Tunneling Generation, Relaxation, and Tunneling Detection of Hole-Electron Imbalance in Superconductors*

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The theory of electron tunneling is extended to treat the generation and detection of an imbalance Q between electronlike and holelike quasiparticle populations in superconductors. The equilibration of injected high-energy quasiparticles owing to inelastic phonon processes is discussed, and it is shown that in tin, Q relaxes in $\tau_Q \approx [\Delta(0)/\Delta(T)](2\times10^{-10})$ sec. It is also shown that even a small ($\sim 1\%$) anisotropy of the gap can lead to significant Q relaxation by elastic scattering processes. This discussion extends our previous brief interpretation of the pair-quasiparticle potential difference measured by Clarke.

Clarke[1] has recently reported experimental observation of a potential difference V between pairs and quasiparticles in a nonequilibrium region of a superconductor, set up by the tunneling injection of quasiparticles. The interpretation of his results involved concepts and processes which have received little previous attention, apart from the work of Phillips.[2] In particular, Q, the difference in quasiparticle populations (per unit volume) of the electronlike and holelike branches of the BCS spectrum, turns out to be the crucial quantity. If the injection voltage is high, the injected quasiparticles go predominantly into one branch, and $Q \approx (I/e\Omega)\tau_Q$, where I is the injection current, Ω is the sample volume, and τ_Q is the relaxation time for the imbalance. Apart from a correction factor $Q^*/Q \approx 1$, the imbalance sets up a zero-voltage tunneling current proportional to Q. As shown by Tinkham and Clarke,[3] the observed potential difference is that required to null this current, namely,

$$V = I\tau_Q/2e^2\Omega N(0) g_{NS} ,$$

where g_{NS} is the normalized tunneling conductance of the normal probe and $N(0)$ is the usual density of states for electrons of one spin. The purpose of the present paper is to provide a more complete discussion of these effects than was possible in that brief report.

First, it is necessary to present the results of conventional tunneling theory for the unconventional case in which the quasiparticle populations are not necessarily in thermal equilibrium, and also distinguishing which branch of the quasiparticle spectrum is involved, rather than simply computing the total current. Second, we consider the relaxation of high-energy injected quasiparticles toward thermal equilibrium by the emission of phonons. Finally, we consider the relaxation of the population imbalance Q to determine τ_Q.

I. GENERALIZED TUNNELING RESULTS

In this section, we review the results of a standard "Golden Rule" calculation of the tunneling of electrons, with special attention to keeping the holelike and electronlike branches of the quasiparticle spectrum distinct. We take the usual[4] tunneling Hamiltonian

$$H' = \sum_{kq} (T_{kq} c_k^\dagger c_q + T_{qk}^* c_q^\dagger c_k) , \tag{1}$$

where the index k refers to the superconductor in question and q refers to the other metal, usually taken to be in the normal state. In (1) we have suppressed spin indices, because we assume no spins flip in the tunneling process. The first term describes a process which transfers an electron into the superconductor, whereas the second refers to the reverse process. We then reexpress (1) in terms of the quasiparticle operators appropriate to the superconducting state, using relations such as

$$c_k^\dagger = u_k \gamma_{ek0}^\dagger + v_k \gamma_{hk1} . \tag{2}$$

Here u_k and v_k are the usual BCS[5] parameters, taken real for simplicity, so that

$$u_k^2(\epsilon_k) = v_k^2(-\epsilon_k) = \tfrac{1}{2}(1 + \epsilon_k/E_k) , \tag{3}$$

where $E_k = (\Delta^2 + \epsilon_k^2)^{1/2}$, ϵ_k being the one-electron energy relative to the chemical potential, and Δ being the energy-gap parameter. Also we have

$$\gamma_{ek0}^\dagger = u_k c_{k\uparrow}^\dagger - v_k S^\dagger c_{-k\downarrow} = S^\dagger \gamma_{hk0}^\dagger$$

and

$$\gamma_{hk1} = u_k S^\dagger c_{-k\downarrow} + v_k c_{k\uparrow}^\dagger = S^\dagger \gamma_{ek1} \tag{4}$$

as the modified Bogoliubov–Valatin[6] quasiparticle operators introduced by Josephson[7] and by Bardeen.[8] In these expressions, $S^\dagger$ adds a pair to the condensate, while S removes one. Thus, $\gamma_{ek0}^\dagger$ definitely adds one electron, while $\gamma_{hk0}^\dagger$ defi-

nitely removes one, as is suggested by the subscripts e and h, for electron and hole. For the purposes of the present paper, it is vital to note that both types of operators exist for k both greater than and less than k_F, with a gradual changeover in character near k_F, where $u_k^2 = v_k^2 = \frac{1}{2}$. By contrast, when we refer to the electronlike and holelike *branches* of the quasiparticle spectrum, we refer to states with $k > k_F$ or $k < k_F$, respectively, with a discontinuous distinction being made. The physical motivation for making the latter dichotomy is that, as we shall show in detail, quasiparticles from one of these branches have little probability of being scattered onto the other (there is *no* probability in the normal state); thus it is useful to treat them as distinct populations.

When (1) is rewritten in terms of the appropriate quasiparticle operators, it becomes

$$H' = \sum_{kq} T_{kq} (u_k \gamma_{ek0}^\dagger + v_k \gamma_{hk1})(u_q \gamma_{eq0} + v_q \gamma_{hq1}^\dagger) + \text{H. c.} \tag{5}$$

All the explicit terms transfer an electron into the superconductor, while all terms in the Hermitian-conjugate expression do the reverse. If we now specialize to the case where the metal q is normal, we have $u_q = 0$ and $v_q = 1$ for $q < q_F$, and $u_q = 1$ and $v_q = 0$ for $q > q_F$, and (5) simplifies to

$$\begin{aligned} H' = \sum_{k, q<q_F} & [T_{kq}(u_k \gamma_{ek0}^\dagger + v_k \gamma_{hk1}) \gamma_{hq1}^\dagger \\ & + T_{qk}^* \gamma_{hq1} (u_k \gamma_{ek0} + v_k \gamma_{hk1}^\dagger)] \\ & + \sum_{k, q>q_F} [T_{kq} (u_k \gamma_{ek0}^\dagger + v_k \gamma_{hk1}) \gamma_{eq0} \\ & + T_{qk}^* \gamma_{eq0}^\dagger (u_k \gamma_{ek0} + v_k \gamma_{hk1}^\dagger)] \ . \end{aligned} \tag{6}$$

