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Numerical Constants for Isolated Vortices in Superconductors*

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For isolated vortex lines in high-κ, type-II superconductors, Abrikosov derived the expressions $B(0)=\kappa^{-2}(\ln\kappa+C_0)H_{c2}$ and $H_{c1}=\frac{1}{2}\kappa^{-2}(\ln\kappa+C_1)H_{c2}$, but the numerical values he provided for the constants C_0 and C_1 were previously found to violate an identity $C_1-C_0-\frac{1}{2}=C_\gamma>0$. The constants are reevaluated, giving $C_0=-0.282$, $C_1=0.497$, $C_\gamma=0.279$. Furthermore, for superconductors containing a high concentration of magnetic impurities, it was previously shown that the electric field generated at the center of an isolated vortex in flux-flow situations is proportional to H_{c2} and the flux-flow velocity v. The proportionality constant C_E in the high-κ limit is numerically evaluated here to be 0.951, which, together with the value for C_γ, determines the flux-flow resistivity $\rho_f=0.381\rho_n\langle B\rangle/H_{c2}$ in the low-applied-field limit when vortices are very far apart.

I. INTRODUCTION

In his famous paper on the magnetic properties of type-II superconductors, Abrikosov[1] studied isolated vortex lines in the high-κ (the Ginzburg-Landau-parameter) limit, and derived the expressions

$$B(0)=\kappa^{-2}(\ln\kappa+C_0)H_{c2}\ , \quad (1)$$

$$H_{c1}=\tfrac{1}{2}\kappa^{-2}(\ln\kappa+C_1)H_{c2}\ . \quad (2)$$

In these equations $B(0)$ is the local field at the vortex center, H_{c2} is the upper critical field for transition from the mixed state to the normal state, and H_{c1} is the lower critical field for initial flux penetration. The numerical values provided by Abrikosov for the two constants C_0 and C_1 are

$$C_1=+0.08\ ,\quad C_0=-0.18\ , \quad (3)$$

which have been widely quoted in books on the subject of superconductivity.[2]

Recently, in studying dynamic structure of vortices in superconductors for applied field $H\ll H_{c2}$, Hu and Thompson[3] derived an identity which in the high-κ limit reduced to the simple relation

$$C_1-C_0-\tfrac{1}{2}=C_\gamma\ , \quad (4)$$

where

$$C_\gamma=\int_0^\infty (df/dr)^2 r\,dr>0 \quad (5)$$

is the constant which Gor'kov and Kopnin[4] called γ. In Eq. (5), $f=f(r)$ is the order parameter normalized to unity far away from the vortex center and r is the radial distance measured from the center. Since Abrikosov's numbers in Eq. (3) make the left-hand side of Eq. (4) negative in contradiction to Eq. (5), one must conclude that at least one of Abrikosov's numbers is seriously in error.

One straightforward way to determine the constants is to solve for $B(0)$, H_{c1}, and C_γ at finite values of κ and then to extrapolate the results to the high-κ limit. This task has been partially accomplished by Harden and Arp,[5] since they have calculated H_{c1} up to $\kappa=50$. Equating their value for $(2\kappa^2H_{c1}/H_{c2})$ at $\kappa=50$ to $(\ln 50+C_1)$ one finds $C_1=0.486$, but its accuracy cannot be confidently determined without another independent calculation. The first purpose of this paper is, therefore, to devise some *direct* methods for calculating the constants C_0, C_1, and C_γ, and to provide our new values for these constants which do indeed satisfy the identity Eq. (4). (The calculation will also show that the value supplied by Gor'kov and Kopnin[4] for C_γ is wrong by about 18%.)

