the generalized Kadanoff-Baym equations (GKB).

Since the KB equations are so lengthy, it will actually be easier to begin the derivation from first principles than to reproduce them here. One eventually wants to calculate real-time correlations such as [note well that our definition of $g^{\gtrless}$, $\sigma^{\gtrless}$, etc., differs slightly from that of KB in that to obtain our "greater than" quantities, one should multiply the quantities of KB by i and to obtain our "lesser than" quantities one should multiply those of KB by $-i$. This distinction holds only before Fourier transforming with respect to the difference-time variable. After such Fourier transformation, our definition is the same as KB. This happens because the KB $g^{\gtrless}(p,\,p_0)$ is not the Fourier transform of their $g^{\gtrless}(r,\,\tau)$ but differs by a factor of $\pm i$.]

$$g^{>}_{\mu\nu}(t,t')\equiv\langle v(t)\,\psi_\mu(t)\,v^{-1}(t)\,v(t')\,\psi^\dagger_\nu(t')v^{-1}(t')\rangle \tag{A1a}$$

and

$$g^{<}_{\mu\nu}(t,t')\equiv\langle v(t')\,\psi^\dagger_\nu(t')v^{-1}(t')v(t)\,\psi_\mu(t)\,v^{-1}(t)\rangle\,, \tag{A1b}$$

where

$$v(t) = T \exp\left[- i \int_{-\infty}^{t} H_1(\bar{t})\, d\bar{t} \right]. \tag{A1c}$$

Here H_1 is the Hamiltonian representing any external field driving the system from equilibrium (it vanishes by assumption at $t = -\infty$), and all operators are taken to propagate in the interaction representation according to the terms remaining in the Hamiltonian without H_1; similarly, the notation $\langle \cdots \rangle$ represents a thermal average assuming the system is in equilibrium with $H_1 = 0$ at $t = -\infty$. Here the symbol T denotes that the terms in the expansion of (A1c) are to be ordered according to their arguments with the earliest times to the right. Finally, $\psi_\nu^\dagger$ and ψ_μ are creation and destruction operators for particles whose properties are labeled by the subscripts μ, and ν, which denote space, spin, and which type of particle is being referred to. Wherever possible we will not write these labels, in which case matrix multiplication will be implied, if appropriate.

Equations (A1) are inconvenient because the standard diagramatic perturbation expansions do not give these functions but rather an imaginary- or complex-time Green's function, defined for fermions by

$$g_{\mu\nu}(t, t') = - i \langle T S \psi_\mu(t)\, \psi_\nu^\dagger(t') \rangle \operatorname{sgn}[\operatorname{Im}(t' - t)] / \langle TS \rangle , \tag{A2a}$$

where

$$S = T \exp\left[- i \int_{t_0}^{t_0 - i\beta} H_1(\bar{t})\, d\bar{t} \right]. \tag{A2b}$$

Here t and t' are both on the line between t_0 and $t_0 - i\beta$ and the operator T now orders all factors with those with time arguments closest to t_0 appearing to the right. The introduction of the real time t_0 is a device introduced by KB to recover (A1) from a knowledge of (A2): Specifically, it follows from straightforward algebra that [except for the factor $- i \operatorname{sgn}(t - t')$] the "greater than" and "lesser than" pieces of (A2a) are equal to (A1a) and (A1b), respectively, in the limit of $t_0 \to -\infty$.

The Green's function satisfies the Dyson equation

$$g(t, t') = g_0(t, t') + \int_{t_0}^{t_0 - i\beta} g_0(t, \bar{t})\, U(\bar{t})\, g(\bar{t}, t')\, d\bar{t} + \int_{t_0}^{t_0 - i\beta} d\bar{t} \int_{t_0}^{t_0 - i\beta} d\bar{t}'\, g_0(t, \bar{t})\, \sigma(\bar{t}, \bar{t}')\, g(\bar{t}', t') , \tag{A3}$$

plus a similar equation with the order of all the factors reversed. The quantity $g_0(t, t')$ is the Green's function for the one-body part of the Hamiltonian with the external field turned off, and σ is the self-energy which possesses a well-defined perturbation expansion in powers of the Green's function g. The quantity U is defined by

$$H_1 = \sum_{\mu\nu} \psi_\mu^\dagger U_{\mu\nu} \psi_\nu . \tag{A4}$$

The Kadanoff-Baym transport equations are simply Eq. (A3) written in terms of the real physical response functions $g^{\gtrless}$ of (A1) and the corresponding self-energy function $\sigma^{\gtrless}$. These equations are derived by separating out the various analytic pieces in (A3) and then letting $t_0 \to -\infty$. To accomplish this simply, we prove a theorem for evaluating integrals of the form

$$D(t, t') \equiv \int_{t_0}^{t_0 - i\beta} d\bar{t}\; A(t, \bar{t})\, B(\bar{t}, t') . \tag{A5}$$

Consider $D^{>}(t, t')$ which is given by

$$- i D^{>}(t, t') = \int_{t_0}^{t'} d\bar{t}\, (-i) A^{>}(t, \bar{t})\, (i)\, B^{<}(\bar{t}, t') + \int_{t'}^{t} d\bar{t}\, (-i) A^{>}(t, \bar{t})\, (-i)\, B^{>}(\bar{t}, t') + \int_{t}^{t_0 - i\beta} d\bar{t}\; (i) A^{<}(t, \bar{t})\, (-i)\, B^{>}(\bar{t}, t'). \tag{A6}$$

On taking the limit to $t_0 \to -\infty$, we see that

$$D^{>}(t, t') = i \int_{-\infty}^{\infty} d\bar{t}\, A^{>}(t, \bar{t}) [B^{>}(\bar{t}, t') + B^{>}(\bar{t}, t')]\, \theta(t' - \bar{t}) - i \int_{-\infty}^{\infty} d\bar{t} [A^{>}(t, \bar{t}) + A^{<}(t, \bar{t})]\, \theta(t - \bar{t}) B^{<}(\bar{t}, t') . \tag{A7}$$

The above suggests defining retarded and advanced functions

$$A_r(t, t') \equiv - i [A^{>}(t, t') + A^{<}(t, t')]\, \theta(t - t') , \tag{A8a}$$

$$A_a(t, t') \equiv + i [A^{>}(t, t') + A^{<}(t, t')]\, \theta(t' - t) , \tag{A8b}$$

and similarly for B or any other function. Then (A7) becomes

$$D^{>} = A^{>} B_a + A_r B^{>} , \tag{A9}$$

where now matrix multiplication is also implied for the time integrals which extend from $-\infty$ to ∞. Similarly, one can prove an analogous relation for $D^{<}$, so that

$$D^{\gtrless} = A^{\gtrless} B_a + A_r B^{\gtrless} . \tag{A10}$$

Finally, an integral involving three functions

$$E(t, t') = \int_{t_0}^{t_0 - i\beta} d\bar{t} \int_{t_0}^{t_0 - i\beta} d\bar{t}'\, A(t, \bar{t})\, B(\bar{t}, \bar{t}')\, C(\bar{t}', t') \tag{A11}$$

becomes

$$E^{\gtrless} = A_r B_r C^{\gtrless} + A_r B^{\gtrless} C_a + A^{\gtrless} B_a C_a . \tag{A12}$$

Equation (A12) is readily followed from the repeated use of (A10) plus the easily verified identities

$$(AB)_r = A_r B_r ; \quad (AB)_a = A_a B_a . \tag{A13}$$

It is then, of course, straightforward to generalize

(A12) by induction to a product of n functions.

We are now in a position to use the above theorems on Eqs. (A3) which we write symbolically as

$$g=g_0+g_0Ug+g_0\sigma g\,,\qquad g=g_0+gUg_0+g\sigma g_0\,, \tag{A14}$$

or alternatively as

$$(g_0^{-1}-U)g=1+\sigma g\,,\quad g(g_0^{-1}-U)=1+\sigma g\,. \tag{A15}$$

Since g_0^{-1} does not have a unique inverse, Eqs. (A15) provide less information than (A14) and must be supplied with an initial condition in order to determine g.

We first apply theorem (A10) to both sides of (A15) yielding

$$(g_0^{-1}-U)g^{\gtrless}=\sigma_r g^{\lessgtr}+\sigma^{\gtrless}g_a\,, \tag{A16a}$$

$$g^{\gtrless}(g_0^{-1}-U)=g_r\sigma^{\gtrless}+g^{\gtrless}\sigma_a\,. \tag{A16b}$$

This can be written in a simple form if one makes the definitions

$$a=g^{>}+g^{<}=-i(g_a-g_r)\,,\qquad \gamma=\sigma^{>}+\sigma^{<}=-i(\sigma_a-\sigma_r)\,, \tag{A17}$$

and

$$g=\tfrac{1}{2}(g_a+g_r)\,,\qquad \sigma=\tfrac{1}{2}(\sigma_a+\sigma_r)\,. \tag{A18}$$

In this nonequilibrium case, g and $\frac{1}{2}a$ play the same role as the real and imaginary parts of the Green's function does in the equilibrium case; σ and $\frac{1}{2}\gamma$ have a similar relation to the self-energy. Now if we subtract (A16b) from (A16a) and use (A17) and (A18) we obtain

$$\begin{aligned}(1/i)[(g_0^{-1}-U-\sigma),\,g^{\gtrless}],\ -(1/i)[\sigma^{\gtrless},g]\\ &=-\tfrac{1}{2}\{\gamma,g^{\gtrless}\}+\tfrac{1}{2}\{\sigma^{\gtrless},a\}\\ &=-\tfrac{1}{2}\{\sigma^{\gtrless},g^{\lessgtr}\}+\tfrac{1}{2}\{\sigma^{\lessgtr},g^{\gtrless}\}\,, \end{aligned}\tag{A19}$$

where the square brackets denote commutators and the curly brackets denote anticommutators. These equations are what we will call the generalized Kadanoff-Baym equations (GKB); note that they are analogous to Eqs. (9.25) of Ref. 19, except that these equations are exact; no assumption of slowly varying disturbances has been made.

Generally, it will be necessary only to consider one of the equations [(A19)], say, for the one for $g^{<}$ because as we will show, there exists a usual type of Dyson equation for g_a and g_r so that $g^{>}$ can be determined by (A17) if $g^{<}$ is known. To determine these Dyson equations, we use the theorem (A12) on (A14), obtaining

$$g^{\gtrless}=g_0^{\gtrless}+g_{0r}Ug^{\gtrless}+g_{0r}(\sigma_r g^{\gtrless}+\sigma^{\gtrless}g_a)$$

$$+g_0^{\gtrless}Ug_a+g_0^{\gtrless}\sigma_a g_a\,, \tag{A20}$$

where the definitions of $g_0^{\gtrless}$, g_{0r}, and g_{0a} are the obvious ones. Upon adding the two equations, (A20) and using the last equalities in each line of (A17), we obtain

$$\begin{aligned}g_a-g_r=(g_{0a}-g_{0r})\,U(g_a-g_r)\\ +g_{0r}[\sigma_r(g_a-g_r)+(\sigma_a-\sigma_r)g_a]\\ +(g_{0a}-g_{0r})\,Ug_a+(g_{0a}-g_{0r})\,\sigma_a g_a\,,\end{aligned}\tag{A21}$$

which upon making cancellations can be rearranged as

$$-g_a+g_{0a}+g_{0a}(U+\sigma_a)g_a=-g_r+g_{0r}+g_{0r}(U+\sigma_r)g_r\,. \tag{A22}$$

Because of the definition of retarded and advanced quantities (A8) plus the theorem (A13), it is clear that depending on the time arguments, either the left- or the right-hand side of (A22) is zero by definition. Therefore, we must have

$$g_a=g_{0a}+g_{0a}(U\sigma_a)\,g_a \tag{A23a}$$

and

$$g_r=g_{0r}+g_{0r}(U+\sigma_r)\,g_r\,. \tag{A23b}$$

Thus the retarded and advanced Green's functions satisfy Dyson's equations as in equilibrium. From their solution one can obtain g and a for use in (A19).

As a consistency check, note that the addition of the two equations, (A19) yields

$$[(g_0^{-1}-U-\sigma),\,a]-[\gamma,g]=0\,. \tag{A24}$$

It is a straighforward matter to show that (A23) identically satisfies (A24).

The equations of KB may be obtained from (A19) by replacing the commutators by Poisson brackets and the anticommutators by simple products, thus making a simple gradient expansion in slowly varying quantities. In our case, however, we want to expand in space and time but not in spin. Consider a quantity C defined by

$$C(t_1,t_2)=\int dt'\,A(t_1,t')\,B(t',t_2)\,. \tag{A25}$$

We illustrate the gradient expansion in the time variables, but the results can be immediately extended to include the space variables as well. It is convenient to introduce sum and difference variables

$$t=\tfrac{1}{2}(t_1+t_2)\;\tau=t_1-t_2, \tag{A26}$$

and, with a slight change in notation, to write C in terms of these variables; that is $C(t_1,t_2)\to C(\tau,t)$, and similarly for A and B. Since we assume the Hamiltonian for the system without external field is independent of time, then with no external field the various quantities would be independent of t (A, B, and C are taken to be Green's functions, self-energies or the like). Similarly, if the external field is slowly varying, one expects these quantities to be a weak function of t, and for, say, $\partial A/\partial t$ to be small, and $\partial A^2/\partial t^2$ or $\partial^2 A/\partial t^2$ to be even smaller. By neglecting terms of the latter type in a gradient expansion, we are making an expansion in the $\hbar$ times the frequency of the external disturbance, divided by an appropriate characteristic energy of the system itself. It is usually not obvious *a priori* what this energy is, but as an intuitive guide, one expects it to be the smallest spacing between energy levels unless $k_B T$ is bigger than this, in which case we expect it to be $k_B T$. Since we consider only the case where $h\gamma H \ll k_B T$ and T much larger than the Kondo temperature, we expect our results to be valid whenever $\hbar\Omega \ll k_B T$.