Note that all these operators are in the excitation representation, so that both $\gamma_{ek}^\dagger$ and $\gamma_{hk}^\dagger$ create excitations with positive energy $E_k = (\Delta^2 + \epsilon_k^2)^{1/2}$, which reduces to $|\epsilon_q|$ in the normal metal. The eight distinct terms in (6) give rise to transitions with relative probabilities as given in Table I. This table also gives the changes in number of electrons and of excitations in the superconductor, and the constraint on E_q imposed by the conservation of energy. Inspection of this table shows that the eight cases can be reduced to four by converting the description of the normal metal from the excitation representation to the particle representation. That is, the expressions for $q > q_F$ can be applied also for $q < q_F$, if they are expressed in terms of ϵ_q, rather than $E_q = |\epsilon_q|$. This simply changes the sign of the energy for $q < q_F$, and thus also takes $f(E_q)$ into $1 - f(\epsilon_q)$. Using the conservation of energy to eliminate ϵ_q, we then can reduce the possibilities to those listed in Table II. In writing this, we have assumed that the normal metal can be characterized by a Fermi distribution, so that the occupation numbers are known functions of energy. This assumption has, however, *not* been made for the superconductor, a distinction made formally by retaining the notation f_k, referring to the actual occupation number, even if it is a nonequilibrium distribution.

TABLE I. Analysis of tunneling processes in excitation representation.

	Term	Probability	Electrons added	Excitations added	E_q
$q < q_F$	$\gamma_{eko}^\dagger \gamma_{hq1}^\dagger$	$u_k^2(1-f_k)(1-f_q)$	1	1	$-E_k + eV$
	$\gamma_{hk1} \gamma_{hq1}^\dagger$	$v_k^2 f_k (1-f_q)$	1	−1	$E_k + eV$
	$\gamma_{eko} \gamma_{hq1}$	$u_k^2 f_k f_q$	−1	−1	$-E_k + eV$
	$\gamma_{hk1}^\dagger \gamma_{hq1}$	$v_k^2 (1-f_k) f_q$	−1	1	$E_k + eV$
$q > q_F$	$\gamma_{eko}^\dagger \gamma_{eq0}$	$u_k^2 (1-f_k) f_q$	1	1	$E_k - eV$
	$\gamma_{hk1} \gamma_{eq0}$	$v_k^2 f_k f_q$	1	−1	$-E_k - eV$
	$\gamma_{eko} \gamma_{eq0}^\dagger$	$u_k^2 f_k (1-f_q)$	−1	−1	$E_k - eV$
	$\gamma_{hk1}^\dagger \gamma_{eq0}^\dagger$	$v_k^2 (1-f_k)(1-f_q)$	−1	1	$-E_k - eV$

TABLE II. Simplified analysis of tunneling processes.

Probability	Electrons added	Excitations added
$u_k^2(1-f_k) f(E_k - eV)$	1	1
$v_k^2 f_k [1 - f(E_k + eV)]$	1	−1
$u_k^2 f_k [1 - f(E_k - eV)]$	−1	−1
$v_k^2 (1-f_k) f(E_k + eV)$	−1	1

Generalized Results for Tunnel Current

We now use the results of Table II to write down several useful results. For simplicity, we assign a common value $|T|^2$ to all relevant $|T_{kq}|^2$. Then, using the Golden Rule formula and cancelling terms which subtract electrons against those that add, the net electric current into the superconductor is given by

$$\begin{aligned} I = & \frac{4\pi e}{\hbar} |T|^2 N_k(0) N_q(0) \\ & \times \sum_{\text{branches}} \int_\Delta^\infty \mathfrak{N}(E_k) \{u_k^2 [f(E_k - eV) - f_k] \\ & + v_k^2 [f_k - f(E_k + eV)]\} \, dE_k \ . \end{aligned} \tag{7}$$

In this,

$$\mathfrak{N}(E_k) = \frac{E_k}{(E_k^2 - \Delta^2)^{1/2}} = \frac{E_k}{|\epsilon_k|} \quad (E_k > \Delta) \tag{8}$$

is the usual normalized BCS density of states.

A factor of 2 has been included in (7) for the two spin orientations. The sum is over the two branches $k>k_F$ and $k<k_F$. Using the symmetry relation (3) between $u_{k>}^2$ and $v_{k<}^2$, the contributions of the two branches in (7) can be combined to give

$$I=\frac{4\pi e}{\hbar}|T|^2N_k(0)N_q(0)\times\int_\Delta^\infty\{(f_{k<}-f_{k>})+\mathfrak{N}(E)\times[f(E-eV)-f(E+eV)]\}dE\ , \quad (9)$$

where we have used the fact that

$$u_{k>}^2-v_{k>}^2=-u_{k<}^2-v_{k<}^2=\frac{|\epsilon_k|}{E_k}=\mathfrak{N}^{-1}(E_k)\ .$$

If f_k is a function only of energy, as it is in thermal equilibrium, then $f_{k<}=f_{k>}$ (since both refer to states of the same E_k), and the usual[4] expression for the normal-superconductor (NS) tunneling current is obtained:

$$I=\frac{G_{NN}}{e}\int_\Delta^\infty\mathfrak{N}(E)[f(E-eV)-f(E+eV)]dE\ , \quad (10)$$

where

$$G_{NN}=\frac{4\pi e^2}{\hbar}|T|^2N_k(0)N_q(0)$$

is the tunnel conductance of the junction if both metals are in the normal state. If $eV\ll kT$, we may expand the Fermi functions about $V=0$ and obtain

$$I=G_{NN}V\int_\Delta^\infty 2\mathfrak{N}(E)\frac{-\partial f}{\partial E}dE\equiv G_{NS}V\ . \quad (11)$$

On the other hand, at $T=0$, (10) reduces to

$$I=(G_{NN}/e)[(eV)^2-\Delta^2]^{1/2}\ . \quad (12)$$

In the novel situation in which there is an imbalance in population between the holelike and electronlike branches, so that $f_{k<}\neq f_{k>}$, there is an additional term in the current. This may be isolated by considering the short-circuited case in which $V=0$, so that the usual part of the current (10) vanishes. Then (9) becomes

$$I(V=0)=(G_{NN}/e)\int_\Delta^\infty(f_{k<}-f_{k>})dE=-G_{NN}Q^*/2N_k(0)e\ , \quad (13)$$

where

$$Q^*\equiv 2N_k(0)\int_\Delta^\infty(f_{k>}-f_{k<})dE. \quad (14)$$

This quantity Q^* differs from the quasiparticle population imbalance in unit volume

$$Q=2N_k(0)\int_\Delta^\infty\mathfrak{N}(E)(f_{k>}-f_{k<})dE \quad (15)$$

only by a factor of $|u_k^2-v_k^2|=\mathfrak{N}^{-1}(E_k)$, which reflects the fact that occupation of quasiparticle states near k_F shifts the expectation value of the charge by less than a full unit. The zero-voltage current (13) is characteristic of the nonequilibrium situation. If it is nulled out by applying a small voltage V ($\ll kT$), it is clear from the above that

$$V=-I(V=0)/G_{NS}=Q^*/2N_k(0)eg_{NS}\ , \quad (16)$$

where $g_{NS}=G_{NS}/G_{NN}$ is the normalized tunneling conductance of the junction. This potential V is what was observed by Clarke[1] in his null measurement.

