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Critical Behavior of the Anisotropic n-Vector Model*

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The critical behavior of a system of n-component classical "spins" with anisotropic pair interactions is discussed using renormalization-group techniques for dimension $d=4-\epsilon$. To first order in ϵ the correlation and susceptibility exponents in the isotropic limit are $2\nu=\gamma=1+[(n+2)/2(n+8)]\epsilon$, while the anisotropy or crossover exponent is $\phi=1+[n/2(n+8)]\epsilon$. When $n\to\infty$ these expansions agree with exact spherical-model results. For $n=3$ and $d=3$ series expansions indicate $\phi\simeq1.2$ (compared with $\gamma\simeq1.38$).

In recent work[1] the critical exponents for generalized classical Ising and XY models were derived for dimension $d=4-\epsilon$ with ϵ small, by using renormalization-group techniques.[2] For $d>4$ the critical exponents assume classical mean-field values independent of ϵ; for $d<4$ the exponents 2ν and γ were calculated[1] exactly to order ϵ (and approximately to order ϵ^2). In this paper we report similar exact first-order calculations for the classical Heisenberg ($n=3$) and general n-component spin models, with anisotropic pairwise coupling. We also present, for the first time, an estimate based on series expansions, for the crossover exponent[3,4] ϕ for the standard anisotropic Heisenberg model ($n=3$, $d=3$).

Let $\vec{\sigma}(\vec{R})$ denote a classical vector "spin" at lattice site $\vec{R}$, with n continuously variable components $\sigma_\alpha(\alpha=1, 2, \ldots, n)$. The phase-space weight factor for a (noninteracting) spin is taken to be $\exp(-\frac{1}{2}|\vec{\sigma}|^2-\frac{1}{4}f|\vec{\sigma}|^4)$ with $f[=O(\epsilon)]$ small so that $\langle\sigma_\alpha^2\rangle_0=1+O(f)$. For simplicity[5] we consider only the anisotropic ferromagnetic-interaction Hamiltonian

$$\mathcal{H}_{\text{int}}=-\tfrac{1}{2}\sum_{\vec{R}}\sum_{\alpha=1}^{n}J_\alpha\mathcal{E}_\alpha(\vec{R}),\qquad \mathcal{E}_\alpha(\vec{R})=\sum_{\vec{\delta}}\sigma_\alpha(\vec{R})\sigma_\alpha(\vec{R}+\vec{\delta}), \tag{1}$$

in which $\vec{\delta}$ runs over the q nearest-neighbor lattice vectors. We introduce *anisotropy parameters* g_α by $J_\alpha=J_0(1+g_\alpha)>0$.

On rewriting $\sigma\sigma'$ as $\frac{1}{2}[\sigma^2+\sigma'^2-(\sigma-\sigma')^2]$, which introduces the square of a (discrete) gradient, and substituting $\sigma_\alpha(\vec{x})=(2dk_BT/qJ_\alpha)^{1/2}s_\alpha(\vec{x})$, we obtain a reduced Hamiltonian[1,6] $\mathcal{H}_0$, of the general form

$$\mathcal{H}_l\{\vec{s}(\vec{x})\}=-\int d\vec{x}\,[\tfrac{1}{2}|\nabla\vec{s}(\vec{x})|^2+Q_l(\vec{s}(\vec{x}))], \tag{2}$$

$$Q_l(\vec{s})=\sum_\alpha r_\alpha^{(l)}s_\alpha^2+\sum_{\alpha,\beta}u_{\alpha\beta}^{(l)}s_\alpha^2s_\beta^2+\cdots, \tag{3}$$

with (for $l=0$) the initial values

$$r_\alpha^{(0)}=d[(k_BT/qJ_\alpha)-1]\approx d(t-g_\alpha), \tag{4}$$

$$u_{\alpha\beta}^{(0)}=d^2f(k_BT)^2/q^2J_\alpha J_\beta\approx d^2f, \tag{5}$$

and $v_{\alpha\beta\gamma}^{(0)}=\cdots=0$. The approximate equalities hold to leading order in the g_α, and in the reduced temperature

$$t=(T-T_0)/T_0\quad\text{with}\quad k_BT_0=qJ_0. \tag{6}$$

The renormalization-group recursion formulas for deriving $\mathcal{H}_{l+1}$ from $\mathcal{H}_l$ by a momentum cutoff reduction factor of $b(>1)$, as found by Wilson,[2] now read[1,2]

$$Q_{l+1}(\vec{y})=-b^d\ln[I_l(b^{1-(d/2)}\vec{y})/I_l(\vec{0})], \tag{7}$$

$$I_l(\vec{z})=\int_{-\infty}^{\infty}dy_1\cdots\int_{-\infty}^{\infty}dy_n\exp[-|\vec{y}|^2-\tfrac{1}{2}Q_l(\vec{z}+\vec{y})-\tfrac{1}{2}Q_l(\vec{z}-\vec{y})]. \tag{8}$$