On the other hand, in Ref. 3, Hu and Thompson studied the flux-flow resistivity ρ_f at applied field $H \ll H_{c2}$ for superconductors containing a high concentration of magnetic impurities. They derived the following general formula for ρ_f:

$$\rho_f = v\langle B\rangle / j_t \quad , \tag{6}$$

$$j_t = \tfrac{1}{2} e\sigma v \int_0^\infty B^2(r)\, r\, dr + (\sigma v / 8e\zeta^2) C_\gamma + \sigma[\mathcal{E}(0) - \tfrac{1}{2} vB(0)] \tag{7}$$

[cf. Eqs. (23), (29) of Ref. 3], where e is the electronic charge, $\sigma = \rho_n^{-1}$ is the normal-state conductivity, v is the flux-flow velocity, $B(r)$ is the local magnetic field at a distance r from the vortex center, C_γ is defined in Eq. (5), ζ is the dynamic screening length for the electric field,[3] and $\mathcal{E}(0)$ is the electric field at the vortex center. Equation (7) written in this form is valid only to first order in v, so $B(r)$ may be regarded as the static value, and $\mathcal{E}(0)$ needs to be calculated only to first order in v. In Ref. 3, Eq. (27), it was shown that for $\kappa \gg 1$

$$\int_0^\infty B^2(r)\, r\, dr = (4e\kappa^2)^{-1} H_{c2} \quad . \tag{8}$$

Thus, if one further defines

$$C_E = (vH_{c2})^{-1}[\mathcal{E}(0) - \tfrac{1}{2} vB(0)] \quad , \tag{9}$$

Eqs. (6) and (7) may be combined into (using $2eH_{c2} = \xi^2$)

$$\rho_f = \left(\frac{1}{4\kappa^2} + \frac{\xi^2}{2\zeta^2} C_\gamma + C_E \right)^{-1} \rho_n \langle B\rangle / H_{c2} \quad . \tag{10}$$

For $\zeta \sim \xi \ll \lambda$ the first term is negligible. The constants C_γ and C_E were evaluated in Ref. 3 only by an approximation method, which amounts to replacing the exact $f(r)$ by the simple function $r/(r^2 + \xi^2)^{1/2}$. (See note added in proof.) Thus, the second purpose of this paper is to combine our more precise value of C_γ with a more accurate calculation of C_E, in order to improve quantitatively the previous prediction of ρ_f in the low-field limit.

In the following, we report the details of these calculations in two sections: Calculations of the constants C_0, C_1, and C_γ are presented in Sec. II, while calculations of C_E and ρ_f are in Sec. III.

II. CALCULATION OF CONSTANTS C_0, C_1, AND C_γ

To calculate the constants C_0, C_1, and C_γ, we may start with the equations[6]

$$\kappa^{-2} \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} f - Q^2 f + f(1-f^2) = 0 \quad , \tag{11}$$

$$\frac{d}{d\rho} \left(\frac{1}{\rho} \frac{d}{d\rho} (\rho Q) \right) = f^2 Q \quad , \tag{12}$$

$$b(\rho) \equiv \frac{\kappa B(\rho)}{H_{c2}} = \frac{1}{\rho} \frac{d}{d\rho} \rho Q \quad , \tag{13}$$

$$h_{c1} \equiv \frac{\kappa H_{c1}}{H_{c2}} = \frac{1}{2} \kappa \int_0^\infty (1 - f^2) \rho\, d\rho \quad , \tag{14}$$

where $\rho = r/\lambda$ and the boundary conditions are

$$f \to 0, \quad Q \to -(\kappa\rho)^{-1} + \tfrac{1}{2} b(0) \rho \quad \text{as } \rho \to 0 \tag{15}$$

and

$$f \to 1, \quad Q \to 0 \quad \text{as } \rho \to \infty \quad . \tag{16}$$

For $\kappa \gg 1$, we expect from Abrikosov's work[1] that

$$b(0) \simeq \kappa^{-1}(\ln\kappa + C_0) \quad , \tag{17}$$

$$h_{c1} \simeq (2\kappa)^{-1}(\ln\kappa + C_1) \quad . \tag{18}$$

We shall first calculate C_1. Defining

$$2\kappa h_{c1} = \mathcal{I}_1 + \mathcal{I}_2 \quad , \tag{19}$$

where

$$\mathcal{I}_1 \equiv \int_0^\infty K_1^2[(\rho^2 + \kappa^{-2})^{1/2}] \rho\, d\rho = \ln\kappa - 0.3841 \quad \text{as } \kappa \gg 1 \tag{20}$$