Creation of Branch Imbalance by Tunnel Injection

Since we have seen that the branch imbalance determines the observed potential, we now turn our attention to computing how this imbalance is created by the tunneling injection of quasiparticles. Referring to Table II, we see that processes proportional to v_k^2 enter with opposite sign in changing the number of excitations compared to their effect in transferring electrons. This is so because these terms deal with the $\gamma_{hk}^\dagger$ operators, where adding an excitation removes an electron. Accordingly, in the expression for $\dot{Q}$ which corresponds to the expression (7) for the current, v_k^2 is replaced by its negative. Also, since Q represents the *difference* between the two branch populations, the summation over the two branches in (7) is replaced by a difference. When these changes are followed through, (9) is replaced by

$$\dot{Q}=\frac{G_{NN}}{e^2\Omega}\int_\Delta^\infty\{[f(E-eV_{inj})-f(E+eV_{inj})]+\mathfrak{N}(E)(f_{k<}-f_{k>})\}dE\ . \quad (17)$$

The second term vanishes in equilibrium, and the departures from equilibrium are always very small compared to the first term, since the injection voltage V_{inj} is typically millivolts, while the nonequilibrium voltage V is typically nanovolts. Thus, we may safely drop the second term in (17). Then, simple analytic results can be obtained in a number of limiting cases:

$$\dot{Q}=\frac{G_{NN}}{e^2\Omega}2f(\Delta)eV_{inj}\quad(eV_{inj}\ll kT)\ , \quad (18a)$$

$$\dot{Q}=\frac{G_{NN}}{e^2\Omega}(eV_{inj}-\Delta)\quad(T=0)\ , \quad (18b)$$

$$\dot{Q}=\frac{G_{NN}}{e^2\Omega}\left\{eV_{inj}-\Delta\left[1-2f\left(\frac{eV_{inj}}{kT}\right)\right]\right\}\times(\Delta\ll kT)\ . \quad (18c)$$

It is also useful to note that the ratio $\Omega eQ/I$, characterizing the degree of imbalance of the injection, is given by

$$\frac{\Omega e\dot{Q}}{I} = \frac{\int_\Delta^\infty [f(E-eV_{inj}) - f(E+eV_{inj})]\,dE}{\int_\Delta^\infty \mathfrak{N}(E)[f(E-eV_{inj}) - f(E+eV_{inj})]\,dE} \leq 1 . \quad (19a)$$

For small injection voltages, this ratio reduces to

$$\Omega e\dot{Q}/I = 2f(\Delta)/g_{NS} \quad (eV_{inj} \ll kT) . \quad (19b)$$

Since both numerator and denominator approach unity at T_c, so does the ratio. A more appropriate limit is that in which the injection voltage is large. An exact result is available at $T = 0$, where

$$\frac{\Omega e\dot{Q}}{I} = \left(\frac{eV_{inj} - \Delta}{eV_{inj} + \Delta}\right)^{1/2} \quad (T = 0) . \quad (19c)$$

This goes to zero if $eV_{inj} \to \Delta$, because in that case the injection goes equally into states just above and below k_F. However, for higher injection voltages, the ratio approaches the limit $1 - (\Delta/eV_{inj}) \approx 1$. Thus, for the high injection voltages used by Clarke,[1] it is quite a good approximation to take $\dot{Q} = I/e\Omega$ under all circumstances.

II. COOLING OF INJECTED QUASIPARTICLES

The injected quasiparticles will be distributed initially rather uniformly in energy from Δ to eV_{inj}, which is typically much greater than kT. The injected population will cool by phonon-emission processes, approaching a distribution characterized by the sample temperature T. Charge neutrality is maintained by drawing from the background equilibrium population a current of electrons equal to the injected current. In the superconducting state, these will be condensed pairs at μ_p; in the normal state, they will be from an energy range of order kT about μ. Since the details of this nonequilibrium process are complex, it is useful to make even a rough estimate of the speed of the process.

To do this, we first restrict attention to the normal state. This will be an excellent approximation near T_c, where $\Delta \to 0$, and it should be qualitatively useful at low temperatures as well. Our second major simplification is to take the temperature of the lattice and the background electrons to be $T = 0$. This will be a good approximation for the initial cooling, but of course will overestimate the cooling rate as equilibrium is approached. In this approximation, only spontaneous emission processes can occur. The probability per unit time of an energy loss between ϵ and $\epsilon + d\epsilon$ for an electron in a state of energy E (relative to μ) can be written

$$[1 - f(E-\epsilon)]\,\alpha\epsilon^2 d\epsilon . \quad (20)$$

The quadratic dependence on ϵ arises from the square of the electron-phonon matrix element and the appropriate density of final states, both proportional to ϵ for ϵ much less than the maximum energy loss $k\Theta$, while the constant α absorbs the absolute normalization factors. The factor $[1 - f(E-\epsilon)]$ takes account of the available density of empty final states. In the present approximation, this factor is 1 for $\epsilon < E$ and 0 for $\epsilon > E$, since all states below μ are filled. Denoting the nonequilibrium population by $g(E)$, we then have for $0 \leq E \leq k\Theta$

$$\frac{dg(E)}{dt} = \int_0^\infty \alpha\epsilon^2 g(E+\epsilon)\,d\epsilon - \int_0^E \alpha\epsilon^2 g(E)\,d\epsilon$$

$$= \alpha g(E)\left[\int_0^\infty \epsilon^2 \left(\frac{g(E+\epsilon)}{g(E)}\right) d\epsilon - \tfrac{1}{3}E^3\right] . \quad (21)$$

Since $g(E)$ starts as a rather square distribution from $E = 0$ to eV_{inj}, but relaxes toward a Fermi distribution for $T \approx 0$, it is clear that $g(E)$ will have no simple exact form. In the absence of numerical integrations of (21) for various initial distributions, we seek a convenient and reasonable analytic approximation. Inspection of (21) shows that near the top of the energy distribution, where downward transitions dominate,

$$\frac{dg}{dt} = -\tfrac{1}{3}\alpha E^3 g ,$$

so that

$$g = g_0 e^{-\alpha E^3 t/3} , \quad E > 0 \quad (22)$$

where $g_0(E)$ would be roughly constant out to some maximum value $E_0 \approx eV_{inj}$. Since the high-energy part of $g(E)$ dominates the relaxation processes (because of the higher transition rates $\sim E^2$), this form of cutoff, appropriate at high energies, will give more accurate results under extreme nonequilibrium conditions than, for example, a Boltzmann exponential factor, which would be appropriate in equilibrium. In its simple form, this distribution does not conserve particles, since it ignores the positive dg/dt term at low energies given by the first term of (21). A simple remedy is to take g_0 to be a time-dependent normalization constant

$$g_0 = (\tfrac{1}{3}\alpha t)^{1/3} / \int_0^\infty e^{-x^3} dx = (\tfrac{1}{3}\alpha t)^{1/3}\,\Gamma^{-1}(\tfrac{4}{3}) , \quad (22a)$$

where $\Gamma(\frac{4}{3}) = 0.893$. This prescription washes out the finite slope of $g(E)$ at $E = 0$, but our results are quite insensitive to the behavior of $g(E)$ at low energy.