Writing (A25) in terms of sum and difference variables gives

$$C(\tau, t) = \int dt' A[t_1 - t'; t + \tfrac{1}{2}(t' - t_2)] \times B[t' - t_2; t - \tfrac{1}{2}(t_1 - t')], \qquad \text{(A27)}$$

which on expansion becomes

$$\begin{aligned} C(\tau, t) = & \int dt' A(t_1 - t'; t) B(t' - t_2; t) \\ & + \int dt' \frac{\partial A}{\partial t} (t_1 - t'; t) \tfrac{1}{2}(t' - t_2) B(t' - t_2; t) \\ & - \int dt' \tfrac{1}{2}(t_1 - t') A(t_1 - t'; t) \frac{\partial B(t' - t_2; t)}{\partial t} + \cdots . \end{aligned} \qquad \text{(A28)}$$

Finally we Fourier transform with respect to the difference variables:

$$C(p_0, t) \equiv \int d\tau\, e^{ip_0\tau} C(\tau, t) \equiv [AB]_{\mathrm{Ft}}, \qquad \text{(A29)}$$

so that (A28) becomes

$$\begin{aligned} C(p_0, t) = [AB]_{\mathrm{Ft}} = & A(p_0, t) B(p_0 t) \\ & + \frac{1}{2i} \left(\frac{\partial A(p_0, t)}{\partial t} \frac{\partial B(p_0, t)}{\partial p_0} - \frac{\partial A(p_0, t)}{\partial p_0} \frac{\partial B(p_0, t)}{\partial t} \right) + \cdots . \end{aligned} \qquad \text{(A30)}$$

This is readily generalized to include the expansion in space:

$$\begin{aligned} C(p, p_0; x, t) \equiv & [AB]_{\mathrm{Ft}} \\ = & A(p, p_0; x, t) B(p, p_0; x, t) \\ & + \frac{1}{2i} \left(\frac{\partial A(p, p_0; x, t)}{\partial t} \frac{\partial B(p, p_0; x, t)}{\partial p_0} - \frac{\partial A(p, p_0; x, t)}{\partial p_0} \frac{\partial B(p, p_0; x, t)}{\partial t} \right) \\ & - \frac{1}{2i} [\nabla_x A(p, p_0; x, t) \cdot \nabla_p B(p, p_0; x, t) \\ & - \nabla_p A(p, p_0; \vec{x}, t) \cdot \nabla_x B(p, p_0; \vec{x}, t)] + \cdots , \end{aligned} \qquad \text{(A31)}$$

where $\vec{x} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$ and $\vec{p}$ in the Fourier-transform variable corresponding to the difference variable $\vec{r} = \vec{x}_1 - \vec{x}_2$.

We conclude this Appendix by working out an example of the gradient expansion of great importance, that is the expansion for the inverse of the bare Green's function g_0^{-1} acting on an arbitrary function F. Note that $g_0^{-1}(p, p_0; x, t) = p_0 - \epsilon_p$, so that according to (A31)

$$\begin{aligned} [g_0^{-1}, F]_{\mathrm{Ft}} = & (p_0 - \epsilon_p) F(\vec{p}, p_0; x, t) \\ & - \frac{1}{2i} \frac{\partial F}{\partial t} (\vec{p}, p_0; x, t) + \frac{1}{2i} \\ & \times (-\nabla_p \epsilon_p) \cdot \nabla_x F(\vec{p}, p_0; x, t) + \cdots , \end{aligned} \qquad \text{(A32)}$$

so that

$$\begin{aligned} \frac{1}{i} [g_0^{-1}, F]_{\mathrm{Ft}} \equiv & \frac{1}{i} [g_0^{-1} F]_{\mathrm{Ft}} - \frac{1}{i} [F g_0^{-1}]_{\mathrm{Ft}} \\ = & \left(\frac{\partial}{\partial t} + (\nabla_p \epsilon_p) \cdot \nabla_x \right) F(\vec{p}, p_0; x, t) + \cdots , \end{aligned} \qquad \text{(A33)}$$

thus generating a pair of the important drift terms of the Boltzmann equation. In turns out that the time-derivative part of (A33) is exact, and does not depend on the gradient expansion. To show this, one calculates the commutator directly; in the space-time picture the time part of g_0^{-1} is $i(\partial/\partial t_1)\, \delta(t_1 - t_2)$. We want the quantity

$$\begin{aligned} \frac{1}{i} \int d\bar{t} \Big(& i \frac{\partial}{\partial t_1} \delta(t_1 - t_2) F(\bar{t}, t_2) - F(t_1, \bar{t}) \\ & \times i \frac{\partial}{\partial t_1} \delta(\bar{t} - t_2) \Big) \\ = & \frac{\partial F(t_1, t_2)}{\partial t_1} + \frac{\partial F(t_1, t_2)}{\partial t_2} . \end{aligned} \qquad \text{(A34)}$$

On writing this in terms of sum and difference variables, (A34) becomes

$$\begin{aligned} & \frac{\partial}{\partial t_1} F[t_1 - t_2; \tfrac{1}{2}(t_1 + t_2)] + \frac{\partial}{\partial t_2} F[t_1 - t_2; \tfrac{1}{2}(t_1 + t_2)] \\ & = \frac{\partial F}{\partial \tau} (\tau, t) + \frac{1}{2} \frac{\partial F}{\partial t} (\tau, t) \\ & \quad - \frac{\partial F}{\partial \tau} (\tau, t) + \frac{1}{2} \frac{\partial F}{\partial t} (\tau, t) = \frac{\partial F}{\partial t} (\tau, t) , \end{aligned} \qquad \text{(A35)}$$

which on Fourier transformation becomes just the first term of (A33). We also mention without proof that the second term in (A33) is also exact for the case $\epsilon_p \propto p^2$, but not in general.

APPENDIX B: LIFETIMES FOR CONDUCTION ELECTRONS

In this Appendix we shall derive the various lifetimes for the conduction electrons given in Table II, namely, $\gamma_0(p_0), \gamma_H(p_0), \gamma_{M_d}(p_0), \sigma^{<}_{M_d}(p_0)$, and $\sigma^{<}_{M_s} \times (p_0)$. The approximations discussed in Sec. II apply.

To begin we reproduce the Eq.(2. 20) for $\sigma^{\gtrless}(p, p_0)$:

$$\sigma^{>}(\vec{p}, p_0) = \left(\frac{J}{n}\right)^2 \mathrm{Tr}_S \sum_{k, k_0} \sum_{q, q_0} \vec{s}\cdot\vec{S} g^{>}(\vec{p}+\vec{k}, p_0+k_0) \times G^{>}(\vec{q}, q_0)\vec{s}\cdot\vec{S} G^{<}(\vec{q}+\vec{k}, q_0+k_0) \tag{2. 20a}$$

and

$$\sigma^{<}(\vec{p}, p_0) = \left(\frac{J}{n}\right)^2 \mathrm{Tr}_S \sum_{k, k_0} \sum_{q, q_0} \vec{s}\cdot\vec{S} g^{<}(\vec{p}+\vec{k}, p_0+k_0) \times G^{<}(\vec{q}, q_0)\vec{s}\cdot\vec{S} G^{>}(\vec{q}+\vec{k}, q_0+k_0). \tag{2. 20b}$$

A systematic utilization of the decompositions for $G^{\gtrless}$ $(g^{\gtrless})$ listed in Table I and discussed in Sec. III leads to the desired results. For convenience we also list the explicit decomposition for $g^{\gtrless}$:

$$g^{<}(\vec{p}, p_0) = 2\pi\delta(p_0 - \epsilon_p + \gamma\vec{s}\cdot\vec{H}) f(p_0 + \vec{s}\cdot\vec{H}) + \vec{m}(\vec{p}, p_0)\cdot\vec{s}, \tag{B1a}$$

$$g^{>}(\vec{p}, p_0) = 2\pi\delta(p_0 - \epsilon_p + \gamma\vec{s}\cdot\vec{H}) \times [1 - f(p_0 + \gamma\vec{s}\cdot\vec{H})] - \vec{m}(\vec{p}, p_0)\cdot\vec{s}, \tag{B1b}$$

and

$$\vec{M}_S = \gamma(2s+1)\,\tfrac{1}{3}s(s+1) \sum_{p, p_0} \vec{m}(\vec{p}, p_0). \tag{B2}$$

The calculation proceeds by explicit expansion to first order in $\vec{H}, \vec{M}_s$ or $\vec{M}_d$.

First, we wish to show that $\gamma(\vec{p}, p_0) = \sigma^{>}(\vec{p}, p_0) + \sigma^{<}(\vec{p}, p_0)$ is independent of $\vec{M}_s$. To do this we may set $\vec{H}$ and $\vec{M}_d$ to zero and observe that the remaining parts of $G^{>}$ and $G^{<}$ are δ functions. Hence, since in both (2. 20a) and (2. 20b) above there is now a common factor of $(2\pi)^2\,\delta(q_0)\,\delta(q_0+k_0)\,F$, we may add the two equations, utilizing the relation $g^{>} + g^{<} = 2\pi\delta(p_0 + k_0 - \epsilon_{\vec{p}+\vec{k}})$ to give

$$\gamma(\vec{p}, p_0) = \left(\frac{J}{n}\right)^2 \mathrm{Tr}_S (\vec{s}\cdot\vec{S})^2 \sum_{k, k_0} \sum_{q, q_0} (2\pi)^3 \times \delta(p_0 + k_0 - \epsilon_{\vec{p}+\vec{k}})\,\delta(q_0)\,\delta(q_0+k_0)\,F(0),$$

which is clearly independent of $\vec{M}_s$. Hence

$$\gamma_{M_s}(\vec{p}, p_0) = 0. \tag{B3}$$

The calculation may now be continued to give the $\vec{H}$- and $\vec{M}_d$-independent part of $\gamma(p, p_0)$—viz., $\gamma_0(p, p_0)$. Since $\sum_{q_0} 2\pi\delta(q_0) = 1$ and $(2S+1)\sum_p F(0) = N$, we have, performing the spin sum

$$\gamma_0(p_0) = 2\pi N(J/n)^2\,\tfrac{1}{3}S(S+1)\,s(s+1)\textstyle\sum_k \delta(p_0 - \epsilon_k). \tag{B4}$$

In writing (B4) we have changed the index of the energy from $p+k$ to k under the k sum. Hence in this model γ_0 is independent of the momentum p which variable has hence been removed. By similar transformations we can show that all the lifetimes are independent of momentum.

The calculation of the field dependent part of the γ—namely, $\gamma_H(p_0)$—is considerably more complicated. We shall expand successively the first, second, and third Green's functions in $\sigma^{>}$ and combine it with a similar expansion of $\sigma^{<}$ to find a piece of γ_H which is proportional to $(\partial/\partial p_0)\delta(p_0 - \epsilon_{\vec{p}+\vec{k}})$. The expansion of $g^{>}$ and $g^{<}$ in $\sigma^{>}$ and $\sigma^{<}$, respectively, is clearly of that form:

$$\left(\frac{J}{n}\right)^2 \mathrm{Tr}_S \vec{s}\cdot\vec{S}\gamma\vec{s}\cdot\vec{H}\vec{s}\cdot\vec{S} \times \sum_{k, k_0} \sum_{q, q_0} (2\pi)^2\,\delta(q_0)\,\delta(q_0+k_0)\,F(0) \times \frac{\partial}{\partial p_0}\left(\delta(p_0+k_0-\epsilon_k)\{[1 - f(p_0+k_0)] + f(p_0+k_0)\}\right). \tag{B5}$$

The pieces arising from $G^{>}$ and $G^{<}$ in $\sigma^{>}$ and $\sigma^{<}$, respectively, can be written

$$\left(\frac{J}{n}\right)^2 \mathrm{Tr}_S \vec{s}\cdot\vec{S}\gamma\vec{s}\cdot\vec{H}\vec{s}\cdot\vec{S} \sum_{k, k_0} \sum_{q, q_0} (2\pi)^3 \times \Big(\delta(p_0+k_0-\epsilon_k)[1 - f(p_0+k_0)]\,\delta(q_0+k_0) \times F(q_0+k_0)\frac{\partial}{\partial q_0}[\delta(q_0)] + \delta(p_0+k_0-\epsilon_k) f(p_0+k_0)\,\delta(q_0+k_0)\frac{\partial}{\partial q_0}[\delta(q_0 F(q_0))]\Big). \tag{B6}$$

First, we integrate by parts and write the resulting derivatives as $-(\partial/\partial k_0)[\delta(q_0+k_0)\,F(q_0+k_0)]$. Then a second integration by parts on the derivative with respect to k_0 allows the term inside the sum of (B6) to be written

$$\frac{\partial}{\partial p_0}\delta(p_0+k_0-\epsilon_k)\,\delta(q_0)\,\delta(q_0+k_0)\,F(0). \tag{B7}$$

Finally, the pieces arising from $G^{<}$ and $G^{>}$ in $\sigma^{>}$ and $\sigma^{<}$, respectively, lead to

$$\left(\frac{J}{n}\right)^2 \mathrm{Tr}_S (\vec{s}\cdot\vec{S})^2\gamma\vec{s}\cdot\vec{H} \sum_{k, k_0} \sum_{q, q_0} (2\pi)^3 \times \Big(\delta(p_0+k_0-\epsilon_k)[1 - f(p_0+k_0)]\,\delta(q_0)$$

$$\times \frac{\partial}{\partial k_0}\left[\delta(q_0+k_0)F(q_0+k_0)\right]$$

$$\left. + \delta(p_0+k_0-\epsilon_k)f(p_0+k_0)\,\delta(q_0)\,F(q_0)\frac{\partial}{\partial k_0}\delta(q_0+k_0)\right) . \tag{B8}$$

In this case clearly integration by parts yields the negative of (B7). Adding the relevant parts of (B5)–(B8) we have, doing the integrals over q, q_0, and k_0,

$$\gamma_H(p,p_0) = 2\pi\left(\frac{J}{n}\right)^2 \sum_k \frac{\partial}{\partial p_0}\delta(p_0-\epsilon_k)$$

$$\times\left[\frac{N}{2S+1}\right]\mathrm{Tr}_S\,\{\vec{s}\cdot\vec{S}\,\gamma\vec{s}\cdot\vec{H}\vec{s}\cdot\vec{S}+\vec{s}\cdot\vec{S}$$

$$\times[\gamma\vec{s}\cdot\vec{H},\ \vec{s}\cdot\vec{S}]\} . \tag{B9}$$

The formulas in Table IV yield for the spin trace $(2S+1)\,\frac{1}{3}\,S(S+1)\,s(s+1)\,\gamma\vec{s}\cdot\vec{H}$ and hence

$$\gamma_H(p_0) = \gamma\vec{s}\cdot\vec{H}\,\frac{\partial}{\partial p_0}\,\gamma_0(p_0) . \tag{B10}$$

The first order in magnetic field we may write that

$$\gamma_0(p_0)+\gamma_H(p_0) = \gamma_0(p_0+\gamma\vec{s}\cdot\vec{H}). \tag{B11}$$

It is interesting to speculate that this result may be true to higher order in J/n, but we know of no proof of this. Equally fascinating is the fact that the lifetime for the local spin has a similar property (see Appendix C).