To leading order in the $u_{\alpha\beta}=O(\epsilon)$ one then finds, with $q_\alpha=1/(1+r_\alpha)$,

$$r'_\alpha=b^2[r_\alpha+2u_{\alpha\alpha}q_\alpha+\textstyle\sum_\gamma u_{\gamma\alpha}q_\gamma+O(\epsilon^2)], \tag{9}$$

$$u'_{\alpha\beta}=b^\epsilon[u_{\alpha\beta}-2u_{\alpha\beta}(u_{\alpha\alpha}q_\alpha^2+2u_{\alpha\beta}q_\alpha q_\beta+u_{\beta\beta}q_\beta^2)-\textstyle\sum_\gamma u_{\gamma\alpha}u_{\gamma\beta}q_\gamma^2+O(\epsilon^3)], \tag{10}$$

where the prime denotes superscript $(l+1)$, while on the right-hand side the superscript (l) has been dropped.

In (*A*) *the isotropic case* ($g_\alpha\equiv0$) the recursion formulas simplify since $r_\alpha=r$ and $u_{\alpha\beta}=u$. As in the Ising case ($n=1$)[1] one then finds a "Gaussian" fixed point, $u=0$, $r=0$, and associated classical exponent values. These exponents apply for $d>4$ ($\epsilon<0$) but when $d<4$ with $u^{(0)}>0$, this fixed point is unstable. Following Ref. 1, the stable n-vector fixed point is found to $O(\epsilon)$, to be

$$u^*=\bar{\epsilon}/(n+8),\quad r^*=u^*(n+2)/(1-b^{-2}), \tag{11}$$

where $\bar{\epsilon}=b^\epsilon-1\approx\epsilon\ln b$. The critical exponent ν is now calculated[1,2] by linearizing (9) and (10) about this fixed point and looking for solutions in which $\Delta r^{(l)}$ diverges as λ^l; one then has $\nu=\ln b/\ln\lambda$. This leads easily to

$$2\nu=1+[(n+2)/2(n+8)]\epsilon+O(\epsilon^2), \tag{12}$$

which is independent of b indicating its exactness to first order.[1,7] To this order we have[1] $\eta=0$ so that $\gamma = 2\nu + O(\epsilon^2)$. For $n=1$ (Ising) and $n=2$ (XY) the previous results are recaptured; for $n\to\infty$, which corresponds to the spherical model,[8] the formula is consistent with the exact result $2\nu=\gamma = (1-\frac{1}{2}\epsilon)^{-1}$ $(0 \le \epsilon < 2)$. The thermodynamic exponents α, $\beta, \cdots$ may be derived to $O(\epsilon)$ from the standard two-exponent scaling relations[4,9] and also agree with spherical-model results when $n\to\infty$.

For (B) the *anisotropic case* we may, with no loss of generality, suppose $g_\alpha = 0$ $(J_\alpha = J_0)$ for $\alpha \le m$, and $g_\alpha < 0$ $(J_\alpha < J_0)$ for $\alpha > m$, i.e., dominant m isotropy. We now study the recurrence relations (9) and (10) with the initial conditions (4) and (5) and vary t (i.e., T) to find the critical value $t_c(g_\alpha; f)$ which will yield an asymptotic fixed point. Now, recalling $u_0 \simeq d^2 f = O(\epsilon)$, the relations (10) give $u'_{\alpha\beta} = u_{\alpha\beta}[1+O(\epsilon)]$, so that the $u^{(l)}_{\alpha\beta}$ vary slowly with l. Conversely, (9) shows that, in general, $r'_\alpha \simeq b^2 r_\alpha$ so that $r^{(l)}_\alpha \sim b^{2l}$, which diverges rapidly (and yields no fixed point). However, if t is chosen so that $r_0 \simeq dt_c$ satisfies