Although temperature has no well-defined meaning for a nonequilibrium distribution such as (22), for qualitative purposes it is useful to define an effective temperature T^* such that the mean energy of particles is T^*, the same as if $g \sim e^{-E/T^*}$. (We shift here to units in which $k_B = 1$, so that T has the dimension of energy.) Then, we have

$$T^* \equiv \bar{E} = \left(\frac{3}{\alpha t}\right)^{1/3} \int_0^\infty x e^{-x^3} dx \Big/ \int_0^\infty e^{-x^3} dx$$

$$= \frac{1}{2}\frac{\Gamma(5/3)}{\Gamma(4/3)}\left(\frac{3}{\alpha t}\right)^{1/3} = 0.506\left(\frac{3}{\alpha t}\right)^{1/3} . \quad (23)$$

Of course by carrying the integrations to ∞ and assuming (22) holds for all energies, we have specialized to times long enough that T^* has fallen by at least a factor of 2 since injection. Since we are interested in $T^*(t) \approx T_c$, while $T^*(0)$ is typically $10T_c$, this condition is satisfied in all cases of interest here. The physical reason for the unusual cooling law (23) is that phonon emission rapidly becomes less likely as the available energy is reduced. Thus, the cooling is very rapid at first, but gets slower and slower as the electrons cool off.

To obtain numerical results from (23), we must evaluate the parameter α. It is convenient to do this in terms of the measured high-temperature limit of the electrical or thermal conductivity. In the presence of thermal phonons, the spontaneous emission rate for downward transitions given by (20) is enhanced by a factor of $(1-e^{-\epsilon/T})^{-1}$, while upward stimulated transitions from E to $(E+\epsilon)$ occur with a factor $[1-f(E+\epsilon)] \times (e^{\epsilon/T}-1)^{-1}$. At $T=0$, the phonon factors reduce to one and zero, as they should. For $T \gg \Theta$, both of these factors reduce to T/ϵ, and the total scattering rate for an electron of energy $E \approx 0$ by phonons is

$$\frac{1}{\tau} = \alpha\int_0^{\Theta} \epsilon^2\left(\frac{T}{\epsilon}\right)\{[1-f(E+\epsilon)]+[1-f(E-\epsilon)]\}d\epsilon$$

$$\approx \alpha T\Theta^2[1-f(E)] \approx \tfrac{1}{2}\alpha T\Theta^2 . \quad (24)$$

It is convenient to rewrite this as

$$\alpha = 2/\tau_\Theta\Theta^3 , \quad (25)$$

where $\tau_\Theta = l_\Theta/v_F (\sim 10^{-14}$ sec) is the scattering time extrapolated back to Θ from the high-temperature limit, in which the resistance is proportional to T.

It is appropriate to emphasize at this point the crude nature of this entire treatment. By neglecting umklapp processes, we undoubtedly overestimate the strength of the ordinary phonon scattering processes. The hope is that this error is largely cancelled if umklapp processes are consistently excluded. This was shown to be true within a factor of about 3 in the case of recombination treated by Rothwarf and Cohen.[9] Accordingly, we can expect only a similar order-of-magnitude numerical reliability for the estimates in this paper.

When (25) is inserted into (23), we find

$$T^* \approx \Theta(3\tau_\Theta/16t)^{1/3} . \quad (26)$$

It is also useful to invert this relation and note that the time required for the injected particles to cool down to T_c (if the sample temperature is much less than T_c) is

$$t(T_c) \approx (\tfrac{3}{16}\tau_\Theta)(\Theta/T_c)^3 . \quad (27)$$

As a numerical example, in the case of tin ($\tau_\Theta = 2\times10^{-14}$ sec, $\Theta = 200\,^\circ$K, and $T_c = 3.8\,^\circ$K) this characteristic cooling time is estimated to be 5.5×10^{-10} sec. Since this estimate takes the sample temperature to be zero and neglects any trapping of the emitted phonons or reexcitation of the electrons by these phonons, it provides a lower limit to the actual cooling time. Note further that this time rapidly increases if cooling to lower temperatures is considered. For example, it would take eight times as long to cool to $\frac{1}{2}T_c$, even if the sample temperature were still much lower than that.

For the following discussion, it is also useful to note that the scattering time analogous to (24) for an electron at the Fermi surface, but computed using the proper phonon factors assuming thermal equilibrium at $T_c \ll \Theta$, is

$$\tau(T_c) = (\tau_\Theta/8.4)(\Theta/T_c)^3 . \quad (28)$$

Note that this scattering time is very similar in magnitude to the cooling time (27), and with our numerical estimates for tin, it is 3.5×10^{-10} sec. Thus, the time scale set by (27) or (28) will be characteristic of phonon processes near T_c, no matter how high the electron injection energy is.

III. RELAXATION OF BRANCH IMBALANCE Q

Having developed some understanding of how the injected quasiparticles cool down toward the sample temperature, let us now consider the relaxation of the imbalance between the populations of the electronlike and holelike branches of the quasiparticle spectrum in a superconductor. For this purpose it is necessary to work in the excitation representation, as we did in Sec. I, since only in this representation is there a simple symmetry between the two branches.

Consider first the inelastic scattering processes in the normal state, which we treated in Sec. II in the particle representation. In the excitation representation, these are described as scattering processes, if the final state is on the same branch as the initial state, but are described as annihilation or creation processes if the final state is on the opposite branch. For example, an electron scattered from $k > k_F$ to $k < k_F$ annihilates a holelike excitation at $k < k_F$, as well as an electronlike excitation at $k > k_F$. On the other hand, the reverse process *creates* both a holelike and an electronlike excitation. Evidently, all such processes, allowed in the normal state, conserve Q, the difference in the numbers of excitations in the two branches. One might then wonder why

the injected Q given by (19) does not just build up without a limit. This does not occur because excess charge builds up only until it sets up the tiny potential gradient necessary to carry away the injected current as a normal current. The situation is drastically modified in the superconductor, because the pairs carry away the injected current without requiring any potential gradient. In addition, over a region of uniform injection, the quasiparticle populations are uniform in space.