The calculation of the $\vec{M}_d$-dependent part of γ—namely, $\gamma_{M_d}(p_0)$—ultilizes for the first time in this Appendix the nonequilibrium part of the decomposition of $G^{\gtrless}$, that part of $g^{\gtrless}$ having already been shown to give no contribution to $\gamma(p_0)$. Clearly only the nonequilibrium part of $G^<$ is required, since the piece of $G^>$ would be multiplied by the $F(0)$ from $G^<$ and hence give a contribution proportional to N. The N-independent part of $\gamma_{M_d}(p_0)$ is given by

$$\gamma_{M_d}(p_0) = \left(\frac{J}{n}\right)^2 \mathrm{Tr}_S \sum_{k,k_0}\sum_{q,q_0}(2\pi)^2$$

$$\times\{\delta(p_0+k_0-\epsilon_{\vec{p}+\vec{k}})[1-f(p_0+k_0)]$$

$$\times\ \delta(q_0)\,(\vec{s}\cdot\vec{S})^2\vec{M}(\vec{q}+\vec{k},q_0+k_0)\cdot\vec{S}$$

$$+\delta(p_0+k_0-\epsilon_{\vec{p}+\vec{k}})f(p_0+k_0)$$

$$\times\ \delta(q_0+k_0)\vec{s}\circ\vec{S}\,\vec{M}(\vec{q},q_0)\cdot\vec{S}\,\vec{s}\cdot\vec{S}\} . \tag{B12}$$

At this stage we introduce a variable transformation which we shall use repeatedly

$$(\vec{q},q_0)\rightarrow(\vec{q}-\vec{k},q_0-k_0)$$

$$\text{and} \tag{B13}$$

$$(\vec{k},k_0)\rightarrow(-\vec{k},-k_0).$$

This transformation carried out inside the sum over all the relevant variable allows the first term of (B12) inside the sum to be written

$$\delta(p_0-k_0-\epsilon_{\vec{p}-\vec{k}})[1-f(p_0+k_0)]\,\delta(q_0+k_0)$$

$$\times(\vec{s}\cdot\vec{S})^2\vec{M}(\vec{q},q_0)\cdot\vec{S}.$$

In this case the switch $\vec{k}\rightarrow-\vec{k}$ is superfluous and may be reversed. Then the two terms of (B12) may be combined, the integral over k_0 performed, and the spin sums executed to yield

$$\gamma_{M_d}(p_0) = 2\pi\left(\frac{J}{n}\right)^2\left[(2S+1)\,\tfrac{1}{3}\,S(S+1)\sum_{q,q_0}\mathrm{M}(\vec{q},q_0)\cdot\vec{s}\right]$$

$$\times\textstyle\sum_k\{-\frac{1}{2}\delta(p_0+q_0-\epsilon_k)[1-f(\epsilon_k)]$$

$$+\tfrac{1}{2}\delta(p_0-q_0-\epsilon_k)f(\epsilon_k)\} . \tag{B14}$$

We observe that the sum over k yields, for a constant density of state,

$$\rho\{\tfrac{1}{2}[f(p_0-q_0)+f(p_0+q_0)]-\tfrac{1}{2}\} . \tag{B15}$$

As we have seen or will see (depending on the orders in which the paper is being read) in Sec. VI of the LHS of GKB $\vec{M}(\vec{q},q_0)$ is proportional to $\delta(q_0-E_q)\rightarrow\delta(q_0)$. Then (B15) is independent of q_0 and the curly brackets in (B14) are equal to $\vec{M}_d\cdot\vec{s}/\gamma$. Accordingly, an instructive form of (B14) is

$$\gamma_{M_d}(p_0) = 2\pi(J/n)^2(\rho/\gamma^2)\gamma\vec{s}\cdot\vec{M}_d[f(p_0)-\tfrac{1}{2}] . \tag{B16}$$

The appropriate self-energy is given by the equation

$$\sigma_{M_d}(p_0) = \int\frac{d\omega}{2\pi}\,\frac{\gamma_{M_d}(\omega)}{p_0-\omega} . \tag{B17}$$

Since we have already assumed a constant density

TABLE IV. Spin commutators and traces.[a]

$\vec{S}\times\vec{S}=i\,\vec{S}$
$[\vec{A}\cdot\vec{S},\ \vec{B}\cdot\vec{S}]=i\,\vec{S}\cdot(\vec{A}\times\vec{B})$[b]
$[\vec{S},\ \vec{S}]=-i\,\vec{S}\times\overleftrightarrow{1}$[c]
$\mathrm{Tr}_S\,\vec{A}\cdot\vec{S}\,\vec{B}\cdot\vec{S}=(2S+1)\,\frac{1}{3}\,S(S+1)\,\vec{A}\cdot\vec{B}$
$\mathrm{Tr}_S\,\vec{s}\cdot\vec{S}\,\vec{C}\cdot\vec{S}\,\vec{s}\cdot\vec{S}=\frac{1}{2}(2S+1)\,\frac{1}{3}\,S(S+1)\,\vec{C}\cdot\vec{s}$[d]
$\mathrm{Tr}_S\,\vec{s}\cdot\vec{S}\,\vec{s}\cdot\vec{S}\,\vec{C}\cdot\vec{S}=-\frac{1}{2}(2S+1)\,\frac{1}{3}\,S(S+1)\,\vec{C}\cdot\vec{s}$
$\mathrm{Tr}_S\,\vec{s}\cdot\vec{S}\,\vec{s}\cdot\vec{C}\,\vec{s}\cdot\vec{S}=(2S+1)\,\frac{1}{3}\,S(S+1)\,[s(s+1)-1]\,\vec{C}\cdot\vec{s}$

[a] Similar formulas hold if $\vec{S}$ is replaced by $\vec{s}$, or, where appropriate, interchanged with $\vec{s}$.

[b] The vectors $\vec{A}$ and $\vec{B}$ are assumed to commute with $\vec{S}$, but not necessarily with each other.

[c] $\overleftrightarrow{1}$ is a unit diagonal tensor.

[d] The vector $\vec{C}$ is assumed to commute with $\vec{s}$ and $\vec{S}$.

of states the only consistent approximation for that part of the integral in (B17) involving the assumed constant factor of $\frac{1}{2}\rho$ in (B16) is to take it to be zero. We shall do this repeatedly and when we do the tag phrase "as always" shall alert the reader to what is going on. For reference we write the following "identity" (by assumption):

$$\int d\omega \, \frac{\text{const}}{\omega - a} = 0. \tag{B18}$$

Now we consider the integral involving $f(p_0)$. Again for reference we write

$$\int d\omega \, \rho(\omega) \frac{f(\omega)}{\omega - p_0} \simeq \rho \int_D^{-k_B T} \frac{d\omega}{\omega - p_0} \simeq \rho \ln \frac{k_B T + p_0}{D} , \tag{B19}$$

where D is an energy the order of the bandwidth. Another form which arises is

$$\int d\omega \, \rho(\omega) \, \frac{1 - f(\omega)}{\omega - p_0}$$

$$\simeq k_B T \rho \int_{k_B T}^{D} \frac{d\omega}{\omega - p_0} \simeq \rho \ln \frac{k_B T - p_0}{D} . \tag{B20}$$

Using (B19) in (B17) we have

$$\sigma_{M_d}(p_0) = -\left[(J/n)^2 (\rho/\gamma^2) \ln(k_B T - p_0)/D\right] \gamma \vec{s} \cdot \vec{M}_d . \tag{B21}$$

This leads us to introduce the symbol

$$\alpha_2 = (J/n)^2 \, (\rho/\gamma^2) \, \ln k_B T/D < 0. \tag{B22}$$

Thus the second-order logarithmic contribution to $\sigma_{M_d}(p, p_0)$ at the Fermi surface is

$$\sigma_{M_d}^{(2)}(0) = -\alpha_2 \gamma \vec{s} \cdot \vec{M}_d . \tag{B23}$$

This completes the calculation of the various contributions to γ and we turn to $\sigma^{<}(p_0)$.

The calculation of $\sigma_{M_d}^{<}(p_0)$ is trivial since the N-independent part comes from the substitution of $G^{>}(\vec{q}+\vec{k}, q_0 + k_0) = \delta(q_0 + k_0)$, $G^{<}(\vec{q}, q_0) = \vec{M}(\vec{q}, q_0) \cdot \vec{S}$, and $g^{<}(\vec{p}+\vec{k}, p_0 + k_0) = \delta(p_0 + k_0 - \epsilon_{p+k}) f(p_0 + k_0)$ and it is given by

$$\sigma_{M_d}^{<}(p_0) = 2\pi \left(\frac{J}{n}\right)^2 \left[(2S+1) \, \tfrac{1}{3} S(S+1) \sum_{q, q_0} \vec{M}(\vec{q}, q_0) \cdot \vec{s}\right]$$

$$\times \textstyle\sum_k \tfrac{1}{2} \, \delta(p_0 - q_0 - \epsilon_k) f(p_0 - q_0) . \tag{B24}$$

Upon assuming $\vec{M}(\vec{q}, q_0) \propto \delta(q_0)$, we get

$$\sigma_{M_d}^{<}(p_0) = \left[2\pi (J/n)^2 \, (\rho/\gamma^2) \, \tfrac{1}{2} f(p_0)\right] \gamma \vec{s} \cdot \vec{M}_d. \tag{B25}$$

We turn to $\sigma_{M_s}^{<}(p_0)$. Since $G^{<}(\vec{q}, q_0) \, G^{>}(\vec{k}+\vec{q}, k_0 + q_0) = (2\pi)^2 F(0) \, \delta(q_0) \, \delta(k_0)$ and $g^{<}(\vec{p}+\vec{k}, p_0 + k_0) = \vec{m}(\vec{p}+\vec{k}, p_0 + k_0) \cdot \vec{s}$, the calculation of the spin sums quickly yields

$$\sigma_{M_s}^{<}(p_0) = (J/n)^2 N \, \tfrac{1}{3} S(S+1) \left[s(s+1) - 1\right]$$

$$\times \textstyle\sum_k \vec{m}(\vec{p}+\vec{k}, p_0) \cdot \vec{s}. \tag{B26}$$

Finally we note that

$$\sigma_0^{<}(p_0) = 2\pi N (J/n)^2 \, \tfrac{1}{3} S(S+1) \, s(s+1)$$

$$\times \textstyle\sum_k \delta(p_0 - \epsilon_k) f(p_0) \tag{B27}$$

or hence that

$$\sigma_0^{<}(p_0) = \gamma_0(p_0) f(p_0) , \tag{B28}$$

which is a general equilibrium property. Furthermore if we turned on a static magnetic field, it follows that, as an explicit calculation would show,

$$\sigma_0^{<}(p_0 + \gamma \vec{s} \cdot \vec{H}) = \gamma_0(p_0 + \gamma \vec{s} \cdot \vec{H}) f(p_0 + \gamma \vec{s} \cdot \vec{H}) . \tag{B29}$$

One further point requires discussion. When the equations for $\sigma^{>}(\vec{p}, p_0)$ and $\sigma^{<}(\vec{p}, p_0)$ were derived in Sec. II C, certain subtleties involved in taking the Fourier transform were neglected. In Appendix A, the identity for the Fourier transform of two Green's functions was derived (A31). A similar equation for the Fourier transform of the product of three Green's functions can be routinely derived and we list just the temporal part of it below for the specific case of $\sigma^{>}$:

$$\left[(1/i) \, g^{>}(t_1, t_2) \, G^{>}(t_1, t_2) \, G^{<}(t_2, t_1)\right]_{\mathrm{Ft}}$$

$$= \sum_{k_0, q_0} \left(\frac{\partial g^{>}}{\partial k_0} \, (p_0 + k_0) \, \frac{\partial G^{>}(q_0)}{\partial t} - \frac{\partial g^{>}(p_0 + k_0)}{\partial t} \, \frac{\partial G^{>}(q_0)}{\partial q_0} \right) G^{<}(k_0 + q_0)$$

plus similar terms involving derivatives on $g^{>} G^{<}$ and on $G^{>} G^{<}$ plus higher-order derivatives. But it is easy to see that each of the new terms has one piece like $\partial G^{>}(p_0)/\partial t$ which is of second order in small quantities—in this case, $\propto \Omega H$. When the self-energies are combined in calculating any part of GKB the resulting term will be of third-order in small quantities and hence by the approximation discussed in Sec. II can be neglected. Accordingly only the first term of the Fourier transform of any self-energy or lifetime expression needs to be retained.

APPENDIX C: LIFETIME FOR LOCAL SPINS

In this Appendix we shall calculate the various lifetimes for the local spins given in Table III, namely, $\Gamma_0(p_0)$, $\Gamma_H(p_0)$, $\Gamma_{M_s}(p_0)$, $\Sigma_{M_s}^{<}(p_0)$ and $\Sigma_{M_d}^{<}(p_0)$. Much of the work is similar to that in Appendix B.

To begin we reproduce Eq. (2.21) for $\Sigma^{\gtrless}(\vec{p}, p_0)$:

$$\Sigma^{>}(\vec{p}, p_0) = \left(\frac{J}{n}\right)^2 \mathrm{Tr}_s \sum_{k, k_0} \sum_{q, q_0} \vec{s} \cdot \vec{S} \, G^{>}(\vec{p}+\vec{k}, p_0 + k_0)$$

$$\times g^{>}(\vec{q}, q_0)\,\vec{s}\cdot\vec{S}\,g^{<}(\vec{q}+\vec{k}, q_0+k_0)$$

and

$$\Sigma^{<}(\vec{p}, p_0)=\left(\frac{J}{n}\right)^2 \mathrm{Tr}_s \sum_{k,k_0}\sum_{q,q_0} \vec{s}\cdot\vec{S}\,G^{<}(\vec{p}+\vec{k}, p_0+k_0)$$

$$\times g^{<}(\vec{q}, q_0)\,\vec{s}\cdot\vec{S}\,g^{>}(\vec{q}+\vec{k}, q_0+k_0).$$

As in Appendix B the calculation proceeds by an explicit expansion to first order in $\vec{H}$, $\vec{M}_s$, or $\vec{M}_d$ of the decomposition of $G^{\gtrless}$ listed in Table I and of $g^{\gtrless}$ given by (B1a), (B1b), and (B2).

First, we wish to show that $\Gamma(p, p_0)=\Sigma^{>}(p, p_0)+\Sigma^{<}(p, p_0)$ is independent of $\vec{M}_d$. To do this we set $\vec{H}$ and $\vec{M}_s$ (i.e., $\Delta g^{<}$) to zero. Applying the variable transformation (B13) to $\Sigma^{<}$ we see that its $g^{<}g^{>}$ is equal to $g^{>}g^{<}$ in $\Sigma^{>}$. Adding $\Sigma^{>}$ and $\Sigma^{<}$ we have

$$\Gamma_{M_s}(\vec{p}, p_0)=J^2\,\mathrm{Tr}_s \sum_{k,k_0}\sum_{q,q_0} \vec{s}\cdot\vec{S}$$

$$\times\left[-\vec{M}(\vec{p}+\vec{k}, p_0+k_0)\cdot\vec{S}+\vec{M}(\vec{p}-\vec{k}, p_0-k_0)\cdot\vec{S}\right]$$

$$\times\vec{s}\cdot\vec{S}\,(2\pi)^2\,\delta(q_0-\epsilon_q)\,[1-f(q_0)]$$

$$\times\delta(q_0+k_0-\epsilon_{\vec{k}+\vec{q}})\,f(q_0+k_0)\,. \qquad \text{(C1)}$$

Since, as shown in Sec. VIII, $\vec{M}(q, q_0)\propto\delta(q_0)$, the first square bracket in (C1) is identically zero and we have the result

$$\Gamma_{M_d}(p, p_0)=0\,. \qquad \text{(C2)}$$

For all other terms we see that $\Sigma^{<}$ is of order N and hence the N-independent part of Γ is given by $\Sigma^{>}$. Setting $\vec{M}_s$ and $\vec{H}$ to zero, we have that

$$\Gamma_0(p, p_0)=\left(\frac{J}{n}\right)^2 \mathrm{Tr}_s\,(\vec{s}\cdot\vec{S})^2 \sum_{k,k_0}\sum_{q,q_0}(2\pi)^3\,\delta(p_0+k_0)$$

$$\times\delta(p_0-\epsilon_q)\,[1-f(q_0)]\,\delta(k_0+q_0-\epsilon_{k+q})\,f(q_0+k_0)\,. \qquad \text{(C3)}$$

The integrals over q_0 and k_0 and the spin sum are easily performed to give

$$\Gamma_0(p_0)=2\pi(J/n)^2(2s+1)\,\tfrac{1}{3}s(s+1)\,S(S+1)$$

$$\times\sum_{k,q}[1-f(\epsilon_q)]f(\epsilon_k)\,\delta(p_0+\epsilon_k-\epsilon_q)\,, \qquad \text{(C4)}$$

where the variable transformation $\vec{k}+\vec{q}$ to $\vec{k}$ has occurred. The momentum dependence Γ_0 has been eliminated since there is none—a result consistent with the infinite-mass limit and true for all Γ and $\Sigma^{<}$. For the record the sum over k and q for a constant density of state is

$$\rho^2\int d\epsilon\,d\epsilon'[1-f(\epsilon')]f(\epsilon)$$

$$\times\delta(p_0+\epsilon-\epsilon')=\rho^2 p_0/(1-e^{-\beta p_0}), \qquad \text{(C5)}$$

which equals kT at $p_0=0$.

