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Time-Dependent Superconductivity and Quantum Dissipation

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A time-dependent Ginzburg-Landau theory is used to calculate the time-varying currents and potentials in a one-dimensional superconductor undergoing quasicontinuous phase slip. To account for the irreversible nature of this process, the concept of superconducting dissipation is introduced. This analysis is consistent with many observations on "weak links" and provides an adequate model for at least their low-frequency behavior.

I. INTRODUCTION

Superconductivity is a thermodynamically stable electronic state which comes about in some metals via a particular electron-lattice interaction. In this interaction some electron-lattice collisions act to "pair" electrons with opposing momenta and spin within a time scale of about 10^{-12} sec. These pairs are sufficiently dense that they overlap and thus create a macroscopic phase-coherent quantum state. On a time scale long relative to the "pairing time" the electrodynamics of superconductivity can be determined from the parameters of this macroscopic quantum state. Supercurrents are determined by the gradient of the quantum phase of the macrostate and any change of this current in time is then guided by the change of phase with time, that is to say, the Josephson frequency-voltage relation.

This paper is concerned with describing the time evolution of a one-dimensional superconductor on a time scale slow with respect to the pairing time so that we may look to the equilibrium state as a starting point. However, we consider this state as being in constant "acceleration." Under this condition there will be only a finite lifetime for a particular superconducting state, since it will eventually reach a velocity at which superconductivity is unstable relative to the normal state. We restrict ourselves here to situations in which this lifetime is also long relative to the pairing time. Our approach is to apply a time-dependent Ginzburg-Landau equation to describe the evolution of a one-dimensional superconductor in an electric field E.

The general characteristics of the accelerating superconducting state as described by this equation are a "free-particle" acceleration at low velocity ($dj/dt = \lambda^{-2}E$) until the current j reaches nearly the critical velocity. As the critical velocity is approached, the amplitude of the superconducting state begins to decrease and at the critical velocity the amplitude becomes unstable. The superconducting state then begins to decay toward zero on a time scale comparable to the pairing time. However, the mathematics based on the equilibrium state is unable to describe what happens next after the destruction of superconductivity without some physical help.

In order to develop a quasicontinuous description, we invoke the concept of one-dimensional phase slip[1] and hide the details of this process behind the introduction of loss into superconducting dynamics. By one-dimensional phase slip we mean essentially the one-dimensional limit of Anderson's vortex motion model.[2] However, here the singularity is assumed to be fixed in space and not to propagate, although the amplitude may be time dependent. We assume that the superconductor remains in a phase-coherent state as long as the amplitude remains finite, and thus the current can be calculated, for example, by a Ginzburg-Landau theory. However, after the system has accelerated to its critical velocity, it becomes unstable and the amplitude decays to zero. The assumption of phase slip is that the system subsequently recovers into a phase-coherent state in which the relative phase difference between any two points separated by the slip region has changed by 2π. In order to establish and maintain these specific boundary conditions on the phase, we assume that our one-dimensional superconductor is a short section connecting two "stronger" superconductors which always remain strongly superconducting during the slip process. How short this "weak" section must be to localize the slip plane will be discussed later. Any voltage V will be measured between these strongly superconducting regions. The new state which appears after the phase-slip transition then accelerates to its critical velocity and decays to repeat the process and produce a kind of quantum-mechanical relaxation oscillation. In this discussion we presume that the phase slip heals quickly enough that we may neglect the dynamics of the healing process.

The result of this approach is a time-dependent state of superconductivity at finite voltage in which the supercurrent "oscillates" at the Josephson frequency. However, to complete the description the Josephson oscillation must be augmented by dissipation. In a Josephson tunnel junction, where the evolution of relative phase is a continuous process, there is no intrinsic dissipation in the oscillation except for possible radiation effects. However, in a one-dimensional phase-slip situation such as we describe here, the amplitude of the macroquantum-state is periodically driven to zero in a small region roughly one coherence length long. And thus a small amount of condensation energy is lost each cycle. As has been pointed out previously,[1] the average lifetime for the periods of acceleration in such a periodic process must be $\tau_0 = h/2eV = \varphi_0 V^{-1}$ and the power lost by this periodic dissipation is $\bar{I}_s(t)\varphi_0\tau_0^{-1}$, where $\bar{I}_s(t)$ is the time-average supercurrent. In this paper we calculate some of the details of $I_s(t)$ based on the Ginzburg-Landau theory.