In the superconducting state, Q can relax by scattering of an excitation from one branch to the other or by annihilation or creation of a pair of excitations on the same branch, all processes which are forbidden in the normal state. The usual probabilities of these processes are reduced by the coherence factors $(uu'-vv')^2$ and $(vu'+uv')^2$, for scattering and annihilation, respectively. It is easy to verify that the sum of these two expressions is one. For example, in an isotropic superconductor, the coherence factor for *elastic* scattering between branches is zero, while that for annihilation between branches of two quasiparticles of the same energy is unity, both exactly as in the normal metal. On the other hand, in the superconductor *both* probabilities are finite for *inelastic* processes; if the superconductor has an anisotropic gap, both are finite even for elastic processes. We first consider the inelastic processes, since these are always present. The elastic processes based on gap anisotropy are treated later.

Q Relaxation by Inelastic Phonon Processes

We initially confine our analysis to the case very near T_c, where it turns out that τ_Q, the relaxation time for the imbalance Q, is large compared with both $t(T_c)$, the time [estimated in (27)] for the injected quasiparticles to cool down to T_c, and $\tau(T_c)$, the time [estimated in (28)] for inelastic scattering to maintain internal equilibrium on the separate branches. (At lower temperatures, Q and T^* relax at similar rates, and the two relaxation processes must be treated simultaneously.) Near T_c, $\Delta \ll T$. Since the coherence factors for Q relaxation differ significantly from zero only when one of the two states involved has an energy between Δ and $\sim 2\Delta$, and since only a fraction $\sim \Delta/T \ll 1$ of the occupied states are in this range, it is permissible to restrict attention to pairs of states one of which has an energy E' near Δ and the other of which has an energy E that is much higher. For definiteness, we take the high-energy state to be on the electronlike branch, where we can take it to have $u=1$ and $v=0$, then the coherence factors are u'^2 for scattering processes and v'^2 for annihilation processes. Noting that Q is reduced by 2 by each scattering process from $k>k_F$ to $k<k_F$, and by each annihilation of two quasiparticles for $k>k_F$, we can write the contribution of these processes to the rate of change of Q as follows:

$$
\begin{aligned}
-2\alpha \int_\Delta^\infty \int u_<'^2 \Bigg(& f_>(E)[1-f_<(E')]\,\frac{(E-E')^2}{1-e^{-(E-E')/T}} \\
& -f_<(E')[1-f_>(E)]\,\frac{(E-E')^2}{e^{(E-E')/T}-1}\Bigg) \\
& \times 2N(0)\mathfrak{N}(E)\mathfrak{N}(E')\,dE\,dE' \\
-2\alpha \int_\Delta^\infty \int v_>'^2 \Bigg(& f_>(E)f_>(E')\,\frac{(E+E')^2}{1-e^{-(E+E')/T}} \\
& -[1-f_>(E)][1-f_>(E')]\,\frac{(E+E')^2}{e^{(E+E')/T}-1}\Bigg) \\
& \times 2N(0)\mathfrak{N}(E)\mathfrak{N}(E')\,dE\,dE' \quad . \qquad (29)
\end{aligned}
$$

In writing this, we have included the enhancement factors due to thermal phonons, and also subtracted the reverse processes, namely, upward scattering and pair creation. It is readily verified that the net rate for each process is zero if $f_<$ and $f_>$ are given by the Fermi function; this is required in equilibrium by detailed balance. To find dQ/dt, we must subtract from (29) the similar expression with the branch indices reversed.

These expressions may be simplified by noting that

$$
u_<'^2 = v_>'^2 = \tfrac{1}{2}\left[1-\left(\left|\epsilon'\right|/E'\right)\right] = \tfrac{1}{2}\left[1-\mathfrak{N}^{-1}(E')\right] . \qquad (30)
$$

This is a highly peaked function such that $u_<'^2\mathfrak{N}(E')$ can be well approximated by $\frac{1}{2}\Delta\delta(E'-\Delta)$. If this is inserted in (29) and the integration over E' performed, a common factor of $\frac{1}{2}\Delta$ appears, and E' is replaced everywhere by Δ. Since we are treating Δ as very small, we may set it to zero inside the integral, retaining only the explicit linear dependence mentioned above. [For consistency, we should also take the lower limit of integration to be zero and drop the factor $\mathfrak{N}(E)$, but we continue to carry the exact forms since they cause no difficulty at this point.]

With these simplifications, (29) reduces to

$$
\begin{aligned}
-2N(0)\alpha\Delta \int_\Delta^\infty \Bigg(& \frac{f_>(E)[1-\delta f(\Delta)]}{1-e^{-E/T}} \\
& -\frac{[1-f_>(E)][1-\delta f(\Delta)]}{e^{E/T}-1}\Bigg) E^2\,\mathfrak{N}(E)\,dE \ , \qquad (31)
\end{aligned}
$$

where we have introduced the notation

$$
\delta f(E) \equiv f_>(E) - f_<(E) \ , \qquad (32)
$$

which is not zero if $Q \neq 0$. Subtracting the term with branch indices reversed, and taking $\bar{f}(E) \equiv \frac{1}{2}[f_>(E)+f_<(E)] = f(E) = (e^{E/T}+1)^{-1}$ since the absolute departures from the equilibrium Fermi

function are small for small injection currents, we find after rearranging terms

$$\frac{dQ}{dt} = -2N(0)\,\alpha\Delta \int_{\Delta}^{\infty} \delta f(E)\left(1+\frac{\delta f(\Delta) f'(E)}{\delta f(E) f'(0)}\right) \times \frac{e^{E/T}+1}{e^{E/T}-1} E^2\, \mathfrak{N}(E)\, dE \ . \quad (33)$$

So long as Δ/T is small, this result should be generally valid for any $\delta f(E)$. But if τ_Q is long compared to $\tau(T_c)$, as it is for Δ sufficiently small, we may assume that both branch populations can be described by Fermi functions with slightly different effective chemical potentials. In that case,

$$\delta f(E) = \frac{-\partial f}{\partial E}(\mu_> - \mu_<) \ . \quad (34)$$

An exception to (34) must be made for $\delta f(\Delta)$, because it refers to the states just above the gap edge for which the coherence factors are essentially equal for processes involving both branches. Thus, after a time of order $\tau(T_c)$, $\delta f(\Delta)$ will have relaxed to zero, and the expression in square brackets in (33) will drop from two to one. In the limit as $\Delta/T \to 0$, $\tau(T_c)$ is negligible compared to τ_Q, and we can simply set $\delta f(\Delta) = 0$ throughout. Thus, if we define a distribution function

$$q(E) = 2N(0)\,\mathfrak{N}(E)\,\delta f(E) \ ,$$

such that $Q = \int q(E)\, dE$, we can write (33) in the form

$$-\frac{dQ}{dt} = \int_{\Delta}^{\infty} q(E)\,\tau_Q^{-1}(E)\, dE \equiv \frac{Q}{\tau_Q} \ , \quad (35)$$

where

$$\frac{1}{\tau_Q(E)} = \alpha\Delta E^2 \frac{e^{E/T}+1}{e^{E/T}-1} \qquad (\Delta \ll T) \ . \quad (36)$$