$$(b^{-2}-1)r_0 \simeq \frac{2u_0}{1+r_0} + \sum_\gamma \frac{u_0(1+g_\gamma)^{-1}}{1+r_0-dg_\gamma}\ , \tag{13}$$

we find from (9) that the $r^{(l)}_\alpha$ for $\alpha \le m$ become slowly varying; but the $r^{(l)}_\beta$ for $\beta > m$ still diverge rapidly, forcing the coupling factors q_β to zero, and hence becoming thermodynamically irrelevant variables. After a relatively few iterations, therefore, we effectively obtain a *reduced*, m-isotropic set of recurrence relations (for α, $\beta \le m$). The appropriate fixed point is thence given by (11) with m replacing n. All the exponents are, likewise, just m-vector-like. This establishes the expected dominance of the m largest J_α $(=J_0)$ in determining the critical behavior. Previously this dominance has been demonstrated only by series-expansion techniques (for $n=3$, $d=3$).[10]

To discuss the case (C) of *weak anisotropy* we linearize the full recursion relations about the n-vector fixed point[11] (11) and look for solutions with $\Delta r^{(l)}_\alpha = c_\alpha \lambda^l$. To leading order this yields the eigenvalue equation

$$\lambda c_\alpha = b^2[(1-2u^*)c_\alpha - u^* \textstyle\sum_\gamma c_\gamma]\ , \tag{14}$$

which has a single nondegenerate root $\lambda^0 \approx b^2 \times[1-(n+2)u^*]$. This represents the dominant temperature instability, having a totally symmetric eigenvector, $c^0_\alpha \equiv 1$, and leads back to (12). In addition, there is an $(n-1)$-fold-degenerate eigenvalue $\lambda^1 \approx b^2[1-2u^*]$, with orthogonal eigenvectors $\sum_\alpha c^j_\alpha = 0$. With $g^j_\alpha = c^j_\alpha g$, these eigenvectors correspond to anisotropic perturbing spin operators of the form, say,

$$\mathfrak{Q}_1 = \sigma_x\sigma'_x - \sigma_y\sigma'_y\ ,$$
$$\mathfrak{Q}_2 = \sigma_z\sigma'_z - \tfrac{1}{2}(\sigma_x\sigma'_x + \sigma_y\sigma'_y)\ ,\ \cdots\ . \tag{15}$$

The exponent $\nu^1 = \ln b/\ln\lambda^1$ now describes the critical-point divergence of the correlation length ξ as $g^{-\nu^1}$ $(g\to 0)$. On introducing the crossover exponent ϕ through the scaling formula

$$\xi(T,\ g) \approx \Delta T^{-\nu} X[g/(\Delta T)^\phi] = g^{-\nu/\phi}\bar{X}[g/(\Delta T)^\phi]\ , \tag{16}$$

with $\Delta T = T - T_c$, we obtain

$$\phi = \nu/\nu^1 = 1 + [n/2(n+8)] + O(\epsilon^2)\ . \tag{17}$$

When $n\to\infty$ this is again consistent with the exact spherical-model result[3] $\phi = (1-\frac{1}{2}\epsilon)^{-1}$. In this limit $\phi=\gamma$ but for $n<\infty$ and small ϵ we evidently have $\phi<\gamma$. On the other hand, Suzuki's conclusion[12] $\phi=1$ is generally incorrect. As suggested by Wegner,[13] the error arises because Suzuki considers only the tensorially mixed operator $\sigma_z\sigma'_z$ in first order in g, rather than introducing the anisotropy through properly symmetrized operators like the $\mathfrak{Q}_j$. In that case $\langle \mathfrak{Q}_j\rangle_{g=0} = 0$ and one must go to second order in g. For completeness we note that the operators $\mathfrak{Q}_j$ all scale as[14] $r^{-\omega_Q}$, where the anomalous dimension[15] is $\omega_Q = d - (\phi/\nu)$, while the corresponding susceptibility χ_Q diverges (when $g=0$) with exponent $\gamma_Q = 2\phi - d\nu \approx \epsilon(n+4)/2(n+8) > \alpha$. A more complete analysis of the isotropic fixed point under all perturbations (of the single-spin weight factor) has been carried out independently by Wegner.[16]