This approach also leads naturally to a time-dependent two-fluid model of superconductivity if we assume that at finite voltage the total dissipation can be described as the sum of the two separate processes, one arising out of the superconducting state and one from the "normal" state. The superconducting loss from phase slip is $\bar{I}_s\varphi_0$ per cycle and thus, under conditions of a quasicontinuous phase slip, dissipates power as $\bar{I}_s V$. We assume the normal dissipation to be V^2/R, where R is the normal resistance of the weak section. And there may be conditions under which the superconducting dissipation exceeds that of the normal process. Thus the total power dissipated by a weak superconductor above its critical current, in terms of voltage and total current I_T, is $I_T V = V^2/R + \bar{I}_s V$ and $I_T = V/R + \bar{I}_s$. Under this assumption, a weak superconductor, when fed current I_T in excess of its critical current, carries a time-average supercurrent $\bar{I}_s = I_T - V/R$. It should also be noted that this excess current (or time-average supercurrent) would be expected to exist at all voltages since it reflects a "superconducting dissipation." And this excess current will not tend to vanish at high voltages, as is the case for a Josephson junction shorted by an external resistor.[3]

This additional superconducting dissipation also means that a weak superconductor dissipates *more* power at a given voltage than if it were in the normal state. This shows up in the expression for total current $I_T = V/R + \bar{I}_s$, where at a given voltage more current flows than in the normal state leading to a higher dissipation (VI_T). Contrariwise, at a given current the dissipation is less for a weak superconductor than in the normal state. Since $V = R(I_T - \bar{I}_s)$ the potential required to drive the current I_T is lower than in the normal state.

We also note that in this model we can expect a time-dependent voltage to accompany the oscillation if the weak superconductor is driven from a current source. From the above, $V = R(I_T - \bar{I}_s)$, and we anticipate that $V(t) = RI - RI_s(t)$. In other words, the energy loss per electron (or chemical potential difference) which is $RI_T = V_n$ in the normal state is lower for the weak superconductor carrying the same total current I_T. The relative potential across the weak superconductor, $V = V_n - v_s(t)$, then varies in time as $v_s = RI_s(t)$, reflecting the periodic destruction of superconductivity in the weak region. This potential v_s has been observed[1] and experimentally fitted to a functional form $v_s = \frac{1}{2}RI_c\,[1 + \cos(2e/\hbar)\int V dt]$. This result is very nearly what is predicted here, lending additional support to this application of Ginzburg-Landau theory.

II. TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

In the superconducting state the complex order parameter Ψ represents an additional thermodynamic variable. According to the Ginzburg-Landau theory,[4] the free energy near T_c and the supercurrent density can be expressed in terms of Ψ, its gradient, and the vector potential $\vec{A}$:

$$F = \int d^3x \left[a\,|\Psi(x)|^2 + \frac{b}{2}\,|\Psi(x)|^4 + \frac{\hbar^2}{2m}\left| \left(\vec{\nabla} - \frac{2ie}{\hbar c}\,\vec{A}(x)\right)\Psi(x)\right|^2 \right], \quad (1)$$

$$\vec{j}_s(x) = -\frac{ie\hbar}{m}\left[\Psi^*(x)\left(\vec{\nabla} - \frac{2ie}{\hbar c}\,\vec{A}(x)\right)\Psi(x) - \text{c. c.}\right]. \quad (2)$$

Phenomenologically the parameters a and b are determined by the critical field $H_c(T)$ and the penetration depth $\lambda(T)$. The equilibrium properties of a superconductor can be obtained by functionally averaging over all order-parameter configurations weighted by the Boltzmann factor with energy given by (1). Outside the immediate vicinity of T_c, the most probable configuration dominates, so that the order parameter is determined by

$$\frac{\delta F}{\delta \Psi^*} = 0 \;, \quad (3)$$

which leads to the well-known Ginzburg-Landau equation for Ψ:

$$\left[a + b\,|\Psi|^2 - \frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{2ie}{\hbar c}\,\vec{A}\right)^2 \right]\Psi(x) = 0\,. \quad (4)$$