To calculate the field-dependent part of the lifetime, $\Gamma_H(p_0)$, we expand each Green's function in $\Sigma^{>}$ to linear order in $\vec{H}$. The resulting expression

$$\Gamma_H(p_0)=\left(\frac{J}{n}\right)^2(2\pi)^3\,\mathrm{Tr}_s$$

$$\times\sum_{k,k_0}\sum_{q,q_0}\vec{s}\cdot\vec{S}\,\gamma\vec{S}\cdot\vec{H}\,\vec{s}\cdot\vec{S}\,\frac{\partial}{\partial p_0}\,\delta(p_0+k_0)$$

$$\times\,\delta(q_0-\epsilon_q)\,\delta(q_0+k_0-\epsilon_{\vec{k},\vec{q}})[1-f(q_0)]f(k_0+q_0)$$

$$+\,\vec{s}\cdot\vec{S}\,\gamma\vec{s}\cdot\vec{H}\,\vec{s}\cdot\vec{S}\,\delta(p_0+k_0)$$

$$\times\,\frac{\partial}{\partial q_0}\{\delta(q_0-\epsilon_q)[1-f(q_0)]\}$$

$$\times\delta(q_0+k_0-\epsilon_{\vec{k}+\vec{q}})f(k_0+q_0)$$

$$+\,(\vec{s}\cdot\vec{S})^2\gamma\vec{s}\cdot\vec{H}\,\delta(p_0+k_0)\delta(q_0-\epsilon_q)[1-f(q_0)]$$

$$\times\,\frac{\partial}{\partial k_0}[\delta(q_0+k_0-\epsilon_{\vec{k}+\vec{q}})f(q_0+k_0)] \qquad \text{(C6)}$$

has three terms each of which is proportional to $(\partial/\partial p_0)\,\delta(p_0+k_0)$. The first is obviously so. The second requires two integration by parts, first to $(\partial/\partial k_0)[\delta(q_0+k_0-\epsilon_{\vec{k}+\vec{q}})\,f(q_0+k_0)]$ and then to the desired form. The third requires only the second integration by parts and hence has a minus sign. Combining the three terms we have, performing the integrals over q_0 and k_0,

$$\Gamma_H(p_0)=2\pi\left(\frac{J}{n}\right)^2\sum_{k,q}[1-f(\epsilon_q)]f(\epsilon_{\vec{k}+\vec{q}})$$

$$\times\,\frac{\partial}{\partial p_0}\,\delta(p_0+\epsilon_{\vec{k}+\vec{q}}-\epsilon_q)$$

$$\times\mathrm{Tr}_s\{\vec{s}\cdot\vec{S}\,\gamma\vec{S}\cdot\vec{H}\,\vec{s}\cdot\vec{S}+\vec{s}\cdot\vec{S}[\gamma\vec{s}\cdot\vec{H},\,\vec{s}\cdot\vec{S}]\}\;. \qquad \text{(C7)}$$

The spin trace gives $(2s+1)\,\frac{1}{3}\,s(s+1)\,S(S+1)\,\gamma\vec{S}\cdot\vec{H}$ and hence we see that

$$\Gamma_H(p_0)=\gamma\vec{S}\cdot\vec{H}\,\frac{\partial}{\partial p_0}\,\Gamma_0(p_0). \qquad \text{(C8)}$$

To first order in the magnetic field we observe that

$$\Gamma_0(p_0)+\Gamma_H(p_0)=\Gamma_0(p_0+\gamma\vec{S}\cdot\vec{H}). \qquad \text{(C9)}$$

The M_s-dependent part of Γ is calculated by inserting the nonequilibrium parts of $g^{\gtrless}$ from (B1):

$$\Gamma_{M_s}(p_0)=\left(\frac{J}{n}\right)^2\mathrm{Tr}_s\sum_{k,k_0}\sum_{q,q_0}(2\pi)^2\,\delta(p_0+k_0)$$

$$\times\{-\vec{s}\cdot\vec{S}\,\vec{m}(\vec{q}, q_0)\cdot\vec{s}\,\vec{s}\cdot\vec{S}\,\delta(q_0+k_0-\epsilon_{\vec{k}+\vec{q}})f(q_0+k_0)$$

$$+ (\vec{s}\cdot\vec{S})^2\, \vec{m}(\vec{q}+\vec{k}, q_0+k_0)\cdot\vec{s}\;\delta(q_0-\epsilon_q)[1-f(q_0)]\} \,. \tag{C10}$$

The variable transformation (B13) is applied to the second term of (C10), the spin sum performed to yield

$$\Gamma_{M_s}(p_0) = -2\pi\,(J/n)^2\{(2s+1)\tfrac{1}{3}s(s+1) \times \sum_{q,q_0} \vec{m}(\vec{q},q_0)\cdot\vec{S}\} \times \sum_k \{\tfrac{1}{2}\delta(p_0-q_0+\epsilon_k)f(\epsilon_k) + \tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)[1-f(\epsilon_k)]\}. \tag{C11}$$

We observe that the sum over k, for a constant density of states is

$$\rho[\tfrac{1}{2}f(q_0-p_0) - \tfrac{1}{2}f(p_0+q_0) + \tfrac{1}{2}] \,. \tag{C12}$$

If the q_0 dependence of (C12) could be neglected, the first curly brackets in (C11) would by (B2) be equal to $\gamma\vec{S}\cdot\vec{M}_d/\gamma^2$. That this is essentially the case follows from Sec. VII where $\vec{m}(\vec{q},q_0)$ is shown to be proportional to $\delta(q_0-\epsilon_q)\,\partial f(q_0)/\partial q_0$. Hence (C10) is well approximated by

$$\Gamma_{M_s}(p_0) = +2\pi(J/n)^2(\rho/\gamma^2)[f(p_0)-1]\,\gamma\vec{S}\cdot\vec{M}_d \,. \tag{C13}$$

As in Appendix B we can calculate $\Sigma_{M_s}(p_0)$. In particular we see that

$$\Sigma_{M_s}(0) = -\alpha_2\gamma\vec{S}\cdot\vec{M}_d \,, \tag{C14}$$

which is analogous to the result (B23).

The calculations for $\Sigma^<_{M_s}$ and $\Sigma^<_{M_s}$ proceed routinely. Upon substitution

$$\Sigma^<_{M_s}(p_0) = \left(\frac{J}{n}\right)^2 \mathrm{Tr}_s \sum_{k,k_0}\sum_{q,q_0} (2\pi)^2\,\delta(p_0+k_0)F(0) \times\{\vec{s}\cdot\vec{S}\;\vec{m}\,(q,q_0)\cdot\vec{s}\;\vec{s}\cdot\vec{S}\,\delta(q_0+k_0-\epsilon_{k+q})[1-f(q_0+k_0)] - (\vec{s}\cdot\vec{S})^2\,\vec{m}(q+k,q_0+k_0)\cdot\vec{s}\;\delta(q_0-\epsilon_q)f(q_0)\}, \tag{C15}$$

which differs from (C10) by a change in signs, the interchange of f and $1-f$, and the presence of $F(0)$. Hence $\Sigma^<_{M_s}$ is proportional to the number of local spins while Γ_{M_s} is not. The evaluation of (C15) proceeds routinely to give

$$\Sigma^<_{M_s}(p_0) = 2\pi[N/(2S+1)](J/n)^2\{(2S+1)\tfrac{1}{3}s(s+1) \times \sum_{q,q_0}\vec{m}(\vec{q},q_0)\cdot\vec{S}\} \times \sum\nolimits_k\{\tfrac{1}{2}\delta(p_0-q_0+\epsilon_k)[1-f(\epsilon_k)] + \tfrac{1}{2}\delta(p_0+q_0-\epsilon_k)f(\epsilon_k)\}. \tag{C16}$$

Under the same approximation that leads to (C13) we see that (C16) reduces to

$$\Sigma^<_{M_s}(p_0) \simeq [N/(2S+1)]\,2\pi(J/n)^2(\rho/\gamma^2)f(p_0)\,\gamma\vec{S}\cdot\vec{M}_s \,, \tag{C17}$$

that is, that

$$\Sigma^<_{M_s}(p_0) = -N\Gamma_{M_s}(-p_0)/(2S+1) \,, \tag{C18}$$

which follows *exactly* from (C11) and (C16).

Finally upon the obvious substitutions

$$\Sigma^<_{M_s}(p_0) = \left(\frac{J}{n}\right)^2 \times \sum_{k,k_0}\sum_{q,q_0} \mathrm{Tr}_s\,\vec{M}(\vec{p}+\vec{k},p_0+k_0)\cdot\vec{S}\,(\vec{s}\cdot\vec{S})^2 \times (2\pi)^2\,\delta(q_0-\epsilon_q)f(q_0)\,\delta(q_0+k_0-\epsilon_{\vec{k}+\vec{q}}) \times[1-f(k_0+q_0)], \tag{C19}$$

which reduces to

$$\Sigma^<_{M_d}(p_0) = 2\pi(J/n)^2\,(2s+1)\tfrac{1}{3}s(s+1)[S(S+1)-1] \times\sum_{k,k_0}\vec{M}(\vec{p}+\vec{k},p+k_0)\cdot\vec{S} \times\sum_q f(\epsilon_q)[1-f(\epsilon_{\vec{k}+\vec{q}})]\,\delta(k_0+\epsilon_q-\epsilon_{\vec{k}+\vec{q}}) \,. \tag{C20}$$

This equation may look slightly forbidding, but when one remembers that $M(\vec{q},q_0)$ has no momentum dependence in the infinite-mass limit we see that

$$\Sigma^<_{M_s}(p_0) = [S(S+1)-1] \times\sum_{k_0}\vec{M}(\vec{p},p_0+k_0)\cdot\vec{S}\,[\Gamma_0(k_0)/S(S+1)], \tag{C21}$$

a relation which plays an important role in calculating the right-hand side of GKB. We observe that, as it must in equilibrium,

$$\Sigma^<_0(p_0+\gamma\vec{S}\cdot H) = \Gamma_0(p_0+\gamma\vec{S}\cdot\vec{H})\,F(p_0+\gamma\vec{S}\cdot\vec{H}) \,. \tag{C22}$$

Finally we observe, as we saw at the end of Appendix C, that any terms beyond the first in the gradient expansion of the Fourier transform of Γ or $\Sigma^<$ may be neglected.

APPENDIX D

Throughout the reduction of the right-hand side of the GBK we use the decompositions

$$g^<(\vec{p},p_0) = a(\vec{p},p_0)f(p_0) + \delta g^<(\vec{p},p_0) \,, \tag{D1}$$

$$G^<(\vec{p},p_0) = A(\vec{p},p_0)F(p_0) + \delta G^<(\vec{p},p_0) \,, \tag{D2}$$

which define $\delta g^<$ and $\delta G^<$, respectively. As always the notation $a(\vec{p},p_0)$ [or $A(\vec{p},p_0)$] refers to the complete nonequilibrium spectral weight function; therefore a has pieces proportional to $\vec{s}\cdot\vec{H}$ and $\vec{s}\cdot\vec{M}_d$ and A has pieces proportional to $\vec{S}\cdot\vec{H}$ and $\vec{S}\cdot\vec{M}_s$. Since by definition

$$\vec{M}_s = \gamma\, \mathrm{Tr}_s \vec{s} \sum_{p,p_0} g^{<}(\vec{p}, p_0)\,, \tag{D3}$$

$$\vec{M}_d = \gamma\, \mathrm{Tr}_S \vec{S} \sum_{p,p_0} G^{<}(\vec{p}, p_0)\,, \tag{D4}$$

it is clear that $\delta g^{<}(\delta G^{<})$ is not proportional to $\vec{M}_s(\vec{M}_d)$ alone. Here we derive expressions for $\delta g^{<}$ and $\delta G^{<}$ analogous to (D3) and (D4).

