$$C_{\alpha\beta}=\sum_{\vec{R}}[\langle\mathcal{E}_\alpha(\vec{0})\mathcal{E}_\beta(\vec{R})\rangle-\langle\mathcal{E}_\alpha\rangle\langle\mathcal{E}_\beta\rangle] \quad (\alpha,\ \beta=x,\ y,\ z)\ . \quad (18)$$

The combination $C_{zz}+2C_{xy}$ is proportional to the specific heat and varies as $\Delta T^{-\alpha}$; but, since $\gamma_Q=2\phi+\alpha-2>\alpha$, all other linear combinations should diverge as $\Delta T^{-\gamma_Q}$. The only available series are for $S=\frac{1}{2}$; although very erratic, they are not inconsistent with $\gamma_Q\simeq 2(1.2)+(-0.1)-2\simeq 0.3$. It is gratifying that our tentative numerical estimates are in accord with the small-ϵ prediction, $1<\phi<\gamma$. However, derivation of longer series (including the XY case) is under way.

Note added in proof. The initial estimates from the longer series indicate $\phi\simeq 1.25$ with a smaller uncertainty. Details of the analysis will be published. An account of the exact calculation of the ϵ^2 terms (Ref. 7) has now appeared: K. G. Wilson, Phys. Rev. Letters 28, 548 (1972).

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Critical Exponents in Isotropic Spin Systems*

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Critical indices for isotropic systems of n-dimensional spins in $(d=4-\epsilon)$-dimensional lattices are calculated to order ϵ. All critical indices corresponding to perturbations of the spin probability distribution are given. Such perturbations might arise from the effects of external or crystal fields on the spin system.

Recently, Wilson and Fisher[1] calculated some critical exponents for the Ising model and the XY model for dimension $d=4-\epsilon$ with ϵ small. This calculation was based on the renormalization-group techniques for critical phenomena by Wilson.[2] Here we use this theory to calculate the critical indices for isotropic systems of n-dimensional spins.[3] $n=1$, 2, 3 corresponds to the Ising, the XY, and the Heisenberg model, respectively. We calculate *all* critical indices corresponding to perturbations of the probability distribution $\exp[-Q_k(\vec{y})]$ for the total normalized spin $\vec{y}$ of a block of length 2^k. These perturbations might arise from the effect of external fields or crystal fields. No perturbations which include space derivatives like the stress tensor are considered. The transformation properties under rotation of the spins are conserved by the renormalization procedure and therefore give the correspondence between the crystal fields and the perturbations of the probability distribution. We start from Wilson's recursion formula[4]

$$Q_{k+1}(\vec{y}) = -2^d \ln[I_k(2^{1-d/2}\vec{y})/I_k(0)], \quad (1)$$

$$I_k(\vec{z}) = \int d\vec{y} \exp[-y^2 - \tfrac{1}{2}Q_k(\vec{y}+\vec{z}) - \tfrac{1}{2}Q_k(-\vec{y}+\vec{z})]. \quad (2)$$

At criticality $Q_k(\vec{y})$ approaches a "fixed point" of the recursion formula $Q^*(\vec{y}) = \lim_{k\to\infty} Q_k(\vec{y})$. To first order in ϵ, one obtains an isotropic solution

$$Q^*(y) = r^* y^2 + u^*(y^2)^2, \quad (3)$$

with

$$r^* = -4(n+2)\epsilon \ln 2\,[3(n+8)]^{-1}, \quad u^* = \epsilon \ln 2\;(n+8)^{-1}. \quad (4)$$