Gor'kov[5,6] derived the Ginzburg-Landau equations from the microscopic BCS theory by expanding the superconducting Green's-function equations in powers of the energy gap and its spatial derivatives. The convergence of this procedure depends upon the size of Δ/kT_c and the ratio of the quasiparticle correlation range to the field penetration depth. In addition, the possibility of a local representation depends upon the exponential decay of spatial quasiparticle correlation. Gor'kov's analysis showed that the order parameter was proportional to the energy gap:

$$\Psi = \left(\frac{7\chi\zeta(3)n}{16\pi^2 T_c^2}\right)^{1/2} \Delta , \tag{5}$$

where n is the electronic density, χ is the Gor'kov impurity function, and $\zeta(x)$ is the Riemann ζ function, so that $\zeta(3) \cong 1.202$; and in addition his analysis related the parameters a and b to the electronic properties of the material. Normalizing Ψ so that $|\Psi|^2 = \frac{1}{2}n\chi\,(1 - T/T_c)$,

$$a = -\frac{\hbar^2}{2m\xi^2(0)}\,\frac{T_c - T}{T_c} , \tag{6}$$

$$b = \frac{\hbar^2}{m\xi^2(0)n\chi} . \tag{7}$$

Here $\xi(T)$ is the Ginzburg-Landau temperature-dependent coherence length

$$\xi(T) = \xi(0)\,(1 - T/T_c)^{-1/2} , \tag{8}$$

with

$$\xi(0) = 0.74\sqrt{\chi}\,\xi_0 . \tag{9}$$

ξ_0 is the BCS coherence length which is equal to $0.18\hbar v_F/kT_c$. Gor'kov's impurity function χ depends upon the ratio of the electronic mean free path l to the BCS coherence length. Its limiting values for clean and dirty materials are

$$\begin{aligned} \chi &\cong 1, & \xi_0 &\ll l \\ &\cong 1.33\,l/\xi_0, & \xi_0 &\gg l . \end{aligned} \tag{10}$$

In an attempt to extend Gor'kov's treatment to time-dependent problems, Abrahams and Tsuneto[7] derived a relaxation equation for the order parameter in which the time rate of change of Ψ is proportional to the deviation of Ψ from its equilibrium value:

$$\hbar\gamma\left(\frac{\partial}{\partial t} + \frac{2i\mu}{\hbar}\right)\Psi = -\frac{\delta F}{\delta\Psi^*} . \tag{11}$$

Here μ is the electronic electrochemical potential and γ is a parameter characterizing the relaxation rate:

$$\gamma = \frac{\pi}{8kT_c}\,\frac{\hbar^2}{2m\xi^2(0)} . \tag{12}$$

In addition to obtaining Eq. (11), Abrahams and Tsuneto obtained the standard expression (2) for the supercurrent and the following expression for the charge density:

$$\delta\rho = \frac{3ie\hbar}{mv_F^2}\left[\Psi^*\left(\frac{\partial}{\partial t} + \frac{2i\mu}{\hbar}\right)\Psi - \text{c. c.}\right] + \frac{3ne^2}{mv_F^2}V . \tag{13}$$

Unfortunately, Eqs. (2) and (13) do not satisfy the continuity equation

$$\vec{\nabla}\cdot\vec{\mathrm{j}} + \frac{\partial\rho}{\partial t} = 0 . \tag{14}$$

The equation for the net charge density $\delta\rho$ is one of the central problems in time-dependent Ginzburg-Landau theory. It appears that the violation of charge conservation arises from the nonlocality of the basic theory for a superconductor with a finite gap. In addition, difficulties arise from the nonequilibrium aspect of the time-dependent problem.

As recently emphasized by Gor'kov and Eliashberg,[8] a derivation of a local time-dependent Ginzburg-Landau theory is thwarted by the long-time oscillations of the quasiparticle correlations. To avoid this difficulty, they considered the case of a paramagnetic alloy in the gapless regime. Then using Green's-function techniques they derived a local charge-conserving time-dependent theory. In their analysis a new function U was introduced which played a central role in calculating the charge density and current:

$$U = V + \frac{mv_F^2}{3ne^2}\,\delta\rho . \tag{15}$$

This implies that eU is the electronic electrochemical potential μ, since μ_0, the zero-field equilibrium value of μ, was chosen to be zero in their work. With the identification of eU as the electrochemical potential, the charge-conserving time-dependent Ginzburg-Landau equations of Gor'kov and Eliashberg are identical to the equations which will be solved here.