Consider the conduction-electron case first. We wish to calculate

$$\mathrm{Tr}_s \gamma \vec{s} \sum_{p,p_0} \delta g^{<}(\vec{p}, p_0) = \vec{M}_s - \mathrm{Tr}_s \gamma \vec{s} \times \sum_{p,p_0} a(\vec{p}, p_0) f(p_0) \tag{D5}$$

to lowest order in $\vec{H}$, $\vec{M}_s$, and $\vec{M}_d$ and to order J^2. The spectral weight function a is given by

$$a(\vec{p}, p_0) = 2\, \mathrm{Im} \frac{1}{p_0 - i0^{+} - \epsilon_p + \gamma \vec{s} \cdot \vec{H} - \sigma(p_0) - \frac{1}{2} i \gamma(p_0)}\;; \tag{D6}$$

note that σ and γ have pieces proportional to $\vec{M}_d$, but that σ and γ are still related by the dispersion relation

$$\sigma(p_0) = -\alpha_1 \gamma \vec{s} \cdot \vec{M}_d + \int \frac{d\omega}{2\pi} \frac{\gamma(\omega)}{p_0 - \omega}\,. \tag{D7}$$

Specifically,

$$\gamma(p_0) = \gamma_0(p_0) + \gamma_H(p_0) + \gamma_{M_d}(p_0)$$
$$\simeq \gamma_0(p_0 + \gamma \vec{s} \cdot \vec{H}) + \gamma_{M_d}(p_0).$$

Therefore, to order J^2 and to linear fields and magnetizations, we may write (D6) as

$$a(\vec{p}, p_0) = a_0(p, p_0 + \gamma \vec{s} \cdot \vec{H}) + \mathrm{Im}\,(p_0 - \epsilon_p - i0^{+})^{-2} [\sigma_{M_d}(p_0) + \tfrac{1}{2} i \gamma_{M_d}(p_0)]\,, \tag{D8}$$

where $a_0(p, p_0 + \gamma \vec{s} \cdot \vec{H})$ is the equilibrium, zero field, value of the spectral weight function but with its energy value augmented by $\gamma \vec{s} \cdot \vec{H}$. To order $\vec{H}$,

$$\mathrm{Tr}_s \gamma \vec{s} \sum_{p,p_0} a_0(p, p_0 + \gamma \vec{s} \cdot \vec{H}) f(p_0)$$
$$= \mathrm{Tr}_s \gamma \vec{s}\;\; \gamma \vec{s} \cdot \vec{H} \sum_{p,p_0} a_0(p, p_0) - \left(\frac{\partial f(p_0)}{\partial p_0}\right) = \chi_{s0} H. \tag{D9}$$

The last equality in (D9) is obviously true only for $J = 0$; to prove it to order J^2, we wish to show that the additional term

$$\sum_{p,p_0} \left(-2\pi \delta'(p_0 - \epsilon_p)\, \sigma_0(p_0) + \frac{\gamma_0(p_0)}{(p_0 - \epsilon_p)^2} \right) f'(p_0) \tag{D10}$$

vanishes. As always we assume a constant density of states, which means that $\gamma_0(p_0)$ and $\sigma_0(p_0)$ are independent of p, a variable that we henceforth suppress for clarity. Since $\sum_p \rightarrow \int \rho\, d\epsilon_p$, it is clear that the second term in (D10) is proportional to $\int d\epsilon_p\, 1/(p_0 - \epsilon_p)^2$ and hence vanishes upon integration. The reason for this is that source of the integral is (D8). Hence

$$\int d\epsilon_p (p_0 - \epsilon_p)^{-2} = \mathrm{Re} \int d\epsilon_p (p_0 - \epsilon_p - i0^{+})^{-2} = 0 \tag{D11}$$

since the contour of integration never passes through the pole. The first term in (D10) becomes after integration over p_0

$$\int \rho\, d\epsilon_p [\sigma'(\epsilon_p) f'(\epsilon_p) + \sigma(\epsilon_p) f''(\epsilon_p)]$$
$$= \int \rho\, d\epsilon_p \frac{d}{d\epsilon_p} [\sigma(\epsilon_p) f'(\epsilon_p)] = 0\,.$$

Therefore the last equality of (D9) follows as stated.

To evaluate (D5) we also need the integral over the last terms in (D8), that is,

$$\gamma \mathrm{Tr}_s \vec{s} \sum_{p,p_0} \left(-2\pi \delta(p_0 - \epsilon_p)\, \sigma_{M_d}(p_0) + \frac{\gamma_{M_d}(p_0)}{(p_0 - \epsilon_p)^2} \right) f(p_0)\,. \tag{D12}$$

The same arguments used on (D10) apply here, so that (D12) becomes

$$\mathrm{Tr}_s \gamma \vec{s} \int \rho\, d\epsilon_p \frac{d}{d\epsilon_p} [\sigma_{M_d}(\epsilon_p) f(\epsilon_p)] = -\rho \sigma_{M_d}(-\infty)$$
$$= -\mathrm{Tr}_s \gamma \vec{s}\, \rho \sigma_{M_d}(-\infty) = \mathrm{Tr}_s \gamma \vec{s}\, \rho \alpha_1 \gamma \vec{s} \cdot \vec{M}_d$$
$$= \alpha_1 \chi_{s0} \vec{M}_d\,, \tag{D13}$$

where the third equality follows from (D7). Finally, therefore, we have to order J^2, by substitution of (D13) and (D9) back into (D5)

$$\mathrm{Tr}_s \gamma \vec{s} \sum_{p,p_0} \delta g^{<}(p, p_0) = \vec{M}_s - \chi_{s0} (\vec{H} + \alpha_1 \vec{M}_d). \tag{D14}$$

Note added in proof. The connection of this result and other of our results to the Anderson model are given following Eq. (D34).

Thus $\delta g^{<}$ expresses the deviation of the magnetization from an instantaneous local magnetization value given by (D14). It is also clear from its definition that $\delta g^{<}$ vanishes in thermal equilibrium, so that a static susceptibility calculation may be done by combining (D14) with its localized spin counterpart to be derived later.

It turns out that (D14) expression can be evaluated to all orders in J, provided that we continue our earlier assumption that the density of states is constant, so that the self-energies are independent of ϵ_p. To establish this result we expand (D5) to lowest order in the fields and magnetizations. Specifically we let $\gamma = \gamma_0 + \delta\gamma$, where $\delta\gamma = \gamma_{M_s} + \gamma_{M_d} + \gamma_H$

(this definition of $\delta\gamma$ and the $\delta\sigma$ is unique to this Appendix, and should not be confused with the definition used elsewhere, e.g., in Sec. VI) and similarly for σ. We include γ_{M_s} in the $\delta\gamma$ above to allow for the possibility that it does not vanish to all orders. Then the last term in (D5) becomes

$$-\sum_{p_0}\int \rho\, d\epsilon_p \operatorname{Tr}_s \gamma\vec{s} \times\frac{\partial}{\partial\epsilon_p}\, 2\,\operatorname{Im}\frac{-\gamma\vec{s}\cdot\vec{H}+\delta\sigma+\frac{1}{2}i\,\delta\gamma}{p_0-\epsilon_p-i0^{+}-\sigma_0(p_0)-\frac{1}{2}i\gamma_0(p_0)}\,f(p_0)\,, \tag{D15}$$

which equals

$$-\sum_{p_0}\rho\int d\epsilon_p \operatorname{Tr}_s\gamma\vec{s}\,\frac{\partial}{\partial\epsilon_p}\,\{a_0(\epsilon_p,p_0)[-\gamma\vec{s}\cdot\vec{H}+\delta\sigma(p_0)] +g_0(\epsilon_p,p_0)\,\delta\gamma(p_0)\}\,f(p_0)\,, \tag{D16}$$

where in a_0 and g_0 we have stressed by our notation the fact that the momentum dependence is solely dependent on ϵ_p. We take the ϵ_p integral over a finite interval $-\epsilon<\epsilon_p<\epsilon$ at first, and then take the limit as $\epsilon\to\infty$. Before taking this limit (D16) becomes

$$-\sum_{p_0}\rho\operatorname{Tr}_s\gamma\vec{s}\{[a_0(\epsilon,p_0)-a_0(-\epsilon,p_0)][-\gamma\vec{s}\cdot\vec{H}+\delta\sigma(p_0)] +[g_0(\epsilon,p_0)-g_0(-\epsilon,p_0)]\,\delta\gamma(p_0)\}\,f(p_0)\,. \tag{D17}$$

Since $a(\infty,p_0)=a(-\infty,p_0)=g(\infty,p_0)=g(-\infty,p_0)=0$ it might seem at first that (D17) vanishes in the implied limit. More careful inspection shows, however, that if a_0 or g_0 is multiplied by a constant or nonvanishing factor as $p_0\to\pm\infty$, then the integral over p_0 will be infinite, yielding an indeterminate form. In fact there are two terms like this: one is $-\gamma\vec{s}\cdot\vec{H}$ which is clearly independent of p_0, and the other is $\sigma^{(1)}=-\alpha_1\gamma\vec{s}\cdot\vec{M}_d$ which according to (D7) is that part of σ which does not vanish at $p_0\to-\infty$. Note that there is never any trouble at $p_0=+\infty$ because $f(p_0)$ vanishes there. Equation (D17), therefore, becomes

$$-\lim_{\epsilon\to\infty}\sum_{p_0}\rho\operatorname{Tr}_s\gamma\vec{s}\,a_0(-\epsilon,p_0)\,(\gamma\vec{s}\cdot\vec{H}+\alpha_1\gamma\vec{s}\cdot\vec{M}_d)\,f(p_0)\,. \tag{D18}$$

Since $a_0(-\epsilon,p_0)$ is, as a function of p_0, peaked in the vicinity of $p_0=-\epsilon$, it will suffice to put $f(p_0)\to f(-\infty)=1$, in the limit. Then the integral over p_0 may be carried out using the sum rule

$$\sum_{p_0}a_0(p,p_0)=1\,,$$

which follows from the definition of a_0. Therefore (D18) becomes

$$-\rho\operatorname{Tr}_s\gamma^2\vec{s}\,\vec{s}\cdot(\vec{H}+\alpha_1\vec{M}_d)=-\chi_{s0}(\vec{H}+\alpha_1\vec{M}_d). \tag{D19}$$

Finally, substitution in (D5) yields (D14) which is now shown to be true to all orders in J.

We now turn to evaluating the right-hand side of the local-spin-electron expression

$$\operatorname{Tr}_S\gamma\vec{S}\sum_{p,p_0}\delta G^{<}(\vec{p},p_0)=\vec{M}_d-\gamma\operatorname{Tr}_S\sum_{p,p_0}A(\vec{p},p_0)\,F(p_0)\,. \tag{D20}$$

Again $A(\vec{p},p_0)$ may be expanded as in (D8). The first term to be evaluated is

$$\operatorname{Tr}_S\gamma\vec{S}\sum A_0(p,p_0+\gamma\vec{S}\cdot\vec{H})\,F(p_0)\,, \tag{D21}$$

where $A_0(p,p_0+\gamma\vec{S}\cdot\vec{H})$ is the equilibrium, zero-field spectral weight function, but with its energy variable p_0 replaced by $p_0+\gamma\vec{S}\cdot\vec{H}$. To lowest order in the field (D21) equals

$$-\operatorname{Tr}_S\gamma\vec{S}\cdot\vec{H}\sum_{p,p_0}A(\vec{p},p_0)\,F'(p_0) =\beta\operatorname{Tr}_S\gamma\vec{S}\,\gamma\vec{S}\cdot\vec{H}\,[\sum_{p,p_0}A(\vec{p},p_0)\,F(p_0)]=\chi_{d0}\vec{H}. \tag{D22}$$

The bracketed quantity above is by definition the number of localized spins;

$$N=\operatorname{Tr}_S\sum_{p,p_0}A(p,p_0)\,F(p_0)\,, \tag{D23}$$

so that (D22) and hence (D21) is equal to the bare Curie susceptibility times H, as indicated in the last equality above. It is interesting to note that the chemical potential μ implied by (D23) is given by $\mu=\mu_0+\Sigma(0)$, where μ_0 is what the chemical potential would have been if $J=0$ (assuming the same number of local spins in either case). To prove this to order J^2, one expands (D23)

$$N\simeq\operatorname{Tr}_S\sum_{p,p_0}\left(2\pi\delta(p_0)-2\pi\delta'(p_0)\,\Sigma_0(p_0)+\frac{\gamma_0(p_0)}{p_0^2}\right)F(p_0) =\operatorname{Tr}_S\sum_{p,p_0}2\pi\delta(p_0)[F(0)+\Sigma_0(0)\,F'(0)] +\left(\Sigma_0'(0)+\sum_{p,p_0}\frac{\Gamma_0(p_0)}{p_0^2}\right)e^{-\beta\mu}\,. \tag{D24}$$

Now using (C4) and (C5) it is easy to see that $e^{-\beta p_0}\Gamma_0(p_0)=\Gamma_0(-p_0)$, so that the dispersion relation (2.6b) implies that

$$\Sigma_0'(0)=-\sum_{p,p_0}\frac{\Gamma_0(p_0)}{p_0^2}\,e^{-\beta p_0}\,. \tag{D25}$$

Therefore, the last bracketed term in (D24) vanishes and

$$N\simeq\sum_{p,p_0}2\pi\delta(p_0)\,F(\Sigma(0))\operatorname{Tr}_S\sum_{p}e^{-\beta[\Sigma(0)-\mu]}\,. \tag{D26}$$

Therefore, the chemical potential shift is precisely $\Sigma(0)$, as promised. One might also notice that the sum rule $\Sigma_p A(p,p_0)=1$ can be easily verified by using the same method to evaluate the right-hand side of (D23) but with $F(p_0)$ replaced by unity (and without summing over $\vec{p}$).

Returning to the evaluation of (D20), we note that as in the conduction-electron case we must also evaluate

$$\mathrm{Tr}_S\,\gamma\vec{S}\sum_{p,p_0}\left[-2\pi\delta'(p_0)\,\Sigma_{M_s}(p_0)+\Gamma_{M_s}(p_0)/p_0^2\right]F(p_0)$$

$$=\mathrm{Tr}_S\,\gamma\vec{S}\sum_{p,p_0}\left[-\Gamma_{M_s}(p_0)\,(1-e^{-\beta p_0})/p_0^2\right]F(0)$$

$$+\,\mathrm{Tr}_S\,\gamma\vec{S}\sum_{p}(-\beta)\,\Sigma_{M_s}(0)\,F(0)\,. \qquad (\mathrm{D}27)$$

To evaluate this we use (C11) and make the usual assumption that the only essential q_0 dependence of $\Sigma_q m_q(\vec{q}, q_0)$ is proportional to $f'(q_0)$, as discussed in Sec. VII and used in Appendix C. Then Γ_{M_s} becomes

$$\Gamma_{M_s}(p_0)=-2\pi(J/n)^2\,\vec{M}_s\cdot\vec{S}$$

$$\times\,\rho\int dq_0\int d\epsilon_k\left[-f'(q_0)\right]\left\{\tfrac{1}{2}\,\delta(p_0-q_0+\epsilon_k)f(\epsilon_k)\right.$$

$$\left.+\,\tfrac{1}{2}\,\delta(p_0+q_0-\epsilon_k)\left[1-f(\epsilon_k)\right]\right\}. \qquad (\mathrm{D}28)$$

The use of this form in the first term of (D27) yields

$$\mathrm{Tr}_S\,\gamma\vec{S}\sum_{p,p_0}-\Gamma(p_0)\,(1-e^{-\beta p_0})/p_0^2\,F(0)$$

$$=(\chi_{d0}M_s/\beta)\,(J/n)^2\,(\rho/\gamma^2)\int dq_0\int d\epsilon_k \qquad (\mathrm{D}29)$$

$$(q_0-\epsilon_k)^{-2}\left[-f'(q_0)\right]\left\{\tfrac{1}{2}(1-e^{-\beta(q_0-\epsilon_k)})\right.$$

$$\left.+\,\tfrac{1}{2}(1-e^{-\beta(\epsilon_k-q_0)})\left[1-f(\epsilon_k)\right]\right\}. \qquad (\mathrm{D}30)$$

Using the three facts: (i) $-f'(q_0)=\beta f(q_0)\left[1-f(q_0)\right]$; (ii) $e^{+\beta\omega}f(\omega)=1-f(\omega)$; and (iii) $e^{-\beta\omega}\left[1-f(\omega)\right]=f(\omega)$, (D30) yields

$$\chi_{d0}\vec{M}_s\left(\frac{J}{n}\right)^2\left(\frac{\rho}{\gamma^2}\right)$$

$$\times\frac{1}{2}\int dq_0\int d\epsilon_k\,\frac{\left[f(q_0)-f(\epsilon_k)\right]\left[1-2f(q_0)\right]}{(q_0-\epsilon_k)^2}$$

$$=\chi_{d0}\vec{M}_s\left(\frac{J}{n}\right)^2\rho\,\frac{1}{2}\int dq_0\int d\epsilon_k\,\frac{\partial f(\epsilon_k)}{\partial\epsilon_k}\,\frac{1-2f(q_0)}{q_0-\epsilon_k}\,, \qquad (\mathrm{D}31)$$

where in the last line we have used (B19).