Finally (D) we have made a *direct estimate of* ϕ *for the standard* ($d=3$, $n=3$) *anisotropic Heisenberg model* using series-expansion data.[17] The appropriate isotropic critical temperatures are known quite reliably[17,18] but the series for the free energy and susceptibility $\chi(g)$, are rather short (only five terms) so that our results must be considered tentative. With $\chi(g) = \sum_k a_k(g)K^k$ and $K = J/k_BT$, it is rewarding[4] to analyze the function[19] $(\partial\chi/\partial g)_0 \sim \Delta T^{-\gamma-\phi}$. Using the T_c and γ $(\simeq 1.38)$ estimates of Ref. 18 and standard ratio techniques, we find $\phi \simeq 1.2$ for fcc, bcc, and sc lattices with $S=\infty$. As usual the series for small S are rather irregular. Furthermore, independently of the estimates for T_c and γ the series $h_k = [d\ln a_k(g)/dg]_0$, yields the direct-ratio slope estimates $\phi_k = [(h_k/h_{k-1})-1](k-1+\Delta) \to \phi$. These are displayed, for stated choice of Δ, in Table I and support our over-all estimate $\phi \simeq 1.21 \pm 0.05$. We have also examined the partial energy fluctuations C_{zz} and C_{xy}, where

TABLE I. Estimates for the crossover exponent ϕ (see text) for the spin ∞ Heisenberg model ($d=3$, $n=3$).

Lattice	Δ	$k=2$	3	4	5
fcc	0.1	1.220	1.191	1.189	1.192
	0.2	1.331	1.248	1.227	1.220
bcc	0.1	1.288	1.175	1.208	1.189
sc	0	1.240	1.188	1.206	1.198

$$C_{\alpha\beta}=\sum_{\vec{R}}[\langle\mathcal{E}_\alpha(\vec{0})\mathcal{E}_\beta(\vec{R})\rangle-\langle\mathcal{E}_\alpha\rangle\langle\mathcal{E}_\beta\rangle] \quad (\alpha,\ \beta=x,\ y,\ z)\ . \quad (18)$$

The combination $C_{zz}+2C_{xy}$ is proportional to the specific heat and varies as $\Delta T^{-\alpha}$; but, since $\gamma_Q=2\phi+\alpha-2>\alpha$, all other linear combinations should diverge as $\Delta T^{-\gamma_Q}$. The only available series are for $S=\frac{1}{2}$; although very erratic, they are not inconsistent with $\gamma_Q\simeq 2(1.2)+(-0.1)-2\simeq 0.3$. It is gratifying that our tentative numerical estimates are in accord with the small-ϵ prediction, $1<\phi<\gamma$. However, derivation of longer series (including the XY case) is under way.

Note added in proof. The initial estimates from the longer series indicate $\phi\simeq 1.25$ with a smaller uncertainty. Details of the analysis will be published. An account of the exact calculation of the ϵ^2 terms (Ref. 7) has now appeared: K. G. Wilson, Phys. Rev. Letters 28, 548 (1972).

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[5]The present results will apply for any short-range potentials $J_\alpha(\vec{R}-\vec{R}')$ and can be extended to long-range power-law potentials.

[6]The reduced Hamiltonian $\mathcal{H}_0$ is the logarithm of the total Boltzmann factor (see Refs. 1 and 2).

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[19]In the notation of Refs. 17 we have $(\partial\chi/\partial g)_0\propto[K(\partial\overline{\chi}/\partial K)-\frac{3}{2}(\partial\overline{\chi}/\partial\eta)]_{\eta=1}$, where $\overline{\chi}=(kT/m^2)\chi_0$.

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Critical Exponents in Isotropic Spin Systems*

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Critical indices for isotropic systems of n-dimensional spins in $(d=4-\epsilon)$-dimensional lattices are calculated to order ϵ. All critical indices corresponding to perturbations of the spin probability distribution are given. Such perturbations might arise from the effects of external or crystal fields on the spin system.

Recently, Wilson and Fisher[1] calculated some critical exponents for the Ising model and the XY model for dimension $d=4-\epsilon$ with ϵ small. This calculation was based on the renormalization-group techniques for critical phenomena by Wilson.[2] Here we use this theory to calculate the critical indices for isotropic systems of n-dimensional spins.[3] $n=1$, 2, 3 corresponds to the Ising, the XY, and the Heisenberg model, respectively. We calculate *all* critical indices corresponding to perturbations of