(K_n being the modified Bessel functions of imaginary argument which vanish exponentially at infinite argument), so that

$$\mathcal{I}_2 \equiv \int_0^\infty \{ \kappa^2(1 - f^2) - K_1^2[(\rho^2 + \kappa^{-2})^{1/2}] \} \rho\, d\rho \,, \tag{21}$$

we expect $\mathcal{I}_2$ to become a finite constant in the limit $\kappa \to \infty$, and

$$C_1 = \lim_{\kappa\to\infty} \mathcal{I}_2 - 0.3841 \quad . \tag{22}$$

The first term is evaluated as follows. First we define $f_0(\rho)$ to satisfy the equation

$$\kappa^{-2} \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} f_0 - \frac{1}{\kappa^2 \rho^2} f_0 + f_0 - f_0^3 = 0 \quad , \tag{23}$$

with the same boundary conditions as for $f(\rho)$. Then using the equations it is not difficult to verify that

$$\begin{aligned}
\lim_{\kappa\to\infty} \mathcal{I}_2 &= \lim_{\kappa\to\infty} \int_0^{\kappa^{-1/2}} \{ \kappa^2(1-f^2) - K_1^2[(\rho^2 + \kappa^{-2})^{1/2}] \} \rho\, d\rho \\
&= \lim_{\kappa\to\infty} \int_0^{\kappa^{-1/2}} \{ \kappa^2(1-f_0^2) - (\rho^2 + \kappa^{-2})^{-1} \} \rho\, d\rho \\
&= \lim_{\kappa\to\infty} \int_0^\infty \{ \kappa^2(1-f_0^2) - (\rho^2 + \kappa^{-2})^{-1} \} \rho\, d\rho \\
&\equiv \lim_{\kappa\to\infty} \mathcal{I}_2' \quad ,
\end{aligned} \tag{24}$$

where errors in each step have been estimated to be of order κ^{-1}. We further change our variable from ρ to $x = \kappa\rho$. Equation (23) then becomes

$$\frac{1}{x}\frac{d}{dx}\,x\frac{d}{dx}f_0-\frac{1}{x^2}f_0+f_0-f_0^3=0\ ,\qquad(25)$$

and $\mathcal{I}_2'$ takes the new form

$$\mathcal{I}_2'=\int_0^\infty[(1-f_0^2)-(x^2+1)^{-1}]x\,dx$$

$$=\int_0^\infty[x^2/(1+x^2)-f_0^2(x)]x\,dx\ ,\qquad(26)$$

showing that $\mathcal{I}_2'$ does not really depend on κ.

Equation (25) may now be solved numerically to find the function $f_0(x)$.[7] Since Eq. (26) is a very slowly convergent integral, we performed the integral numerically only up to a cutoff point x_c, beyond which we used the asymptotical solution for f_0[8]:

$$f_0=1-\tfrac{1}{2}x^{-2}-\tfrac{9}{8}x^{-4}-\tfrac{161}{16}x^{-6}+\dots\ .\qquad(27)$$