This defines a Q-relaxation rate for quasiparticles of energy E. Averaging over the distribution (34), and noting that as $\Delta \to 0$, $\mathfrak{N}(E) \to 1$, we find the limiting value

$$1/\tau_Q = 2\alpha\,\Delta T^2 \int_0^{\infty} (e^x - e^{-x})^{-1} x^2 dx = 4.2\,\alpha\,\Delta T^2 = 8.4\Delta T^2/\tau_\Theta \Theta^3 \quad (37)$$

after using (25). Noting that $\Delta(0) = 1.76 T_c$, and that (37) is only valid near T_c, where Δ is small, it can be rewritten

$$\tau_Q = 0.068\,\tau_\Theta \left(\frac{\Theta}{T_c}\right)^3 \frac{\Delta(0)}{\Delta(T)} \ . \quad (38)$$

With the numerical values for tin used above, the coefficient of $\Delta(0)/\Delta(T)$ in this expression is 2×10^{-10} sec, compared with the experimentally estimated[1] value of 4×10^{-10} sec. Considering the crudeness of the model, this order-of-magnitude agreement is quite satisfactory. Moreover, the experimental data appear to follow the predicted temperature dependence quite closely.

In view of the fact that our derivation was made under the restriction that T was very near T_c, so that we could assume internal equilibrium of the two branches, it is somewhat surprising that our result appears to account for the temperature dependence of the data (i.e., τ_Q constant below $\sim 0.8 T_c$) all the way down to the lowest temperature ($\sim 0.4 T_c$) at which data were taken. Let us see whether our theory can account for this.

As an extreme test, we consider the limiting case in which the sample temperature $T \ll T_c$. The calculation of τ_Q is then simplified. Noting that $E \geq \Delta \gg T$, in (29) we may drop all terms in $e^{-E/T}$ compared to unity. We again approximate the coherence factor by a δ function, although the justification is now less complete, and obtain

$$\frac{dQ}{dt} = -2N(0)\,\alpha\,\Delta \int_{\Delta}^{\infty} \left(\frac{\delta f(E)}{1-e^{(E-\Delta)/T}} + \frac{\delta f(\Delta)}{e^{(E-\Delta)/T}-1}\right) \times (E-\Delta)^2\, \mathfrak{N}(E)\, dE \ . \quad (39)$$

If T is taken to be strictly zero, there are no stimulated-phonon processes, and the quantity in the large parenthesis in (39) reduces simply to $\delta f(E)$. Apart from the replacement of E^2 by $(E-\Delta)^2$, (39) then agrees exactly with the low temperature limit of (33). We can again write this in terms of $\tau_Q(E)$ as in (35), but now

$$1/\tau_Q(E)\big|_{T=0} = \alpha\Delta(E-\Delta)^2 \ . \quad (40)$$

No large error should result, if the weighting function $\delta f(E)\,\mathfrak{N}(E)$ is taken to be that of the nonequilibrium distribution (22) used in Sec. II to treat the cooling of the quasiparticles in the normal state. Since the gap Δ provides a minimum energy, $E-\Delta$ is equivalent to E in (22). Carrying out the average, we find

$$\frac{1}{\tau_Q} = \frac{4}{3}\frac{\Gamma(\frac{4}{3})}{\Gamma^2(\frac{5}{3})}\alpha\Delta T^{*2} = \left(\frac{\alpha}{3}\right)^{1/3} \frac{\Delta}{\Gamma(\frac{4}{3})}\, t^{-2/3} \ . \quad (41)$$

Since τ_Q is a function of T^*, which in turn is a function of time as a given group of injected quasiparticles cools down, integration of $dQ/dt = -Q/\tau_Q$ no longer leads to simple exponential relaxation. Rather, we have

$$\ln\left(\frac{Q(t)}{Q(0)}\right) = -\int_0^t \tau_Q^{-1}\, dt' = \left(\frac{t}{t_1}\right)^{1/3} \ , \quad (42)$$

where $t_1 = \Gamma^3(\frac{4}{3})/9\alpha\Delta^3$. The effective relaxation time is then

$$\tau_{Q,\mathrm{eff}} = \int_0^{\infty} \left(\frac{Q(t)}{Q(0)}\right) dt = 6t_1 = \frac{0.475}{\alpha\Delta^3}$$

$$= \frac{0.155}{\alpha \Delta T_c^2} = 0.044 \tau_\Theta \left(\frac{\Theta}{T_c}\right)^3 . \quad (43)$$

With our numerical values for tin, this is 1.3×10^{-10} sec. Comparing with (38), we see that this value for $T = 0$ is about $\frac{2}{3}$ as large as would be obtained by applying the high-temperature expression all the way down to $T = 0$.

Considering the rough approximations we have made, this degree of consistency is entirely satisfactory. Some feeling for the sensitivity of this value to the form of the nonequilibrium population can be obtained by noting that (43) is increased by a factor of about 2 if the population $g(E)$ is assumed to be proportional to $(1 - E/E_{max})$, and by a factor of about 20 if a Boltzmann exponential form is assumed. This comparison illustrates the vital importance of recognizing the rather sharp cutoff of the nonequilibrium $g(E)$. Numerical estimates of the error incurred by omitting the density of states factor $\mathfrak{N}(E)$ in obtaining (41) suggest that correcting this factor alone would increase $\tau_{Q,eff}$ by a factor of about 1.7, which would bring $\tau_{Q,eff}$ to a value about 10% *above* the extrapolation of the high-temperature form. In the absence of consistent detailed numerical calculations of all related corrections, we do not feel justified in making this single correction, but it suggests that there is no reason to think that the present model is incapable of quantitatively explaining the observed temperature dependence of τ_Q. The overall numerical discrepancy of about a factor of 2 in absolute value with the experimental determination is certainly insignificant in view of the possible errors in the simple method used to evaluate the parameter $\tau_\Theta \Theta^3$, and in the various steps separating the experimentally measured voltage V from the inferred τ_Q.

