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Theory of Spin Waves in Nonferromagnetic Metals

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A simple quantum-mechanical theory of spin waves in nonferromagnetic metals has been developed. This theory consists of a description of the spin-wave excitations superimposed on the paramagnetic Hartree–Fock ground state of an interacting electron gas subjected to a constant uniform magnetic field. Use has been made of a gauge-independent density-matrix formalism. It is shown that to first order in the magnetic field the transport equation for the transverse spin magnetization has the same form as the corresponding phenomenological Fermi-liquid equation proposed by Silin and by Platzman and Wolff. It is also shown that for large fields, such that the cyclotron radius is smaller than the interelectronic distance, the term in the transport equation related to the cyclotron motion of the electrons is not affected by the exchange interactions. The theory presented here lends additional credence to the validity of Platzman and Wolff's phenomenological analysis of the experimental spin-wave spectra in the alkali metals.

I. INTRODUCTION

In 1958 Silin predicted the existence of spin-wave excitations in nonferromagnetic metals.[1] These excitations were subsequently observed by Schultz and Dunifer as a series of spin-wave sidebands in conduction-electron spin-resonance experiments performed on sodium and potassium.[2] Platzman and Wolff interpreted the data successfully by means of a phenomenological theory of transport for the transverse spin magnetization in a Landau Fermi liquid.[3] It was shown that this type of experiment provides a means to measure the important Fermi-liquid parameters B_0, $B_1, \ldots$, i.e., those parameters which determine the spin-dependent part $\zeta(\vec{k},\vec{k}')$ of the Fermi-liquid interaction function. Since it is the transport equation for the transverse magnetization itself that will be the subject of this paper, we will reproduce here Platzman and Wolff's transport equation for ease of discussion:

$$\frac{\partial\mu}{\partial t} + \left(\vec{v}\cdot\vec{\nabla} - \frac{e}{\hbar c}(\vec{v}\times\vec{H}_0)\cdot\vec{\nabla}_k - i\Omega_0\right)(\mu+\delta\epsilon_2) = \tfrac{1}{2}\gamma_0^2(\vec{v}\cdot\vec{\nabla} - i\Omega_0)\,h_+ \,. \quad (1)$$

Here, $\vec{v}$ is the quasiparticle velocity, $\vec{H}_0$ is the applied constant magnetic field (taken in the z direction),

$$\gamma_0 \equiv ge\hbar/2mc, \qquad \Omega_0 = \gamma_0 H_0/(1+B_0)\,,$$

h_+ is the rf magnetic field $h_x + ih_y$, and $\delta\epsilon_2$ is the change in the quasiparticle energy resulting from a change in the distribution function. The quantity μ is defined as $m_+ = -(\partial f_0/\partial\epsilon_k)\mu$, where m_+ is the transverse spin magnetization $m_x + im_y$, f_0 is the equilibrium distribution, and ϵ_k is the unperturbed quasiparticle energy. For simplicity we have omitted in Eq. (1) terms due to orbital and spin-flip collisions, since these will not play a role in the following considerations.

Equation (1) is very important, because it describes the propagation of spin waves in nonferromagnetic metals. In particular, significant information about the electron-electron interaction in sodium and potassium has already been obtained by analyzing the experimental spin-wave spectra of these metals with the help of this equation. However, Eq. (1) is also phenomenological. A first-principles quantum-mechanical derivation is highly desirable, so that one may have added confidence in the experimentally determined values of the Fermi-liquid parameters. Van Zandt, in a recent series of papers,[4] has attempted to justify Eq. (1) by generalizing Thomas's theory[5] of transport in a magnetic field to an interacting electron gas. This author has reached the conclusion that in Eq. (1) the term

$$(e/\hbar c)(\vec{v}\times\vec{H}_0)\cdot\vec{\nabla}_k(\mu + \delta\epsilon_2)$$

is incorrect and should be replaced by

$$(e/mc)(\vec{k}\times\vec{H}_0)\cdot\vec{\nabla}_k\,\mu.$$

This is rather puzzling, since this fact would seem to indicate that the cyclotron motion of a quasiparticle (which gives rise to the $\vec{k}\times\vec{H}_0$ as well as of course to the $\vec{v}\times\vec{H}_0$ term) is not affected by the electron-electron interaction.[6] Other authors, however, have argued that it is the quasiparticle mass that is measured in a cyclotron-resonance experiment,[7,8] apparently implying the correctness of the $\vec{v}\times\vec{H}_0$ term in Eq. (1).