The result is $\mathcal{I}_2'=0.8809$, which gives

$$C_1=0.4968\ .\qquad(28)$$

To calculate C_0, first we write

$$\kappa b(0)=-\int_0^\infty\kappa(db/d\rho)\,d\rho\ .\qquad(29)$$

For $\rho\gg\kappa^{-1/2}$, we expect $b(\rho)\cong\kappa^{-1}K_0(\rho)$, $\kappa db(\rho)/d\rho\cong K_1(\rho)$. So, we define

$$\kappa b(0)=\mathcal{I}_3+\mathcal{I}_4\ ,\qquad(30)$$

with

$$\mathcal{I}_3\equiv\int_0^\infty(\rho^2+\kappa^{-2})^{-1/2}K_1[(\rho^2+\kappa^{-2})^{1/2}]\rho\,d\rho$$

$$=\ln\kappa+0.1159\ ,\qquad(31)$$

$$\mathcal{I}_4\equiv-\int_0^\infty\{\kappa(db/d\rho)+\rho(\rho^2+\kappa^{-2})^{-1/2}\times K_1[(\rho^2+\kappa^{-2})^{1/2}]\}d\rho$$

$$=-\int_0^\infty\{\kappa f^2Q+\rho(\rho^2+\kappa^{-2})^{-1/2}\times K_1[(\rho^2+\kappa^{-2})^{1/2}]\}d\rho\ ,\qquad(32)$$

where Eqs. (12) and (13) have been used. We again expect $\mathcal{I}_4$ to approach a constant as $\kappa\to\infty$ and

$$C_0=\lim_{\kappa\to\infty}\mathcal{I}_4+0.1159\ .\qquad(33)$$

We may then verify as before that

$$\lim_{\kappa\to\infty}\mathcal{I}_4=\mathcal{I}_4'\ ,\qquad(34)$$

where

$$\mathcal{I}_4'=\int_0^\infty[f_0^2(x)-x^2/(1+x^2)]x^{-1}dx\ .\qquad(35)$$

Equation (35) is a rapidly convergent integral and may be evaluated by direct numerical integration using the $f_0(x)$ found in this work. The result is $\mathcal{I}_4'=-0.3982$, giving

$$C_0=-0.2823\ .\qquad(36)$$

To check our numerical results with the identity Eq. (4), we need an independent evaluation of C_γ. This is easily done using our solution for f_0 since the integral in Eq. (5) is rapidly convergent. The result is

$$C_\gamma=0.2791\ ,\qquad(37)$$

which should be compared with the value 0.247 provided by Gor'kov and Kopnin.[4]

Equations (28), (36), and (37) constitute the main results of this section. The identity Eq. (4) is seen satisfied to all four significant digits. For the rest of this section, we wish to present a direct proof of Eq. (4) by using the expressions we derived for the constants. First we integrate Eq. (5) by parts and use Eq. (24) to write

$$C_\gamma=\int_0^\infty[f_0^2-x^{-2}f_0^2-f_0^4]x\,dx\ .\qquad(38)$$

Combining this with Eqs. (22), (24), (26), (33)–(35), and realizing the exact relation $(-0.3841)-(0.1159)=\mathcal{I}_1-\mathcal{I}_3=-\frac{1}{2}$, we find that proving Eq. (4) is equivalent to proving the simpler identity

$$\int_0^\infty(1-f_0^2)^2x\,dx=1\ ,\qquad(39)$$

which can be easily derived by multiplying both sides of Eq. (25) by $2x^2df_0/dx$ and then integrating from zero to infinity. Our numerical calculation satisfied Eq. (39) to within 10^{-6}

III. CALCULATION OF CONSTANT C_E AND FLUX-FLOW RESISTIVITY ρ_f

To calculate the constant C_E defined by Eq. (8), we start with the equations derived previously[9]:

$$\xi^2\nabla^2P=f^2P\quad\text{(valid to first order in } v)\ ,\qquad(40)$$

$$P=p(r)\sin\theta\ ,\qquad(41)$$

$$p(r)\to v/2er-[\mathcal{E}(0)-\tfrac{1}{2}vB(0)]r\quad\text{as } r\to0\ ,\qquad(42)$$

where P is the scalar potential in the gauge in which the order parameter is real. (r,θ) is a cylindrical coordinate system with its origin always located at the center of the moving vortex under consideration. The flux-flow velocity $\vec{v}$ is pointing in the $\theta=0$ direction, and the transport current $\vec{j}_t$ is in the $\theta=\frac{1}{2}\pi$ direction.