Q Relaxation by Gap Anisotropy

Although elastic scattering processes are much more frequent than inelastic ones at low temperatures, they have been ignored in the above discussion because the coherence factor $(uu' - vv')^2$ is zero for k and k' on different branches if $E = E'$ and $\Delta = \Delta'$. But real superconductors are anisotropic, so that in general Δ differs from Δ'. Thus elastic scattering processes may contribute significantly to $1/\tau_Q$, particularly at the lower temperatures. We now estimate this contribution.

By inserting the appropriate values for u and v, the coherence factor for elastic scattering can be put into the form

$$(uu' - vv')^2 = \tfrac{1}{2}\left[1 - \frac{(\Delta\Delta')}{E^2} \pm \left(1 - \frac{\Delta^2}{E^2}\right)^{1/2}\left(1 - \frac{\Delta'^2}{E^2}\right)^{1/2}\right] , \quad (44)$$

where the minus sign refers to branch-mixing transitions. If the gap anisotropy is small, this may be expanded in powers of $\delta\Delta = \frac{1}{2}(\Delta' - \Delta)$, with the result

$$(uu' - vv')^2 \approx \frac{(\delta\Delta)^2}{E^2 - \bar{\Delta}^2} \leq \frac{\delta\Delta}{\Delta} \ll 1 , \quad (45)$$

where $\bar{\Delta} = \frac{1}{2}(\Delta' + \Delta)$, and $E \geq \bar{\Delta} + \delta\Delta$. Since (45) gives the probability of branch crossing in each inelastic scattering of a particle of energy E, and since each such event changes Q by two units, it follows that

$$\frac{1}{\tau_Q(E)} \approx \frac{2}{\tau_1} \frac{(\delta\Delta)^2}{E^2 - \bar{\Delta}^2} , \quad (46)$$

where $\tau_1 = l_1/v_F$ is the scattering time for elastic processes as inferred from the residual resistance of the sample at low temperature. Because for purely elastic processes, $\delta f(E)$ for each energy must relax independently, the observed Q will be proportional to $\langle \tau_Q \rangle$, not to $\langle 1/\tau_Q \rangle^{-1}$, as is the case if rapid inelastic scattering processes are maintaining internal equilibrium. The exact average of $(E^2 - \bar{\Delta}^2)$ will depend on the form of $\delta f(E)$, but a reasonable estimate is $T^*(\bar{\Delta} + T^*)$, where T^* characterizes the energy distribution of the nonequilibrium population δf. Thus, we estimate

$$\langle \tau_Q \rangle = \tfrac{1}{2}\tau_1 T^* (\bar{\Delta} + T^*)/(\delta\Delta)^2 . \quad (47)$$

To use this formula, we must have an estimate of $(\delta\Delta)^2$. This can be expressed in terms of $\langle a^2 \rangle$, a normalized mean square anisotropy introduced by Markowitz and Kadanoff.[10] For a rectangular distribution of gap values,

$$\langle (\delta\Delta)^2 \rangle = \tfrac{1}{2}\bar{\Delta}^2 \langle a^2 \rangle = \tfrac{1}{24}(\Delta_{max} - \Delta_{min})^2 . \quad (48)$$

Presumably these relations are a reasonable approximation for other distributions. Rewriting (47) in terms of $\langle a^2 \rangle$, we have

$$\tau_Q = \frac{\tau_1}{\langle a^2 \rangle} \frac{T^*}{\bar{\Delta}} \left[1 + \left(\frac{T^*}{\bar{\Delta}}\right)\right] . \quad (49)$$

For pure tin,[10] $\langle a^2 \rangle_0 \approx 0.02$, but this value is reduced by the Anderson[11] averaging effect in dirty samples such as the films in which Clarke's experiments were carried out. The extent of this reduction is not entirely clear, but from the work of Markowitz and Kadanoff on the behavior of Δ near T_c it seems reasonable to reduce the value $\langle a^2 \rangle_0$ for the pure metal by a factor of $[1 + (\hbar/2\tau_1\Delta)^2]^2$, which at low temperatures is equivalent to $[1 + (\pi\xi_0/2l)^2]^2$. With this *assumption*

$$\tau_Q = \frac{\tau_1}{\langle a^2\rangle_0}\left[1+\left(\frac{\hbar}{2\tau_1\Delta}\right)^2\right]^2\left(\frac{T^*}{\Delta}\right)\left[1+\left(\frac{T^*}{\Delta}\right)\right] . \tag{50}$$

Then τ_Q has a minimum value

$$\tau_{Q,\min} = \frac{\sqrt{3}}{2}\frac{T^*}{\langle a^2\rangle_0\Delta^2}\left(1+\frac{T^*}{\Delta}\right) , \tag{51}$$

when $\tau_1\Delta/\hbar = \frac{1}{2}\sqrt{3}$. In films, $\tau_1\Delta/\hbar$ is typically smaller than that, and τ_Q approaches a dirty limit given by

$$\tau_Q = \frac{\hbar^4 T^*}{16\tau_1^3\Delta^5}\left(1+\frac{T^*}{\Delta}\right) . \tag{52}$$

For the typical case of a tin film with $l = 1000$ Å, $\tau_1 \approx 1.3\times10^{-13}$ sec. Taking $T^* = T_c$ and $\Delta = \Delta(0) = 1.76\,T_c$, we find using (50) that $\tau_Q \approx 1.5\times10^{-9}$ sec, an order of magnitude longer than for the phonon process under the same conditions.

Considering all the uncertainties, it is not clear how realistic this numerical comparison is. More reliable is the qualitative aspect of the temperature dependence. This elastic process is negligible when the sample is near T_c, and Δ is therefore small. Thus, the inelastic phonon result (38) should be correct near T_c, where its temperature dependence and order of magnitude have been confirmed experimentally. On the other hand, when the sample temperature is low and Δ is large, the elastic relaxation process may be of significant size, and more importantly, it becomes faster as T^* decreases, whereas the phonon process becomes slower. Thus, if our estimates are of the right order of magnitude, the elastic process would eventually dominate, if T^* should fall much below T_c before Q had relaxed by phonon processes. It is also possible that spatial inhomogeneity of the gap due to the surfaces causes significant branch mixing.[12] Further work might resolve whether these elastic relaxation processes play a significant role under actual experimental conditions.

IV. CONCLUDING SUMMARY AND DISCUSSION

Under many circumstances [see Eq. (19)] quasiparticles created in a superconductor by tunnel injection of electrons are primarily on either the electronlike ($k > k_F$) or the holelike ($k < k_F$) branch of the spectrum. In that case, Q, the population imbalance between the two branches, tends to increase at a rate $I/e\Omega$, where I is the injection current and Ω is the volume of the sample [sufficiently small compared to the appropriate diffusion length[13] $\lambda = (l_0 v_F \tau_Q)^{1/2}$ that populations are spatially uniform]. This increase in Q is resisted by relaxation processes, characterized by a time τ_Q. It is shown [see Eq. (38)] that inelastic scattering and annihilation or creation processes involving phonons lead to

$$\tau_Q \approx \frac{0.07\tau_\Theta(\Theta/T_c)^3\Delta(0)}{\Delta(T)} ;$$

for tin, the coefficient is estimated to be 2×10^{-10} sec. Given gap anisotropy, elastic scattering can also relax Q, but since little anisotropy is expected in dirty film samples, this process is expected to be slower than the phonon one, except possibly at low temperatures [see Eq. (50)].