The purpose of this paper is to describe an alternative attempt to justify Eq. (1). Our method consists of studying spin-wave excitations superimposed on the paramagnetic Hartree–Fock ground state of an interacting electron gas by means of a gauge-independent density-matrix technique (Sec. II). If the deviations from this ground state due to the spin-wave excitations are sufficiently small, one should obtain a reasonably good approximation to the predictions of the more accurate Fermi-liquid theory. We shall find that the terms in the equation of motion for the transverse magnetization which result from the cyclotron motion of the electrons, are negligibly affected by the exchange interactions when the applied field H_0 is of the order of or higher than ϵ_F/μ_B, ϵ_F being the Fermi energy. This is understandable, since this condition implies that the free-particle cyclotron radius (mv_F/eH_0) is of the order of or smaller than the interparticle distance. If the field is this large, each individual electron is spiraling about $\vec{H}_0$, without ever being influenced by the exchange interactions with the other electrons. This simple argument of course tacitly assumes that the electrons are not diffusing away from their respective field lines due to orbital scattering. However, in this paper we are not interested in the effects of orbital scattering, but rather in the orbital driving term in the transport equation for the transverse magnetization. If, now, the magnetic field is lowered, the cyclotron orbits will become larger and the electron wave functions will begin to overlap. We should expect (perhaps naively) that then the cyclotron motion of an individual electron will be affected by the exchange interactions with the other electrons. Our theory shows that this is precisely what happens. Moreover, we shall demonstrate that our transport equation for the quantity μ has, to first order in H_0, exactly the same form as Eq. (1). Higher-order terms are expected to become important only at hitherto experimentally unattainable fields. We believe therefore that our theory lends additional credence to the values of the Fermi-liquid parameters for sodium and potassium obtained from experimental spin-wave data employing Platzman and Wolff's method of analysis.

Since our formalism is much simpler than the elaborate (representation-free) operator treatment of Ref. 4, a detailed comparison of the two approaches would be a rather complicated matter and will not be attempted in this paper.

II. THEORY OF THE TRANSVERSE SPIN MAGNETIZATION

Our model consists of a uniform interacting electron gas subjected to a uniform constant magnetic field $\vec{H}_0$ (taken in the z direction). We also apply a small x-directed field $\vec{H}_1(\vec{r}, t)$ to this system. This latter field depends on the spatial coordinate $\vec{r}$ as well as on time. We write the vector potential describing the constant magnetic field as

$$\vec{A}(\vec{r}) = H_0(sy,\, x+sx,\, 0) + \vec{\nabla}\lambda(\vec{r})\,, \tag{2}$$

where s is an arbitrary real number and $\lambda(\vec{r})$ is an arbitrary twice differentiable function (so that $\vec{\nabla}\times\vec{A}$ is defined). The equation of motion for the single-particle density matrix $\rho(\vec{x},\vec{x}')$ is then[9]

$$i\hbar\frac{\partial}{\partial t}\rho(\vec{x},\vec{x}') = [\mathcal{H}_1(x) - \mathcal{H}_1^\dagger(\vec{x}')]\rho(\vec{x},\vec{x}') + N\int [V(\vec{r}-\vec{r}_1) - V(\vec{r}'-\vec{r}_1)]\rho_{(2)}(\vec{x},\vec{x}';\vec{x}_1,\vec{x}_1)\,d^3x_1\,, \tag{3}$$

where

$$\mathcal{H}_1(\vec{x}) = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\vec{A}(\vec{r})\right)^2 1 + \mu_e\vec{\sigma}\cdot(\vec{H}_0+\vec{H}_1)\,. \tag{4}$$

1 and $\{\sigma_i\}$ are the unit matrix and the Pauli spin matrices, respectively. μ_e is the intrinsic magnetic moment of the free electron. $V(\vec{r}-\vec{r}\,')$ is the Coulomb interaction energy between two electrons, and $\rho_{(2)}$ is the two-particle density matrix. The coordinate $\vec{x}$ includes both the spatial coordinate $\vec{r}$ and the spin coordinate. The integral over $\vec{x}_1$ implies an integration over $\vec{r}_1$ and a sum over spin. In Eq. (3) we have neglected (as is usually done) the effect of the field $\vec{H}_1$ on the orbital motion of the electrons. Otherwise, Eq. (3) is exact. We now make the Hartree–Fock approximation by setting[10]

$$\rho_{(2)}(\vec{x},\vec{x}\,';\vec{x}_1,\vec{x}_1)=\rho(\vec{x},\vec{x}\,')\,\rho(\vec{x}_1,\vec{x}_1)-\rho(\vec{x},\vec{x}_1)\,\rho(\vec{x}_1,\vec{x}\,')\,. \tag{5}$$