Substituting Eq. (41) into (40) and introducing $x=r/\xi$, the following equation for p results:

$$\frac{d^2p}{dx^2}+x^{-1}\frac{dp}{dx}-x^{-2}p=\left(\frac{\xi}{\zeta}\right)^2f_0^2p\ ,\qquad(43)$$

while the boundary condition becomes (using $2eH_{c2}=\xi^{-2}$)

$$p(x)\to(\xi vH_{c2})(x^{-1}-C_Ex)\quad\text{as } x\to0\ .\qquad(44)$$

In Eq. (43) we have replaced the exact $f(x)$ by $f_0(x)$ of Eq. (25), since we shall consider only the limit $\kappa\to\infty$, when the difference between f_0 and f is nowhere important in solving Eq. (43). Since $f_0(x)$ no longer depends on κ, one finds ζ/ξ the only parameter in Eq. (43). The constant C_E is, therefore, only a function of ζ/ξ, and stays finite

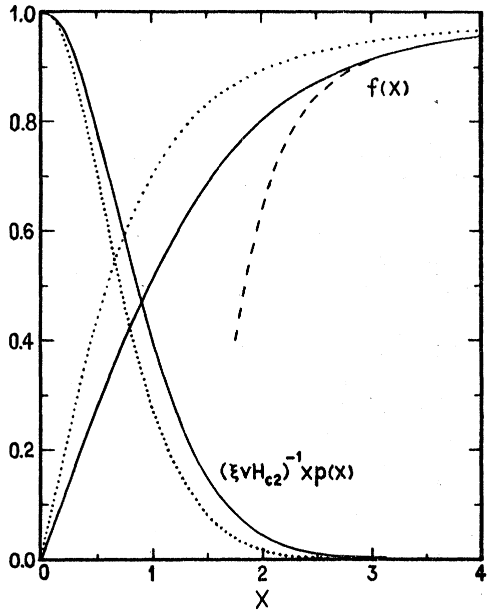

FIG. 1. Rising curves are plots of the order-parameter function $f(x)$ at $\kappa \gg 1$, normalized to 1 at infinite argument, where $r = x\xi$ is the radial distance from the center of an isolated vortex. The falling curves are plots of the function $(\xi\, vH_{c2})^{-1}xp(x)$, where $p(r/\xi)\sin\theta$ is the scalar potential generated around an isolated vortex of flux-flow velocity $\vec{v}$ in the $\theta = 0$ direction, in a gauge in which the order parameter is real. Solid lines, present numerical calculation. Dotted lines, analytic approximations obtained in Ref. 3 following an original idea of Schmid. Dashed line, asymptotic expression for $f(x)$ [cf. Eq. (27)], which is indistinguishable from the numerical result of $f(x)$ in this plot for $x \geq 3$.

in the limit $\kappa \to \infty$, as long as $\zeta \sim \xi \ll \lambda$. Putting this information and Eq. (1) into Eq. (9), we find that in the limit $\kappa \gg 1$

$$\mathcal{E}(0) = C_E\, vH_{c2} \gg vB(0) \ , \tag{45}$$

as long as $\zeta \sim \xi$, as has been deduced through an approximate analysis in Ref. 3.

By studying the combination $(\xi\, vH_{c2})^{-1}xp$ for normalization and to avoid the singularity at origin, Eqs. (43) and (44) are easily solved numerically to give the ζ/ξ dependence of C_E. Equation (9) is then used to determine the flux-flow resistivity in the low-applied-field limit. So far, we have treated the dynamic screening length ζ as an independent length. It has been pointed out in Ref. 3 that as far as we presently know, the only physical value of ζ is $\xi/\sqrt{12}$, corresponding to superconductors containing a high concentration of magnetic impurities. For this physical case, we find

$$C_E = 0.9513 \ , \tag{46}$$

$$\rho_f = 0.3808\ \rho_n\langle B\rangle / H_{c2} \ , \tag{47}$$

and therefore

$$(H_{c2}/\rho_n)(d\rho_f/d\langle B\rangle)_{\langle B\rangle \to 0} = 0.3808 \ , \tag{48}$$

which should replace the approximate estimate 0.33 found in Ref. 3.