We must now calculate the last term of (D27). For this we use (D28) plus the local-spin analog of the dispersion relation (D7) to obtain

$$\Sigma(0)=-\alpha_1\gamma\vec{S}\cdot\vec{M}_s$$

$$+\int d\epsilon_k\int dq_0\left(\frac{J}{n}\right)^2\gamma^{-2}\vec{M}_s\cdot\gamma\vec{S}\,\frac{1}{2}\,\frac{1}{\epsilon_k-q_0}$$

$$\times\left[1-2f(\epsilon_k)\right]\left[-f'(q_0)\right]$$

$$=-(\alpha_1+\alpha_2)\,\gamma\vec{S}\cdot\vec{M}_s\,. \qquad (\mathrm{D}32)$$

Finally, substitution of (D31) and (D32) into (D27) yields

$$(\alpha_1+2\alpha_2)\,\chi_{d0}\vec{M}_s\,, \qquad (\mathrm{D}33)$$

and substitution of this and (D22) back into (D20) yields

$$\mathrm{Tr}_S\,\gamma\vec{S}\sum_{p,p_0}\delta G^{<}(\vec{p},p_0)$$

$$=\vec{M}_d-\chi_{d0}\left[\vec{H}+(\alpha_1+2\alpha_2)\vec{M}_s\right], \qquad (\mathrm{D}34)$$

which is the desired relation.

Note added in Proof. Here we show the connection between the s and d magnetization in the Anderson model and the s-d exchange model used in this paper, Eq. (2.1). The Anderson Hamiltonian [Phys. Rev. 124, 41 (1961)] is

$$H^A=\sum_{\vec{k}s}\epsilon_{\vec{k}}\,c^{\dagger}_{\vec{k}s}c_{\vec{k}s}+\sum_{s}\epsilon_d N_s$$

$$+\,\tfrac{1}{2}\sum_{s}UN_sN_{-s}+\sum_{\vec{k}s}V(c^{\dagger}_{\vec{k}s}d_s+d^{\dagger}_s c_{\vec{k}s})\,, \qquad (\mathrm{D}35)$$

where $c^{\dagger}_{\vec{k}s}$ and $d^{\dagger}_s$ are the appropriate creation operators for the s and d electrons, respectively, and $N_s=d^{\dagger}_s d_s$. In the s-d mixing term we have taken the potential V to be a real constant. We consider only the case of a single extra d orbital which implies $S=\frac{1}{2}$ in the s-d exchange model analogue.

The connection between the Anderson model and the s-d exchange model

$$H=\sum_{\vec{k}s}\epsilon_k\,c^{\dagger}_{\vec{k}s}c_{\vec{k}s}-\sum_{\vec{k}\vec{k}'}(J_{\vec{k}\vec{k}'}/n)\sum_{j\alpha\beta}c^{\dagger}_{\vec{k}\alpha}c_{\vec{k}'\beta}\,\vec{s}_{\alpha\beta}\cdot\vec{S}_j \qquad (\mathrm{D}36)$$

has been most definitively discussed by Schrieffer and Wolff [Phys. Rev. 149, 491 (1966)]; for the case of several impurity orbitals, see B. Mühlschlegel [Z. Physik 208, 94 (1968)]. They showed that under a unitary transformation

$$e^S(H^A)\,e^{-S}\simeq H+\sum_{\vec{k}\vec{k}'s}W_{\vec{k}\vec{k}'s}\,c^{\dagger}_{\vec{k}s}c_{\vec{k}'s}+\mathrm{const}\,, \qquad (\mathrm{D}37)$$

where

$$S=\sum_{\vec{k}s}V\,\frac{c^{\dagger}_{\vec{k}s}d_s-d^{\dagger}_s c_{\vec{k}s}}{\epsilon_{\vec{k}}-\epsilon_d-UN_{-s}}\,. \qquad (\mathrm{D}38)$$

In writing S we have deviated slightly from Schrieffer and Wolff by keeping N_{-s} an operator in the denominator of (D38) instead of forcing it to take on the values of zero and unity. This shall prove quite useful where we are taking expectation values to calculate the magnetization. In general, we shall require the exchange interaction $J_{\vec{k}\vec{k}'}$ and the potential $W_{\vec{k}\vec{k}'s}$ only for $\epsilon_{\vec{k}}=\epsilon_{\vec{k}'}$ and in the case of the potential summed over spin, that is,

$$J(\epsilon) = -2V^2\left(\frac{1}{\epsilon - \epsilon_d} - \frac{1}{\epsilon - \epsilon_d - U}\right) \tag{D39}$$

and

$$W(\epsilon) = \tfrac{1}{2}V^2\left(\frac{1}{\epsilon - \epsilon_d} + \frac{1}{\epsilon - \epsilon_d - U}\right) . \tag{D40}$$

We observe that $J(\epsilon)$ is negative for electron energies lying between the "occupied" and "unoccupied" orbitals, $\epsilon_d < \epsilon < \epsilon_d + U$.

Since the total magnetization (more accurately, the total spin) commutes with every term of H^A and hence with S, it will clearly be unaffected by the Schrieffer-Wolff unitary transformation. That is not the case for the magnetization of conduction electrons or impurity spins, separately. We write the conduction electron magnetization in the Anderson model as

$$\vec{\mathrm{M}}_s^A = \langle \gamma (n_\uparrow - n_\downarrow)\rangle^A , \tag{D41}$$

where

$$n_s = \sum_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} . \tag{D42}$$

We can calculate (D41) by the obvious identity

$$\langle n_s\rangle^A = \langle e^{-S} | e^S n_s e^{-S} | e^S\rangle^A = \langle e^S n_s e^{-S}\rangle , \tag{D43}$$

where the final expression is evaluated in the s-d exchange model. To second order in V we find that

$$\langle n_s\rangle^A - \langle n_s\rangle = -\sum_{\vec{k}} V^2 \left\langle \frac{n_{\vec{k}s} - N_s}{(\epsilon_{\vec{k}} - \epsilon_d - UN_{-s})^2}\right\rangle . \tag{D44}$$

In the numerator of (D44) we observe that

$$\langle n_{\vec{k}s}\rangle = f(\epsilon_{\vec{k}}) \pm A_{\vec{k}}\, \partial f(\epsilon_{\vec{k}})/\partial\epsilon_{\vec{k}} , \tag{D45}$$

where the + and − refer to up and down spins, respectively. Further,

$$2\sum_{\vec{k}} \gamma A_{\vec{k}} \frac{\partial f}{\partial \epsilon_{\vec{k}}} = \vec{\mathrm{M}}_s . \tag{D46}$$

Similarly,

$$\gamma\langle N_\uparrow - N_\downarrow\rangle = \vec{\mathrm{M}}_d . \tag{D47}$$

The denominator of (D44) plays a very important role in the calculation. For example, to leading order in $\vec{\mathrm{M}}_d$,

$$\begin{aligned} V^2 \left\langle \frac{f(\epsilon) + A_{\vec{k}} f'(\epsilon)}{(\epsilon - \epsilon_d - UN_\downarrow)^2} - \frac{f(\epsilon) - A_{\vec{k}} f'(\epsilon)}{(\epsilon - \epsilon_d - UN_\uparrow)^2}\right\rangle &= f(\epsilon)\, V^2 \left(\frac{1}{(\epsilon - \epsilon_d)^2} - \frac{1}{(\epsilon - \epsilon_d - U)^2}\right) \langle N_\uparrow - N_\downarrow\rangle \\ &\quad + V^2\left(\frac{1}{(\epsilon - \epsilon_d)^2} + \frac{1}{(\epsilon - \epsilon_d - U)^2}\right) A_{\vec{k}} f'(\epsilon) \\ &= \tfrac{1}{2} f(\epsilon)\, \frac{\partial J(\epsilon)}{\partial\epsilon} \langle N_\uparrow - N_\downarrow\rangle - 2\, \frac{\partial W(\epsilon)}{\partial \epsilon} A_{\vec{k}} f'(\epsilon) . \end{aligned} \tag{D48}$$

Using (D48) and the identifications (D46) and (D47), and the assumption that the density of states is a constant [which eliminates the N_s term in the numerator of (D44)], we find that

$$\vec{\mathrm{M}}_s^A = \vec{\mathrm{M}}_s - \alpha_1 \chi_{s0} \vec{\mathrm{M}}_d + \frac{\partial W(0)}{\partial\epsilon} \vec{\mathrm{M}}_s . \tag{D49}$$

The first term on the right-hand side is followed by integration by parts and the identification $\frac{1}{2}\rho J(0) = \alpha_1\chi_{s0}$; see (1.13) and (1.18). Equation (D49) differs from the result quoted in the note added in proof to Sec. I only by the term $\partial W(0)/\partial\epsilon$, which arises from potential scattering. Finally, we observe that

$$\vec{\mathrm{M}}_d^A = \vec{\mathrm{M}}_d (1 + \alpha_1\chi_{s0}) - \frac{\partial W(0)}{\partial\epsilon} \vec{\mathrm{M}}_s \tag{D50}$$

follows immediately from (D49) and the fact noted earlier that the total magnetization is conserved under the Schrieffer-Wolff transformation, i.e.,

$$\vec{\mathrm{M}}_s^A + \vec{\mathrm{M}}_d^A = \vec{\mathrm{M}}_s + \vec{\mathrm{M}}_d .$$

We now discuss the so-called "compensation" theorem which applies in thermal equilibrium. Originally proved by Anderson in the Hartree-Fock approximation, but straightforwardly proved to all orders, the result essentially states that

$$\vec{\mathrm{M}}_s^A = \chi_{s0} \vec{\mathrm{H}} , \tag{D51}$$

provided that V and the density of states are constant. Similarly, we have shown (D14) to all orders in J in thermal equilibrium, that

$$\vec{\mathrm{M}}_s = \chi_{s0} (\vec{\mathrm{H}} + \alpha_1 \vec{\mathrm{M}}_d) . \tag{D52}$$

If we insert (D52) into (D49) and neglect the potential scattering term, it is clear that (D52) is the s-d model analogue of the compensation theorem. A word of warning, however: Although (D51) and (D52) are both exact, (D49), the bridge between them, is not. This apparent anomaly is resolved by noting that (D51) assumes V is independent of $\vec{k}$ while (D52) assumes that J is independent of ϵ; examination of (D39) shows that the two assumptions are incompatible. We mention also that if potential scattering had been included in our derivation of (D14) and hence (D52) to lowest order in W, a term $(\partial W/\partial\epsilon)\vec{\mathrm{M}}_s$ would have been obtained, in complete consistency to that order with (D49).

Finally, we must make the connection between our Bloch equations (1.25) and (1.26) and those of Sasada and Hasegawa [Progr. Theoret. Phys. (Kyoto) 45, 1072 (1971), Eq. (4.1)] for the case $\alpha_2 = 0$ and $1/T_{dl} = 0$. Noting that (i) the magnetic field terms in the cross relaxation add to zero, (ii) $\delta\vec{\mathrm{M}}_s + \chi_{s0}\vec{\mathrm{H}} = \vec{\mathrm{M}}_s^A$, and (iii) $(1 + \alpha_1\chi_{s0})(\delta\vec{\mathrm{M}}_d + \chi_{d0}\vec{\mathrm{H}}) = \vec{\mathrm{M}}_d^A(1 - \alpha_1^2\chi_{s0}\chi_{d0}) - \alpha_1\chi_{s0}\vec{\mathrm{M}}_s^A$, we see that (1.25) minus $\alpha_1\chi_{s0}$ times (1.26) combines to give

$$\frac{d\vec{M}_s^A}{dt} = \vec{M}_s^A \times \gamma(\vec{H} + \alpha_1 \vec{M}_d^A) - \frac{1}{T_{sl}}(\vec{M}_s^A - \chi_{s0}\vec{H})$$

$$-\frac{1}{T_{sd}^A}\vec{M}_s^A + \frac{1}{T_{ds}^A}\vec{M}_d^A , \tag{D53}$$

where

$$\frac{1}{T_{sd}^A} = \frac{1}{T_{sd}}(1+\alpha_1\chi_{s0})^2 \tag{D54}$$

and

$$\frac{1}{T_{ds}^A} = \frac{\chi_{s0}}{\chi_{d0}}\frac{1}{T_{sd}}(1-\alpha_1^2\chi_{s0}\chi_{d0}) . \tag{D55}$$

As we observed, the bridge between the two models, (D49), with potential scattering set equal to zero, is only good to first order in J. Accordingly, the parentheses in (D54) and (D55) are accurate only to that order. Then the ratio of the two relaxation times is

$$\frac{T_{ds}^A}{T_{sd}^A} = \frac{\chi_{d0}}{\chi_{s0}}(1+2\alpha_1\chi_{s0}) = \frac{\chi-\chi_{s0}}{\chi_{s0}} \tag{D56}$$

by (1.24). But in the Anderson model, $\chi-\chi_{s0}$ is the susceptibility of the d orbital, while χ_{s0} is the s susceptibility even in the interacting system. Hence (D56) is the result of Sasada and Hasegawa. [The following identification of terms in their Eq. (4.10) may be useful: $\chi_s \to \chi_{s0}$, $\chi'_d \to \chi - \chi_{s0}$, $2\lambda \to \alpha_1$, $T_{2sd} \to T_{sd}^A$, $T_{2ds} \to T_{ds}^A$, $\Delta n \to 1+\alpha_1\chi_{s0}$, $cQ/\Delta n \to (2T_{sd})^{-1}$, $N(0)\to\rho$.]

APPENDIX E: TERMS IN LEFT-HAND SIDE OF GKB FOR CONDUCTION ELECTRONS

In this appendix terms arising from (6.11)–(6.13) will be evaluated. First, we will discuss the results of applying the time-frequency gradient expansion but at the end of the Appendix the position-momentum expansion shall also be applied. The first term of (6.11) leads to

$$\mathrm{Tr}_s\gamma\vec{s}\sum_{p,p_0} -\frac{1}{i}[\sigma_0, \Delta g^<]_{\mathrm{Ft}}$$

$$= \mathrm{Tr}_s\gamma\vec{s}\int \rho\, d\epsilon_p \sum_{p_0} -\frac{1}{i}\left[+\gamma\vec{s}\cdot\vec{H}\frac{\partial\sigma_0}{\partial p_0},\ \vec{m}(\vec{p},p_0)\cdot\vec{s}\right]$$

$$-\mathrm{Tr}_s\gamma\vec{s}\sum_{p_0}\frac{1}{2}\left\{\frac{\partial\sigma_0}{\partial p_0},\ \frac{\partial}{\partial t}\vec{m}(\vec{p},p_0)\cdot\vec{s}\right\} . \tag{E1}$$

In the first term we have used the fact that σ_0 is a function of $p_0+\gamma\vec{s}\cdot\vec{H}$ and in the second the gradient expansion (A33). The two terms can be written

$$+\left(\gamma\vec{H}\times+\frac{\partial}{\partial t}\right)\left\{\gamma(2s+1)\frac{1}{3}s(s+1)\int \rho\, d\epsilon_p\sum_{p_0}\vec{m}(\vec{p},p_0)\right\}$$

$$\times\int\frac{d\omega}{2\pi}\frac{\gamma_0(p,\omega)}{(p_0-\omega)^2} , \tag{E2}$$

which is identical zero by (D11) since γ_0 is proportional to a (assumed) constant density of states.