Before proceeding it proves convenient to express the single-particle density matrix in a somewhat different form. Wilson has shown that in the absence of $\vec{H}_1$ and electron-electron interactions the equilibrium density matrix has the form

$$\rho(\vec{r},\vec{r}\,')=\exp[-ie\vec{H}_0\cdot(\vec{r}\,'\times\vec{r})/2\hbar c]F(\vec{r}-\vec{r}\,')\,,$$

where F is a translationally invariant function.[11] We also introduce a function $F(\vec{x},\vec{x}\,')$ defined as

$$\rho(\vec{x},\vec{x}\,')=e^{i\theta(\mathbf{r},\mathbf{r}')}F(\vec{x},\vec{x}\,')\,, \tag{6}$$

where the phase

$$\theta(\vec{r},\vec{r}\,')=(e/\hbar c)\left\{\tfrac{1}{2}[\vec{A}_0(\vec{r}\,'+\vec{r})]\cdot(\vec{r}\,'-\vec{r})+\lambda(\vec{r}\,')-\lambda(\vec{r})\right\} \tag{7}$$

is the appropriate generalization of Wilson's phase conforming with the class of vector potentials given in Eq. (2). The function $F(\vec{x},\vec{x}\,')$ is not translationally invariant, since $\vec{H}_1$ depends on position. It should be noted that the diagonal elements of ρ and F are identical since $\theta(\vec{r},\vec{r})=0$.

At this point the introduction of the function F seems rather arbitrary. However, the importance of the function becomes apparent when one substitutes Eqs. (6) and (7) into Eq. (3) in order to obtain the equation of motion for F. Using Eqs. (4) and (5) we find

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}F(\vec{x},\vec{x}\,')=&\left(-\frac{\hbar^2}{2m}(\vec{\nabla}^2_r-\vec{\nabla}^2_{r'})+\frac{e\hbar}{2imc}\vec{H}_0\cdot[(\vec{r}-\vec{r}\,')\times(\vec{\nabla}_r-\vec{\nabla}_{r'})]\right)F(\vec{x},\vec{x}\,')\\&+\mu_e[\vec{\sigma}\cdot(\vec{H}_0+\vec{H}_1),F(\vec{x},\vec{x}\,')]+N\int[V(\vec{r}-\vec{r}_1)-V(\vec{r}\,'-\vec{r}_1)]F(\vec{x},\vec{x}\,')F(\vec{x}_1,\vec{x}_1)\,d^3x_1\\&-N\int e^{i\gamma}[V(\vec{r}-\vec{r}_1)-V(\vec{r}\,'-\vec{r}_1)]F(\vec{x},\vec{x}_1)F(\vec{x}_1,\vec{x}\,')\,d^3x_1\,,\end{aligned} \tag{8}$$

where γ is given by

$$\gamma=(e/2\hbar c)\vec{H}_0\cdot[(\vec{r}\,'-\vec{r}_1)\times(\vec{r}-\vec{r}_1)]\quad. \tag{9}$$

The first integral in Eq. (8) is due to the Hartree field, while the second integral results from the Fock-exchange interactions among the electrons.

Equation (8) is the key equation of this paper, since we will be able, as we shall show below, to obtain information from it about the transverse magnetization m_+. We try to solve it by means of a perturbation method. We set

$$F(\vec{x},\vec{x}\,')=F_0(\vec{x}-\vec{x}\,')+F_1(\vec{x},\vec{x}\,')\,, \tag{10}$$

where F_0 is the exact solution of Eq. (8) in the absence of the perturbing field $\vec{H}_1$. F_1 is the small deviation of the function F from its equilibrium value F_0, caused by the small field $\vec{H}_1$. We observe that the Fourier component $m_+(\vec{q})$ of the transverse magnetization m_+ can be expressed in terms of F_1 as

$$m_+(\vec{q})=-\frac{2\mu_e}{V}\int e^{i(\vec{k}+\vec{q})\cdot\vec{r}\,'}\,e^{-i\vec{k}\cdot\vec{r}}F_1(\vec{r}\downarrow,\vec{r}\,'\uparrow)\,d^3r\,d^3r'\,, \tag{11}$$

which follows from the fact that the total transverse magnetization density of electron gas is $M_+(\vec{r})=-2N\mu_e F_1(\vec{r}\downarrow,\vec{r}\uparrow)$. Here, we have made use of the fact that $\theta(\vec{r},\vec{r})=0$. This shows that indeed the transverse magnetization is related to the solutions of Eq. (8).