In Fig. 1, we have plotted the present numerical result for the function $(\xi vH_{c2})^{-1}xp(x)$ at $\zeta = \xi/\sqrt{12}$, its analytic approximation found in Ref. 3,[10]

$$(x^2+1)^{1/2} K_1[2\sqrt{3}(x^2+1)^{1/2}]/K_1(2\sqrt{3}),$$

as well as our numerical result for the function $f_0(x)$, the Schmid's approximate form $x/(1+x^2)^{1/2}$, and the asymptotic expression Eq. (27). In Fig. 2, the function $C_E(\zeta/\xi)$, its analytic approximation[11] $(\xi/2\zeta)K_0(\xi/\zeta)/K_1(\xi/\zeta)$, the normalized initial slope of flux-flow resistivity $(H_{c2}/\rho_n)(d\rho_f/d\langle B\rangle)_{\langle B\rangle\to 0}$, and its analytic approximation[12] $[(\xi^2/8\zeta^2) + (\xi/2\zeta)K_0(\xi/\zeta)/K_1(\xi/\zeta)]^{-1}$, are plotted versus ζ/ξ in a semilogarithmic scale.[13] The normalized initial slope of the flux-flow resistivity is seen to reach 1 for a ζ slightly above $\frac{1}{2}\xi$, at which point the normalized slope near H_{c2} is equal to $1.72 > 1$.[14] Apparently for a narrow range of ζ/ξ ratios near $\frac{1}{2}$ the flux-flow resistivity-versus-$\langle B\rangle$ curves will exhibit a shallow S shape, but these cases are not observable in practice, since these ζ/ξ values do

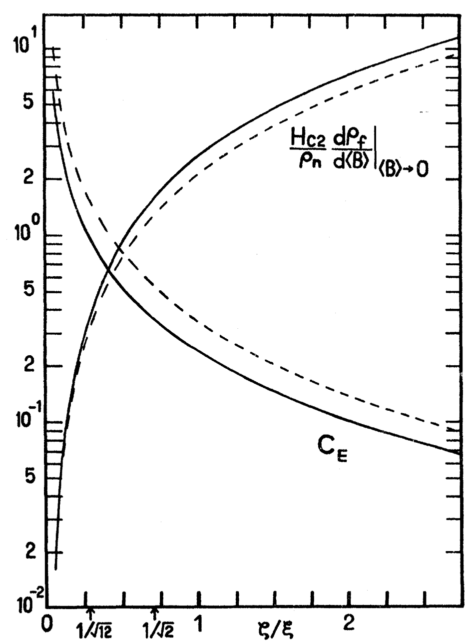

FIG. 2. ζ/ξ dependence of the constant C_E, defined through Eq. (9) (the monotonically decreasing curves), and of the normalized initial slope of the flux-flow resistivity with respect to average magnetic field $(H_{c2}/\rho_n)(d\rho_f/d\langle B\rangle)_{\langle B\rangle\to 0}$ (the monotonically increasing curves). Solid lines, present numerical calculation. Dashed lines, analytic approximation obtained in Ref. 3 following an original idea of Schmid.

not correspond to physical systems, at least not according to our present knowledge.