The second term in (6.11) can be treated similarly to give zero

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0} -\frac{1}{i}[\sigma_{M_s}^<, g]_{\mathrm{Ft}}$$

$$= \mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0}\left(-\frac{1}{i}\right)\left[\sigma_{M_s}^<(\vec{p},p_0),\ \gamma\vec{s}\cdot\vec{H}\frac{\partial g}{\partial p_0}\right]$$

$$+\mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0}\frac{1}{2}\left\{\frac{\partial}{\partial t}\sigma_{M_s}^<(\vec{p},p_0),\ \frac{\partial g}{\partial p_0}\right\} \tag{E3}$$

is proportional via (B26) to

$$\left(\gamma\vec{H}\times+\frac{\partial}{\partial t}\right)\sum_{k,p_0}\vec{m}(\vec{k},p_0)\int\rho\, d\epsilon_p\frac{1}{(p_0-\epsilon_p)^2} , \tag{E4}$$

which by (D11) is zero.

The first term in (6.13) can be written

$$\mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0}\left(-\frac{1}{i}\right)[\sigma_{M_d}, g^< - \Delta g^<]$$

$$= \mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0}\left(-\frac{1}{i}\right)$$

$$\times\left[\sigma_{M_d}, \gamma\vec{s}\cdot\vec{H}\frac{\partial}{\partial p_0}\{\delta(p_0-\epsilon_p)f(p_0)\}\right] ,$$

$$+\mathrm{Tr}_s\gamma\vec{s}\int\rho\, d\epsilon_p\sum_{p_0}\frac{1}{2}$$

$$\times\left\{\left(\frac{\partial}{\partial t}\right)\sigma_{M_d},\ \frac{\partial}{\partial p_0}[\delta(p_0-\epsilon_p)f(p_0)]\right\} . \tag{E5}$$

We integrate by parts and use (B14) for γ_{M_d} to get

$$2\pi\left(\frac{J}{n}\right)^2(2s+1)\frac{1}{3}s(s+1)\left(\frac{\partial}{\partial t}+\gamma\vec{H}\times\right)\{\gamma(2S+1)\frac{1}{3}S(S+1)\sum_{q,q_0}\vec{M}(q,q_0)\}$$

$$\times\int\rho\, d\epsilon_p f(\epsilon_p)\int\frac{dp_0}{2\pi}\frac{\Sigma_k\{-\frac{1}{2}\delta(p_0+q_0-\epsilon_k)[1-f(\epsilon_k)]+\frac{1}{2}\delta(p_0-q_0-\epsilon_k)f(\epsilon_k)\}}{(\epsilon_p-p_0)^2} . \tag{E6}$$

The integral over p_0 gives, after use of $(\epsilon_p - \epsilon_k \pm q_0)^{-2} = (\partial/\partial\epsilon_k)(\epsilon_p - \epsilon_k \pm q_0)^{-1}$ and integration by parts,

$$= \frac{1}{4\pi}\int \rho\, d\epsilon \sum_k \frac{-\partial f(\epsilon_k)}{\partial\epsilon_k}\left(\frac{1}{\epsilon_p + q_0 - \epsilon_k} + \frac{1}{\epsilon_p - q_0 - \epsilon_k}\right).$$

At this point we note that we can neglect the q_0 dependence since $M(\vec{q}, q_0)$ is proportional to $\delta(q_0)$. Hence we have

$$\left(\frac{J}{n}\right)^2 (2s+1)\,\frac{1}{3}\, s(s+1)\left(\frac{\partial}{\partial t} + \gamma\vec{H}\times\right)\vec{M}_d$$
$$\times\int \rho\, d\epsilon_p \sum_k \left(-\frac{\partial f(\epsilon_k)}{\partial\epsilon_k}\right)\frac{f(\epsilon_p)}{\epsilon_p - \epsilon_k}. \tag{E7}$$

Since these two integrals give $\rho^2 \ln k_B T/D$ according to (B19), we find that

$$\mathrm{Tr}_s \gamma\vec{s}\int \rho\, d\epsilon_p \sum_{p_0}\left(-\frac{1}{i}\right)[\sigma^<_{M_d}, g^< - \Delta g^<]$$
$$= \alpha_2\, \chi_{s0}\left(\frac{\partial\vec{M}_d}{\partial t} + \gamma\vec{H}\times\vec{M}_d\right). \tag{E8}$$

The last term in (6.13) is

$$\mathrm{Tr}_s \gamma\vec{s}\int \rho\, d\epsilon_p \sum_{p_0}\left(-\frac{1}{i}\right)[\sigma^<_{M_d}, g]_{\mathrm{Ft}}$$
$$= \mathrm{Tr}_s \gamma\vec{s}\int \rho\, d\epsilon_p \sum_{p_0}\left(-\frac{1}{i}\right)$$
$$\times\left[\sigma^<_{M_d}(p, p_0),\ \gamma\vec{s}\cdot\vec{H}\frac{\partial g}{\partial p_0}\right]$$
$$+ \mathrm{Tr}_s \gamma\vec{s}\int \rho\, d\epsilon_p \sum_{p_0} + \frac{1}{2}\left\{\left(\frac{\partial}{\partial t}\right)\sigma^<_{M_d}(p, p_0)\,\frac{\partial g}{\partial p_0}\right\}. \tag{E9}$$

Using (B25) we obtain

$$-\tfrac{1}{2}(2s+1)\,\tfrac{1}{3}\, s(s+1)\, J^2\left(\frac{\partial}{\partial t} + \gamma\vec{H}\times\right)$$
$$\times\left\{\gamma(2S+1)\,\tfrac{1}{3}\, S(S+1)\sum_{q, q_0}\vec{M}(q, q_0)\right\}$$
$$\times\int \rho\, d\epsilon_p \sum_k \frac{f(\epsilon_k)}{(\epsilon_k + q_0 - \epsilon_p)^2}, \tag{E10}$$

which by (D11) is zero.

Finally we consider (6.12):

$$\mathrm{Tr}_s \gamma\vec{s}\int \rho\, d\epsilon_p \sum_{p_0} - \frac{1}{i}\,[\sigma_{M_d}, \Delta g^<]_{\mathrm{Ft}}$$
$$= \mathrm{Tr}_s \gamma\vec{s}\left(\frac{1}{i}\right)\left[\left\{\gamma(2S+1)\,\tfrac{1}{3}\, S(S+1)\sum_{q, q_0}\vec{M}(q, q_0)\cdot\vec{s}\right\}, \int \rho\, d\epsilon_p \sum_{p_0}\gamma\vec{m}(p, p_0)\cdot\vec{s}\right]$$
$$\times\left(\frac{J}{n}\right)^2\left(\frac{1}{\gamma^2}\right)\sum_k (-1)\left\{\frac{-\frac{1}{2}(1 - f(\epsilon_k))}{p_0 + q_0 - \epsilon_k} + \frac{\frac{1}{2}f(\epsilon_k)}{p_0 - q_0 - \epsilon_k}\right\}, \tag{E11}$$

where (B14) has been used. Since $\vec{M}(\vec{q}, q_0)$ is proportional to $\delta(q_0)$ we can neglect q_0 in the k sum, which according to (B18) and (B19) is $\rho \ln(k_B T + p_0)/D$). Neglecting this weak p_0 dependence, we see that (E11) is

$$\alpha_2 \gamma\vec{M}_d \times \vec{m}(\hat{p}). \tag{E12}$$

We turn now to the implications of the position-momentum gradient expansion for (6.11)–(6.13). As has already been observed in the text [following (6.22)] the expansion for (6.11) is zero. The contribution from (6.12) is small; see (6.28) and (6.30). As for (6.13) the first-order part of σ_{M_d} gives rise to a term (6.26) but as we shall show the second-order parts of σ_{M_d} and $\sigma^<_{M_d}$ give nothing. The first commutator of (6.13) has the gradient expansion

$$\mathrm{Tr}_s \gamma\vec{s}\int p\, d\epsilon_p \textstyle\sum_{p_0}\left(-\tfrac{1}{2}\right)\{\nabla_x\sigma_{M_d}, \nabla_p\,\delta(p_0 - \epsilon_p)\}\, 2\pi f(p_0).$$

From the expression (B14) for $\gamma_{M_d}(p_0)$ we see that $\nabla_x\sigma_{M_d}$ can be written, using the fact that $\vec{M}(\vec{q}, q_0)$ is proportional to $\delta(q_0)$, in the form

$$\nabla_x\vec{M}_d A(p_0),$$

where $A(p_0)$ goes to zero like $|p_0|^{-1}$ for large p_0. Hence the integrals over ϵ_p and p_0 reduce to

$$\int \rho\, d\epsilon_p \int dp_0\, \delta'(p_0 - \epsilon_p)\, f(p_0) A(p_0),$$

which is identically zero in the approximation that we neglect the energy dependence of the density of states. The gradient expansion applied to the second commutator of (6.13) leads to a term proportional to $\int d\epsilon_p (p_0 - \epsilon_p)^{-2}$ which is zero via (D11).

APPENDIX F: TERMS IN LEFT-HAND SIDE OF GKB FOR LOCAL SPINS

In this appendix terms arising from (6.41)–(6.43) will be evaluated. The first term in (6.42) leads to

$$\mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\,[\Sigma_0, \Delta G^<]_{\mathrm{Ft}}$$

$$= \mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\left[\gamma\vec{S}\cdot\vec{H}\,\frac{\partial\Sigma_0}{\partial p_0}\,,\ \vec{M}(p,p_0)\cdot\vec{S}\right]$$

$$- \mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} \left\{\frac{\partial\Sigma_0}{\partial p_0}\,,\ \frac{\partial}{\partial T}\,\vec{M}(p,p_0)\cdot\vec{S}\right\}. \qquad \text{(F1)}$$

In the first term we have used the fact that Σ_0 is a function of $p_0+\gamma\vec{S}\cdot\vec{H}$ and in the second the gradient expansion (A31). The two terms can be written, with (C17) and the dispersion relation (2.6b)

$$\left(\gamma\vec{H}\times+\frac{\partial}{\partial t}\right)\left\{\gamma(2S+1)\,\tfrac{1}{3}S(S+1)\sum_{p,p_0}\vec{M}(\vec{p},p_0)\right\}$$

$$+\left(\frac{J}{n}\right)^2 (2s+1)\,\tfrac{1}{3}s(s+1)\,S(S+1)$$

$$\times\sum_{k,q}\frac{f(\epsilon_k)[1-f(\epsilon_q)]}{(p_0-\epsilon_q+\epsilon_k)^2}\ . \qquad \text{(F2)}$$

Again we observe that $(p_0-\epsilon_q+\epsilon_k)^{-2} = (\partial/\partial\epsilon_q)(p_0-\epsilon_q+\epsilon_k)^{-1}$, so that integration by parts gives the integral

$$-\sum_k f(\epsilon_k)\sum_q \frac{-\partial f}{\partial\epsilon_q}\,(p_0-\epsilon_q+\epsilon_k)^{-1}\,,$$

which equals, via (B19), $-\rho^2\ln(k_BT-p_0)/D$, the p_0 dependence of which is suppressed by $\vec{M}(\vec{p},p_0)$'s proportionality to $\delta(p_0)$. The curly bracket in (F2) is then equal to $\vec{M}_d$ and (F2) reduces to

$$-S(S+1)\,\alpha_2\,\chi_{s0}\left(\frac{\partial\vec{M}_d}{\partial t}+\gamma\vec{H}\times\vec{M}_d\right). \qquad \text{(F3)}$$

The second term in (6.41) can be treated similarly:

$$\mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\,[\Sigma^<_{M_d}, G]_{\mathrm{Ft}}$$

$$= \mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\left[\Sigma^<_{M_d}.\ \gamma\vec{S}\cdot\vec{H}\,\frac{\partial G}{\partial p_0}\right]$$

$$+ \mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} \tfrac{1}{2}\left\{\frac{\partial\Sigma^<_{M_d}}{\partial t}\,,\ \frac{\partial G}{\partial p_0}\right\}. \qquad \text{(F4)}$$

Substitution of (C17) leads to

$$-\left(\gamma\vec{H}\times+\frac{\partial}{\partial t}\right)\{\gamma(2S+1)\,\tfrac{1}{3}S(S+1)$$

$$\times\sum_{p,p_0}\vec{M}(\vec{p}+\vec{k},p_0+k_0)\}$$

$$\times\left(\frac{J}{n}\right)^2 (2s+1)\,\tfrac{1}{3}s(s+1)[S(S+1)-1]$$

$$\times\sum_{k,q} f(\epsilon_q)[1-f(\epsilon_{k+q})]\sum_{k_0} 2\pi\,\frac{\delta(k_0+\epsilon_q-\epsilon_{\vec{k}+\vec{q}})}{p_0^2}\,, \qquad \text{(F5)}$$

where the minus sign comes from $\partial G/\partial p_0$. For convenience in understanding the subsequent algebra the summation signs have been reorganized prior to doing the obvious variable switches $(\vec{p}+\vec{k}, p_0+k_0)\to(\vec{p},p_0)$ and $\vec{k}+\vec{q}\to k$. The sum over k_0, including the factor of $(p_0-k_0)^2$ now leads to the sum

$$\sum_{k,q} f(\epsilon_q)[1-f(\epsilon_k)](p_0-\epsilon_k+\epsilon_q)^{-2} = -\rho^2\ln\frac{k_BT-p_0}{D}\,, \qquad \text{(F6)}$$

the p_0 dependence of which can neglect due to $M(p_0)$'s proportionality to $\delta(p_0)$. The curly bracket in (F5) is then $\vec{M}_d$ and (F5) reduces to

$$+[S(S+1)-1]\,\alpha_2\,\chi_{so}\left(\frac{\partial\vec{M}_d}{\partial t}+\gamma\vec{H}\times\vec{M}_d\right). \qquad \text{(F7)}$$

When (F3) and (F7) are added, we see that (6.41) yields just

$$-\alpha_2\,\chi_{so}\left(\frac{\partial\vec{M}_d}{\partial t}+\gamma\vec{H}\times\vec{M}_d\right), \qquad \text{(F8)}$$

which is identical with (E8) but has the opposite sign and hence will cancel it when the conduction electrons and local spin Bloch equations are added.