A small perturbation $Q_k = Q^* + \delta Q_k$ gives rise to the linear response

$$\ln I_k(\vec{z}) - \ln I^*(\vec{z}) = -\langle \delta\tilde{Q}_k\rangle + \langle \tilde{A}\delta\tilde{Q}_k\rangle - \langle\tilde{A}\rangle\langle\delta\tilde{Q}_k\rangle, \quad (5)$$

with

$$\delta\tilde{Q} = \tfrac{1}{2}\delta Q(\vec{y}+\vec{z}) + \tfrac{1}{2}\delta Q(-\vec{y}+\vec{z}), \quad \tilde{A} = u^*[(y^2+z^2)^2 + 4(\vec{y}\vec{z})^2], \quad (6)$$

$$\langle\tilde{B}\rangle = \int d\vec{y}\,\tilde{B}\exp[-(1+r^*)y^2] \times\{\int d\vec{y}\exp[-(1+r^*)y^2]\}^{-1}. \quad (7)$$

The eigenfunctions

$$\lambda\delta Q_k(\vec{z}) = -2^d[\ln I_k(2^{1-d/2}\vec{z}) - \ln I^*(2^{1-d/2}\vec{z})] \quad (8)$$

lead to solutions

$$Q_k = Q^* + \lambda^k \delta Q_0. \quad (9)$$

If λ is smaller than unity, then the perturbation vanishes for $k\to\infty$, and the corresponding operator is thermodynamically irrelevant. If λ is larger than unity, then the perturbation leads away from the "fixed point" Q^* either to another fixed point $Q^{*\prime}$ or away from criticality. If $\lambda = 1$, then $Q^* + a\delta Q_0$ (with a infinitesimally small) might be a fixed point[5] too. We expect that for Baxter's eight-vertex model[6] the fixed points form a "fixed line." The exponent x of an operator O scaling like[7] r^{-x} is related to λ by

$$\log_2\lambda = d - x. \quad (10)$$

Therefore, a perturbation δQ_0 with $\lambda = 1$ (yielding a fixed line) corresponds to an operator scaling like r^{-d} in agreement with the prediction from operator algebra.[8]

The singular contribution to the expectation value of the operator O is proportional to $\tau^{\nu x}$, the singular contribution to the "susceptibility" $\int dr \times(\langle O\,O(r)\rangle - \langle O\rangle^2)$ is proportional to $\tau^{-\nu(d-2x)}$, and the conjugate field to O scales like $\tau^{\nu(d-x)}$, with $\tau = (T-T_c)/T_c$. Within the theory of scaling the exponents νx, $\nu(d-2x)$, $\nu(d-x)$ are commonly called[9] β, γ, Δ, respectively, if O is the magnetization, and they are defined $1-\alpha$, α, 1, respectively, if O is the energy density.

Now we consider the eigenvalue problem, Eqs. (5) and (8). Expanding $\delta\tilde{Q}$ in powers of y and evaluating the expectation values, we obtain

$$\langle\delta\tilde{Q}\rangle = \exp[\Delta/4(1+r^*)]\delta Q, \quad (11)$$

where Δ is the Laplace operator. Similar expressions can be derived for $\langle y^2\delta\tilde{Q}\rangle$, $\langle y^2y^2\delta\tilde{Q}\rangle$, and $\langle(\vec{y}\vec{z})^2\delta\tilde{Q}\rangle$. Substituting these expressions in Eqs. (5) and (8), we obtain, to order ϵ,

$$\lambda\delta Q(2^{d/2-1}\vec{z}) = 2^d\{1 - u^*[\tfrac{1}{2}z^2\Delta + z_i z_j \partial_i\partial_j + \tfrac{1}{4}(n+2)\Delta + \tfrac{1}{16}\Delta^2]\} \times\exp[\Delta/4(1+r^*)]\delta Q(\vec{z}). \quad (12)$$

The solutions of Eq. (12) are polynomials in z, since the operator on the right-hand side gives only contributions z^p, z^{p-2}, ... upon application on z^p. Since the operator is rotationally invariant, the solutions are of type

$$\delta Q_{ml} = P_{ml}(z^2)H_l(\vec{z}). \quad (13)$$

Here $H_l(\vec{z})$ is a harmonic polynomial of degree l in $\vec{z}$ (compare p. 237 of Ref. 10) defined by $\Delta H_l(\vec{z}) = 0$ and $H_l(\mu\vec{z}) = \mu^l H^l(\vec{z})$, whereas $P_{ml}(z^2)$ is a polynomial of degree m in z^2. Matching the highest power in z we find