The steady-state value of $Q \approx I\tau_Q/e\Omega$ gives rise to a potential difference between quasiparticles and pairs which may be observed as a zero-current voltage V [see Eq. (16)] between a normal tunneling probe and a superconducting contact, as in the experiment of Clarke.[1] Strictly speaking, V is proportional not to Q, but to $Q^* \leq Q$, where in computing Q^* [see Eq. (14)], excitations near k_F are given reduced weight, reflecting less than unit shift in the expectation value of the electron number. Since $Q^* \to Q$ when $\Delta \ll T^*$ (an effective temperature of the imbalance population), $Q^*/Q \approx 1$ near T_c, and it is not much less at low temperatures in Clarke's experiment, since Q relaxes while $T^* \gtrsim T_c$. Thus we have made no correction for Q^*/Q in our results. As indicated in the detailed discussion in the body of the paper, if the quasiparticle injection were made at voltages just above the gap and at low temperatures, the simple results summarized here would be extensively modified. In particular, Q would be created at a rate less than $I/e\Omega$, Q^* would be significantly less than Q, and τ_Q due to inelastic-phonon processes would be much longer than (43), since T^* could be as low as the sample temperature $T \ll T_c$.

A brief account[3] of this theory has been published previously. Although our results superficially resemble those of Rieger *et al.*,[14] the characteristic relaxation time in their theory is the Ginzburg–Landau τ_{GL}, typically an order of magnitude shorter than τ_Q and having a different temperature dependence. Also, the nature of the imbalance referred to in that theory is quasiparticles versus pairs, while in our theory it is the hole-electron branch imbalance in the quasiparticle populations.

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[1]J. Clarke, Phys. Rev. Letters 28, 1363 (1972).

[2]W. A. Phillips, Proc. Roy. Soc. (London) A309, 259 (1969).

[3]M. Tinkham and J. Clarke, Phys. Rev. Letters 28, 1366 (1972).

[4]D. H. Douglass, Jr. and L. M. Falicov, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1964), Vol. IV.

[5]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[6]N. N. Bogoliubov, Nuovo Cimento 7, 794 (1958); J. G. Valatin, *ibid.* 7, 843 (1958).

[7]B. D. Josephson, Phys. Letters 1, 251 (1962).

[8]J. Bardeen, Phys. Rev. Letters 9, 147 (1962).

[9]A. Rothwarf and M. Cohen, Phys. Rev. 130, 1401 (1963).

[10]D. Markowitz and L. P. Kadanoff, Phys. Rev. 131, 563 (1963).

[11]P. W. Anderson, J. Phys. Chem. Solids 11, 26 (1959).

[12]W. L. McMillan and P. W. Anderson, Phys. Rev. Letters 16, 85 (1966).

[13]A. B. Pippard, J. G. Shepard, and D. A. Tindall, Proc. Roy. Soc. (London) A324, 17 (1971).

[14]T. J. Rieger, D. J. Scalapino, and J. E. Mercereau, Phys. Rev. Letters 27, 1787 (1971).

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Numerical Constants for Isolated Vortices in Superconductors*

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For isolated vortex lines in high-κ, type-II superconductors, Abrikosov derived the expressions $B(0)=\kappa^{-2}(\ln\kappa+C_0)H_{c2}$ and $H_{c1}=\frac{1}{2}\kappa^{-2}(\ln\kappa+C_1)H_{c2}$, but the numerical values he provided for the constants C_0 and C_1 were previously found to violate an identity $C_1-C_0-\frac{1}{2}=C_\gamma>0$. The constants are reevaluated, giving $C_0=-0.282$, $C_1=0.497$, $C_\gamma=0.279$. Furthermore, for superconductors containing a high concentration of magnetic impurities, it was previously shown that the electric field generated at the center of an isolated vortex in flux-flow situations is proportional to H_{c2} and the flux-flow velocity v. The proportionality constant C_E in the high-κ limit is numerically evaluated here to be 0.951, which, together with the value for C_γ, determines the flux-flow resistivity $\rho_f=0.381\rho_n\langle B\rangle/H_{c2}$ in the low-applied-field limit when vortices are very far apart.

I. INTRODUCTION

In his famous paper on the magnetic properties of type-II superconductors, Abrikosov[1] studied isolated vortex lines in the high-κ (the Ginzburg-Landau-parameter) limit, and derived the expressions

$$B(0)=\kappa^{-2}(\ln\kappa+C_0)H_{c2}\ , \tag{1}$$

$$H_{c1}=\tfrac{1}{2}\kappa^{-2}(\ln\kappa+C_1)H_{c2}\ . \tag{2}$$

In these equations $B(0)$ is the local field at the vortex center, H_{c2} is the upper critical field for transition from the mixed state to the normal state, and H_{c1} is the lower critical field for initial flux penetration. The numerical values provided by Abrikosov for the two constants C_0 and C_1 are

$$C_1=+0.08\ ,\quad C_0=-0.18\ , \tag{3}$$

which have been widely quoted in books on the subject of superconductivity.[2]

Recently, in studying dynamic structure of vortices in superconductors for applied field $H\ll H_{c2}$, Hu and Thompson[3] derived an identity which in the high-κ limit reduced to the simple relation

$$C_1-C_0-\tfrac{1}{2}=C_\gamma\ , \tag{4}$$

where

$$C_\gamma=\int_0^\infty (df/dr)^2 r\,dr>0 \tag{5}$$

is the constant which Gor'kov and Kopnin[4] called γ. In Eq. (5), $f=f(r)$ is the order parameter normalized to unity far away from the vortex center and r is the radial distance measured from the center. Since Abrikosov's numbers in Eq. (3) make the left-hand side of Eq. (4) negative in contradiction to Eq. (5), one must conclude that at least one of Abrikosov's numbers is seriously in error.

One straightforward way to determine the constants is to solve for $B(0)$, H_{c1}, and C_γ at finite values of κ and then to extrapolate the results to the high-κ limit. This task has been partially accomplished by Harden and Arp,[5] since they have calculated H_{c1} up to $\kappa=50$. Equating their value for $(2\kappa^2H_{c1}/H_{c2})$ at $\kappa=50$ to $(\ln 50+C_1)$ one finds $C_1=0.486$, but its accuracy cannot be confidently