Equation (6.43) is more readily, if tediously, evaluated. For example, the first term of (6.40) is

$$\mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\,[\Sigma_{M_s}, (G^<-\Delta G^<)]_{\mathrm{Ft}}$$

$$= \mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\left[\Sigma_{M_s},\ \gamma\vec{S}\cdot\vec{H}\,\frac{\partial}{\partial p_0}\,2\pi\,\delta(p_0)\,F(p_0)\right]$$

$$+ \mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} \frac{1}{2}\left\{\frac{\partial\Sigma_{M_s}}{\partial t}\,,\ \frac{\partial}{\partial p_0}2\pi\,\delta(p_0)\,F(p_0)\right\}, \qquad \text{(F9)}$$

which, upon substitution of (C16) and integration by parts, becomes

$$\left(\gamma\vec{H}\times+\frac{\partial}{\partial t}\right)\{\gamma(2s+1)\,\tfrac{1}{3}s(s+1)\sum_{q,q_0}\vec{m}(q,q_0)\}$$

$$\left(\frac{J}{n}\right)^2\int dp_0\,\delta(p_0)\sum_p F(p_0)\,(2S+1)\,\tfrac{1}{3}S(S+1)$$

$$\times\sum_k\left[\frac{\tfrac{1}{2}[1-f(\epsilon_k)]}{(p_0-q_0+\epsilon_k)^2}+\frac{\tfrac{1}{2}f(\epsilon_k)}{(p_0+q_0-\epsilon_k)^2}\right]. \qquad \text{(F10)}$$

The δ function on p_0 reduces the sum of k to a single term $\sum_k(\epsilon_k-q_0)^2$ which is zero by (D18).

Likewise the second term of (6.43) can be shown to zero. The algebra starts with

$$\mathrm{Tr}_S\, \gamma\vec{S} \sum_{p,p_0} -\frac{1}{i}\,[\Sigma^<_{M_s}, G]_{\mathrm{Ft}}$$

$$=\mathrm{Tr}_S\gamma\vec{\mathrm{S}}\sum_{p,p_0}-\frac{1}{i}\left[\Sigma^{<}_{M_s}(p_0),\,\gamma\vec{\mathrm{S}}\cdot\vec{\mathrm{H}}\,\frac{\partial G(p_0)}{\partial p_0}\right]$$

$$+\mathrm{Tr}_S\gamma\vec{\mathrm{S}}\sum_{p,p_0}\frac{1}{2}\left\{\frac{\partial\Sigma^{<}_{M_s}(p_0)}{\partial t},\frac{\partial G(p_0)}{\partial p_0}\right\}. \quad \text{(F11)}$$

But by the identity (C18) $\Sigma^{<}_{M_s}(p_0)=-N\Gamma_{M_s}(-p_0)/(2S+1)$,

$$\sum_{p,p_0}\Sigma^{<}_{M_s}(p_0)\,\frac{\partial G}{\partial p_0}=-\frac{N}{(2S+1)}\sum_{p_0}\frac{\Gamma_{M_s}(+p_0)}{{p_0}^2}, \quad \text{(F12)}$$

which we have just shown in zero in (F10).

Finally we consider (6.42):

$$\mathrm{Tr}_S\gamma\vec{\mathrm{S}}\sum_{p,p_0}-\frac{1}{i}\,[\Sigma_{M_s},\Delta G^{<}]_{\mathrm{Ft}}$$

$$=-\mathrm{Tr}_S\gamma\vec{\mathrm{S}}\left(-\frac{1}{i}\right)\Big[\Big\{\gamma(2s+1)\,\tfrac{1}{3}s(s+1)$$

$$\times\sum_{q,q_0}\vec{\mathrm{m}}(q,q_0)\cdot\vec{\mathrm{S}}\Big\},\ \sum_{p,p_0}\gamma\vec{\mathrm{M}}(p,p_0)\cdot\vec{\mathrm{S}}\Big]$$

$$\times\left(\frac{J}{n}\right)^2\left(\frac{1}{\gamma^2}\right)\sum_k\left[\frac{\frac{1}{2}f(\epsilon_k)}{p_0-q_0+\epsilon_k}+\frac{\frac{1}{2}[1-f(\epsilon_k)]}{p_0+q_0-\epsilon_k}\right]. \quad \text{(F13)}$$

The proportionality of $\vec{\mathrm{m}}(\vec{\mathrm{q}},q_0)$ to $\partial f(q_0)/\partial q_0$ and $\vec{\mathrm{M}}(\vec{\mathrm{p}},p_0)$ to $\delta(p_0)$ reduce the $\vec{\mathrm{k}}$ sum to $\sum_k f(\epsilon_k)/\epsilon_k=\rho\ln(k_BT/D)$ by (B19). Hence (F13) is

$$\alpha_2\gamma\vec{\mathrm{M}}_s\times\vec{\mathrm{M}}_d. \quad \text{(F14)}$$

---

*Work supported in part by the National Science Foundation under Grant No. GP-14855.

†Work supported in part by the Office of Naval Research through Contract No. N00014-67-A-0077-0010, Technical Report No. 15; Alfred P. Sloan Foundation Fellow.

[1] We thus give the details of the derivation of the equations which we published previously: D. C. Langreth, D. L. Cowan, and J. W. Wilkins, Solid State Commun. 6, 131 (1968); except that we correct a slight error in the second-order terms. Also, to simplify the analysis here, we neglect the hyperfine interaction.

[2] We apologize in advance for the fact that some appropriate references have surely been omitted. We hoped that enough principal ones have been included that the reader can find his way around the literature.

[3] The basic form of this equation is due to M. Y. Azbel, V. I. Gerasimenko, and I. M. Lifshitz, Zh. Eksperim. i Teor. Fiz. 32, 1212 (1957) [Sov. Phys. JETP 5, 986 (1957)]. It may be derived by specializing the results of Ref. 1. Since then it has been derived under certain assumptions by P. Fulde and A. Luther, Phys. Rev. 175, 337 (1968); W. F. Brinkman and S. Engelsberg, Phys. Rev. Letters 21, 1187 (1968). The equation has been written in this precise form by W. F. Brinkman, S. Engelsberg, and M. B. Walker, Phys. Rev. B 3, 30 (1971).

[4] See, for example, C. P. Slichter, *Principles of Magnetic Resonance* (Harper and Row, New York, 1963), p. 29 (see also p. 157); C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1971), pp. 583–585; G. Pake, *Paramagnetic Resonance* (Benjamin, New York, 1962), pp. 117–118; N. Bloembergen, *Nuclear Magnetic Resonance* (Martenus Nijhoff, The Hague, 1948), Eq. (2.61) also reissued by (Benjamin, New York, 1961). However, some have, of course, caught the term such as A. Redfield, Phys. Rev. 98 (1955), see the second paragraph on p. 1793; A. Abraham, *The Principles of Nuclear Magnetism* (Oxford U. P., London, 1961), p. 53, and implicitly Azbel, Gerasimenko, and Lifshitz, Ref. 3. Recently it has been realized by others—see Refs. 3, 15, and 20.

[5] The kinds of difficulties encountered are illustrated in the work of B. Giovannini, M. Peter, and S. Koide, Phys. Rev. 149, 251 (1966); Spencer and Orbach, *ibid.* 179, 683 (1939); 179, 690 (1969); but see also Ref. 3.

[6] H. C. Torrey, Phys. Rev. 104, 563 (1956).

[7] M. B. Walker, Phys. Letters 32A, 230 (1970); Phys. Rev. B 3, 30 (1971); and J. H. Pifer and R. T. Longo, Phys. Rev. B 4, 3797 (1971).

[8] For a comprehensive review of experimental and theoretical aspects of CESR see M. M. Walsh, in *Solid State Physics*, edited by J. F. Cochran and R. R. Haering (Gordon and Breach, New York, 1968), Vol. I, p. 127.

[9] S. Schultz and G. Dunifer, Phys. Rev. Letters 18, 283 (1967).

[10] P. M. Platzman and P. A. Wolff, Phys. Rev. Letters 18, 280 (1967).

[11] P. Nozières, in *Polarization, Matière et Rayonnement, volume julibaire à l'honneur A. Kastler* (Presses Universitaires de France, Paris, 1969).

[12] J. Korringa, Physica 16, 601 (1950) ($1/T_{ds}$); A. Overhauser, Phys. Rev. 89, 689 (1953) ($1/T_{sd}$); see also Ref. 17, Eqs. (4.14) and (4.15).

[13] J. R. Asik, M. A. Ball, and C. P. Slichter, Phys. Rev. 181, 645 (1969); 181, 662 (1969); R. J. Elliott, *ibid.* 96, 266 (1954).

[14] S. Schultz, M. R. Shanabarger, and P. M. Platzman, Phys. Rev. Letters 19, 749 (1967); P. Monod and S. Schultz, Phys. Rev. 173, 645 (1968).

[15] The correct expressions to linear order in the "ion-electron exchange constant" were phenomenologically written down by H. Cottet, P. Donźe, J. Dupraz, B. Giovannini, and M. Peter, Z. Angewandte, Z. Phys. 24, 249 (1968), which appeared about the same time as our letter (Ref. 17) announcing our first-principles derivation of this result to linear order in J. To second order in J, however, our results had a slight error.

[16] K. Yosida and A. Okiji, Progr. Theoret. Phys. (Kyoto) 34, 505 (1965).

[17] H. Hasegawa, Progr. Theoret. Phys. (Kyoto) 21, 483 (1959).

[18] In addition to Ref. 4, the reader might also consult the following authors: P. A. Fedders, Phys. Rev. 158, 288 (1967); H. J. Spencer and S. Doniach, Phys. Rev. Letters 18, 994 (1967); M. B. Walker, Phys. Rev. 176, 432 (1968); Phys. Rev. B 1, 3690 (1970).

[19] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962), pp. 100–101.

[20] This method was used several years ago to derive the equations of Ref. 1. More recently the methods of Kadanoff and Baym have been invoked in an attempt to derive the relaxation rates to infinite order in J: (a) W. Brenig, W. Gotze, and P. Wölfle, Phys. Letters 30A, 448 (1969); (b) P. Wölfle, W. Brenig, and W. Gotze, Z. Physik 235, 59 (1970); (c) W. Brenig, W. Gotze, and P. Wölfle, Phys. Rev. B 2, 4533 (1970); (d) P. Wölfle (unpublished). Unfortunately these authors do not give the details of their derivation. Their equations do not contain torque terms and hence cannot be used to describe spin resonance even to zeroth order in J, although they add them, apparently phenomenologically later. Secondly, although purporting to deal with the Kondo effect, they completely miss the leading order Kondo logarithmic term. We therefore will not discuss this work further.

[21] We find this technique more convenient in the present calculations than the pseudofermion representation of A. A. Abrikosov, Physics 2, 5 (1965). This technique was apparently first used independently by P. A. Fedders (Ref. 18) and about the same time by us in the derivation announced in Ref. 1. It was later used by Brenig and Götze [Z. Physik 217, 188 (1968)] and again in Ref. 20.

[22] D. C. Langreth and J. W. Wilkins, Phys. Rev. (to be published).

[23] Henceforth in this paper we choose units such that $\hbar = 1$.

[24] It has just been pointed out to us that a similar notation was used by V. Keldysh, Zh. Eksperim. i Teor. Fiz. 47, 1515 (1964) [Sov. Phys. JETP 20, 1018 (1965)]. In fact one form of the generalized Kadanoff-Baym equations is given there. Very recently, equations similar to those used here and in Ref. 1 have been applied to the interacting Fermi Liquid [P. Wölfle, Z. Physik 232, 38 (1970)].

[25] See the Appendix in Ref. 19. A few distinctions and clarifications are in order. First, we are calculating the self-energy and not the Green's function, and hence the initial and final Green's functions are omitted along with two time integrals. Second, the vertices are ordered in the way they appear in Fig. 1 in which the earlier time is on the right (while the reverse is true in Ref. 19). Lines entering (leaving) a vertex multiply it on the right (left).

PHYSICAL REVIEW B VOLUME 6, NUMBER 9 1 NOVEMBER 1972

Correlated Electron Paramagnetic Resonance and Optical Study of CdF_2: Er^{3+}. I. C_{2v} Local-Site Symmetry

T. C. Ensign and N. E. Byer
Research Institute for Advanced Studies,
Martin Marietta Corporation, 1450 *South Rolling Road, Baltimore, Maryland* 21227

(Received 10 April 1972)

A correlated EPR and optical study has been performed on CdF_2: (Er^{3+}, U) ($U \equiv$ unintentionally compensated) and CdF_2: (Er^{3+}, M^+) ($M^+ = Li^+$, Na^+, Ag^+, or K^+) crystals with the following objectives: (i) generating a high concentration of a specific (i.e., C_{2v}) Er^{3+} site through the addition of monovalent cations and characterizing this site by EPR; (ii) unambiguously determining selected optical properties of Er^{3+} in C_{2v} symmetry; and (iii) determining the crystal field splitting of the $^4I_{15/2}$ ground site of Er^{3+} for C_{2v} symmetry. The orthorhombic (C_{2v}) symmetry, produced at the erbium site when M^+ ions are introduced for charge compensation, has been identified through the angular dependence of the Er^{3+} EPR spectrum (at 4.2 K). Moreover, the EPR results reveal that the C_{2v} (Er^{3+}, M^+) site accounts for nearly all (> 98%) of the noncubic sites recorded for (Er^{3+}, M^+) specimens. This result has permitted an unambiguous determination of the emission, excitation, absorption, lifetime, and efficiency properties of Er^{3+} in C_{2v} symmetry. These characteristics have been found to be similar for each of the M^+ ions listed, but different from those obtained from (Er^{3+}, U) crystals. In particular for (Er^{3+}, Na^+) the green ($^4S_{3/2} \rightarrow {}^4I_{15/2}$) quantum yield is observed to increase from 2.3 to 19.7%, whereas the red ($^4F_{9/2} \rightarrow {}^4I_{15/2}$) quantum yield decreases from 26 to 2.9%. The large variation in these radiative-quantum yields is analyzed in terms of multiphonon decay processes ($^4S_{3/2} \rightarrow {}^4F_{9/2}$), which are seen to be sensitive to Er^{3+}-site symmetry, but relatively insensitive to the exact nature of the compensating species. The crystal field splitting of the $^4I_{15/2}$ ground state of Er^{3+} in C_{2v} symmetry is in good agreement with that expected from the cubic-field approximation of Lea, Leask, and Wolf, with crystal field parameters, $A_4 \langle r^4 \rangle = -245\ cm^{-1}$ and $A_6 \langle r^6 \rangle = 40\ cm^{-1}$. The noncubic portion of the total orthorhombic field may be accounted for in terms of an axial distortion along the $Er^{3+} - M^+$ direction, as verified by the excellent agreement obtained between the EPR g values and the optical splitting of the $\Gamma_8^{(1)}$ state of the $^4I_{15/2}$ multiplet.

I. INTRODUCTION

The optical spectra of Er^{3+} in crystals exhibiting the fluorite structure, space group O_h^5 ($Fm3m$), very often exhibit complex structure. Electron-paramagnetic-resonance (EPR) studies of these crystals have shown that this structure arises because the Er^{3+} may reside in a variety of noncubic