$$\log_2\lambda_{ml} = d - x_{ml} = 4 - 2m - l + \epsilon(m + \tfrac{1}{2}l - 1) - \epsilon g_{ml}/(n+8), \quad (14)$$

$$g_{ml} = m(2m-2+n+2l) + (2m+l)(2m+l-1).$$

We note that for $\epsilon\to 0$ the eigenfunctions δQ_{ml} are the polynomials of the harmonic oscillator

$$\delta Q_{ml} = L_m^{(n/2+l-1)}(\tfrac{3}{4}z^2)H_l(\vec{z}),$$

where $L_m^{(j)}$ are the Laguerre polynomials (see p. 188 of Ref. 10). The degree of the polynomial δQ_{ml} is $2m+l$. The exponents $d - x_{ml}$ are listed in Table I for $2m+l \le 4$. We use the spectroscopic notations s, p, d, f, g for $l = 0$, 1, 2, 3, 4. The perturbation with the quantum numbers $ml = 0s$ corresponds to the operator 1, $0p$ corresponds to the magnetization, $1s$ corresponds to the energy density. From the corresponding exponents x one obtains, within the theory of scaling the critical exponents,

$$\alpha = (4-n)\epsilon/2(n+8) + O(\epsilon^2),$$

TABLE I. Exponents $d-x_{ml}$.

ml	$2m+l$	$d-x_{ml}$
$0s$	0	$4-\epsilon$
$0p$	1	$3-\frac{1}{2}\epsilon$
$1s$	2	$2-(n+2)\epsilon/(n+8)$
$0d$	2	$2-2\epsilon/(n+8)$
$1p$	3	$1-\frac{1}{2}\epsilon$
$0f$	3	$1+(n-4)\epsilon/[2(n+8)]$
$2s$	4	$-\epsilon$
$1d$	4	$-8\epsilon/(n+8)$
$0g$	4	$(n-4)\epsilon/(n+8)$

$$\beta=\tfrac{1}{2}-3\epsilon/2(n+8)+O(\epsilon^2)\,, \qquad (15)$$

$$2\nu=\gamma=1+(n+2)\epsilon/2(n+8)+O(\epsilon^2)\,.$$

For $n\to\infty$ one obtains the critical exponents $\alpha=(d-4)/(d-2)$, $\beta=\frac{1}{2}$, $\gamma=2/(d-2)$ for the spherical model to order ϵ. This checks against Stanley's proof[11] of the equivalence of the spherical model with a system of infinite-dimensional spins. A crystal field of type $H_l(\vec{y})$ gives rise to perturbations of type δQ_{ml}. Since $\lambda_{0l}\geq\lambda_{ml}$, the most singular contribution comes from $ml=0l$. Therefore the corresponding field scales like τ^{ϕ} with $\phi=\nu(d-x_{0l})$. An anisotropic interaction of type $(y_i^2-y_j^2)$ gives rise to a perturbation δQ_{0d}. The critical exponent ϕ of the corresponding field[12] is obtained from

$$\phi=\nu(d-x_{0d})=1+n\epsilon/2(n+8)+O(\epsilon^2)\,. \qquad (16)$$

According to this result the conjecture $\phi=1$ by Suzuki[13] is an underestimation, whereas the estimation $\phi\approx\gamma$ by Riedel and Wegner[12] is an overestimation. A more detailed discussion of ϕ will be given in the accompanying paper by Fisher and Pfeuty.[14] The perturbation $0g$ corresponds to a crystal field of cubic symmetry of type $y_1^4+y_2^4+y_3^4-3(y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2)$ for an isotropic Heisenberg model ($n=3$). The exponent $d-x_{0g}=-\frac{1}{11}\epsilon$ is exceptionally small. If higher-order terms in ϵ raise $d-x_{0g}$ to or above 0 for $d=3$, then such a crystal field is thermodynamically relevant; that is, the critical exponents may be changed by such a field.

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