In the remainder of this section we shall explore the properties of the equation of motion for the function F, i.e., Eq. (8), in somewhat more detail. First of all we notice that Eq. (8) is independent of s and $\lambda(\vec{r})$, which implies that the function F is gauge independent, at least for those gauges defined by Eq. (2). Therefore, the magnetization m_+ will also be gauge independent. Next, we would like to draw attention to the factor $e^{i\gamma}$ occurring in the exchange term of Eq. (8), where γ is the field-dependent quantity given in Eq. (9). Suppose that the potential energy $V(\vec{r}-\vec{r}_1)$ is large in an interval $0<|\vec{r}-\vec{r}_1|<l_s$, where l_s is a length typically of the order of the screening length. If the magnetic field H_0 is so large that $e^{i\gamma}$ oscillates rapidly in this interval, then the exchange term will be negligibly small, thus confirming our speculation in the Introduction that the cyclotron motion is not affected by exchange interactions at sufficiently high fields. In order to assure a significant contribution of the exchange term we must require that $(eH_0l_s^2/2\hbar c)\ll 1$, which corresponds to the condition that the cyclotron orbits be larger than

the interelectronic distance. Finally, we wish to make contact with the transport equation for the magnetization [Eq. (1)]. To this end we expand $e^{i\gamma}$ to first order in the field and take matrix elements of Eq. (8) with respect to the plane-wave states $[\exp i\vec{k}\cdot\vec{r}]\downarrow$ and $\{\exp[i(\vec{k}+\vec{q})\cdot\vec{r}\,']\}\uparrow$, where $\downarrow$ and $\uparrow$ indicate the spin direction. Making use of Eqs. (10) and (11), we find to first order in $\vec{q}$

$$\frac{\partial\mu}{\partial t}+\left(-i\vec{v}\cdot\vec{q}-\frac{e}{\hbar c}\,(\vec{v}\times\vec{H}_0)\cdot\vec{\nabla}_k-i\Omega_0\right)(\mu+\delta\epsilon_2)$$

$$=\tfrac{1}{2}\gamma_0^2(-i\vec{v}\cdot\vec{q}-i\Omega_0)\,h_+\,. \qquad (12)$$

Here, $\vec{v}=(1/\hbar)\vec{\nabla}_k\epsilon_k$, ϵ_k being the Hartree–Fock single-particle energy.[12] Ω_0 is the exchange-enhanced spin-precession frequency,[13] and $\delta\epsilon_2$ is given by

$$\delta\epsilon_2=[V/(2\pi)^3]\int d^3k'\,v(\vec{k}-\vec{k}')\,\mu(\vec{k}')\,\delta[\epsilon(\vec{k}')-\epsilon_F]\,. \qquad (13)$$

Here, $v(\vec{k})$ is defined as

$$V(\vec{r})=\textstyle\sum_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}\,v(\vec{k})\,.$$

Equation (12) has the same form as Eq. (1). They are not identical, because quantities such as v, Ω_0, and $\delta\epsilon_2$ are to be calculated with the methods of the Fermi-liquid theory when used in Eq. (1). However, when substituted into Eq. (12) they must be evaluated in the Hartree–Fock approximation. Nevertheless, predictions based on Eqs. (1) or (12) are qualitatively the same. Since the spin-wave excitations occur close to the Fermi surface, the velocity $\vec{v}$ in Eq. (12) is to be taken at the Fermi surface. A drawback of the Hartree–Fock approximation is that this velocity diverges. However, the divergence can be avoided by replacing (as is customarily done) the bare Coulomb interaction $V(\vec{r}-\vec{r}\,')$ by a screened interaction. Predictions of Eqs. (1) and (12) can then actually be made quantitatively the same by chosing an appropriately screened interaction, and we may reiterate our assertion in the Introduction that our theory provides additional confirmation of the validity of Platzman and Wolff's analysis of the spin-wave experiments in the alkali metals. In closing we remark that the (field-dependent) phase factor in the exchange term of Eq. (8) has led to the appearance of the $\vec{v}\times\vec{H}_0$ term in the transport equation for the magnetization. In the absence of this phase factor, the exchange term would not have contributed to the orbital term in the transport equation, and a $\vec{k}\times\vec{H}_0$ term would have resulted.

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