Finally, in Ref. 3, Sec. IV, the current at the center of a moving vortex, $j(0)$, has been studied in the limit $\kappa \gg 1$ and compared with the average transport current j_t which causes the flux flow. The main results of that section are Eqs. (37) and (38) of that work and the value of ζ at which $j(0) = j_t$. These results must also be slightly modified due to the present more precise values of the constants. Omitting the details, we report the corrected relations:

$$j(0) = (1 + 6C_\gamma/C_E)^{-1} j_t = 0.362 j_t \quad \text{when } \zeta = \xi/\sqrt{12}\,, \tag{49}$$

$$j(0) = [B(0)/H_{c1}] j_t \approx 2 j_t \quad \text{when } \zeta = \lambda \;; \tag{50}$$

and $j(0) = j_t$ when

$$\zeta = C_\gamma^{1/2} \lambda / (\ln\kappa + C_0 - \tfrac{1}{2})^{1/2} = 0.528\, \lambda/(\ln\kappa - 0.782)^{1/2} \tag{51}$$

All qualitative conclusions made in Ref. 3, Sec. IV, are obviously still valid after these modifications, while Eq. (49) is slightly more consistent with the extrapolation from the high-field results[15] than the corresponding relation in Ref. 3, Eq. (37).

Note added in proof. More generally, one may use $r/(r^2 + \delta\xi^2)^{1/2}$. In our previous paper (Ref. 3), we used the expression $4e\xi^2\lambda^2 H_{c1} = \int_0^\infty (1 - f_0^2) r\, dr$ to evaluate C_1, which requires the choice $\delta = 1$ to ensure the correct coefficient in front of the $\ln\kappa$ term. Schmid has pointed out to us that if one calculates H_{c1} via the original Ginzburg-Landau free-energy function, then a variational principle exists and it can be shown that $\delta = 2$ is the best choice. Since the choice $\delta = 2$ better approximates the behavior of $f(r)$ near the origin, it actually gives more accurate estimates to the constants discussed in this paper than if $\delta = 1$ is used. However, since the asymptotic behavior of $f(r)$ for large r is not correctly described by any choice other than $\delta = 1$, care must be exercised to avoid using the $\delta = 2$ approximation to evaluate any quantity which depends emphatically on the large-r behavior of $f(r)$.

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[6]See, for example, Ref. 2a, Chap. 14, Eqs. (150)–(153) and (167)–(170).

[7]Although we have the asymptotic behavior of f_0 at large x [See Eq. (27) of the main text], we have tried in vain to integrate Eq. (25) from a large x back to origin. We thus integrated the equation from origin and used a trial-and-error method to pinpoint the correct initial slope. It was discovered that the solution depended critically on the initial slope. Thus, for a trial slope slightly too large or too small, the solution would follow closely the physical behavior only for a while, and then it would either diverge to infinity or plunge down to oscillate about $f_0 = 0$. The best solution we have found approaches the asymptotic expression Eq. (27) from below to within 10^{-7} at $x \approx 16.5$ before deviating away from it again, and the initial slope for this solution is 0.583189.... Although to find such a solution we had to pinpoint all 16 digits of the initial slope in a double-precision calculation, we realize that the later digits do not characterize the physical solution, but depend on the step size and step error of our numerical integration.

[8]Our numerical study shows that this asymptotic expression starts to give (3, 4, 5, 6) significant digits at $x \approx (3, 7, 10.5, 12)$, respectively.

[9]See Ref. 3, Eq. (16) and the following paragraphs of discussion.

[10]Compare Eq. (41) with the expression for P in Ref. 3, Eq. (24), and use $2eH_{c2} = \xi^{-2}$, $\xi/\zeta = (12)^{1/2}$.

[11]Compare Eq. (45) with Ref. 3, Eq. (26).

[12]Ref. 3, Eq. (32) with the term $1/(4\kappa^2)$ ignored.

[13]This figure shows that the errors in the approximate forms are roughly 67–31% for C_E, and 0–23% for flux-flow resistivity, in the range $0.05 \le \zeta/\xi \le 2.75$ studied. The latter error is smaller because the underestimation of C_γ is partially cancelled by the overestimation of C_E in Eq. (10).

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