In order to fully understand them, it seemed physically more transparent to derive these equations phenomenologically as follows[9]: A simple relaxation equation for Ψ, similar in structure to Eq. (11), is introduced to describe the dynamics of the order parameter:

$$\left(\frac{\partial}{\partial t} + \frac{2i\mu}{\hbar}\right)\Psi = \frac{1}{\tau}\left[(1 - \eta|\Psi|^2) + \xi^2\left(\vec{\nabla} - \frac{2ie}{\hbar c}\vec{\mathrm{A}}\right)^2\right]\Psi . \tag{16}$$

Here τ and ξ are the temperature-dependent relaxation time and correlation length, respectively, and η arises from the $b|\Psi|^4$ term in the original Ginzburg-Landau form of the free energy. In the microscopic theory, the ratio ξ^2/τ is proportional to the electron diffusion constant. In this work we will treat these parameters phenomenologically. In general, they can depend upon position such as in the case of a boundary between normal and superconducting metals. The form of the supercurrent given in Eq. (2) is kept, but in addition a normal current is also allowed to flow so that the total current density is given by

$$\vec{j}_{tot} = -\frac{ie\hbar}{m}\left[\Psi^*(x)\left(\vec{\nabla} - \frac{2ie}{\hbar c}\vec{A}\right)\Psi(x) - \text{c. c.}\right] - (\sigma/e)\vec{\nabla}\mu \ , \quad (17)$$

where σ is the electrical conductivity. The charge density is then found from the continuity equation (14), thus ensuring charge conservation. This set of equations combined with Maxwell's equations form the electrodynamics which will be used to discuss superconducting weak links. This scheme provides a simple charge-conserving description and, as mentioned above, is equivalent to the equations derived by Gor'kov and Eliashberg for a gapless superconductor.

A superconducting weak link is essentially a one-dimensional object; all of the features of interest occur as a function of position along the length of the link. So to describe the behavior of a weak link, a one-dimensional model is appropriate. In addition, restricting the analysis to one spatial dimension implies that the vector potential may be ignored. An applied magnetic field depresses the magnitude of the order parameter,[1] but plays no essential role in determining the dynamics of the system, so setting $\vec{A}=0$ is acceptable as long as it is kept in mind that the equilibrium ratio of the order parameter in the strongly superconducting region to that in the weak link reflects the effect of an applied magnetic field, if any.

If Ψ is written in terms of a magnitude and a phase $\Psi = fe^{i\varphi}$, the time-dependent Ginzburg-Landau equation (16) becomes

$$\frac{\partial f}{\partial t} = \frac{1}{\tau}\left\{(1-\eta f^2) + \xi^2\left[\frac{\partial^2}{\partial x^2} - \left(\frac{\partial\varphi}{\partial x}\right)^2\right]\right\}f \ , \quad (18)$$

$$\frac{\partial\varphi}{\partial t} = -\frac{2\mu}{\hbar} + \xi^2\tau^{-1}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{2}{f}\frac{\partial f}{\partial x}\frac{\partial\varphi}{\partial x}\right) \ , \quad (19)$$

and the current (17) is

$$j_{tot} = \frac{2e\hbar}{m}f^2\frac{\partial\varphi}{\partial x} - \frac{\sigma}{e}\frac{\partial\mu}{\partial x} \ . \quad (20)$$

The electrochemical potential in the presence of a net charge density $\delta\rho$ and a potential V is

$$\mu = \mu_0 + eV + (mv_F^2/3en)\delta\rho \ . \quad (21)$$

The charge density is found from the continuity equation

$$\frac{\partial}{\partial t}\delta\rho = -\frac{\partial}{\partial x}j_{tot} \ . \quad (22)$$

Using Eqs. (20), (21), and one of Maxwell's equations, $\nabla^2 V = -4\pi\delta\rho$, the continuity equation becomes a differential equation for $\delta\rho$:

$$\frac{\partial}{\partial t}\delta\rho + 4\pi\sigma\,\delta\rho - 4\pi\sigma\,\lambda_s^2\frac{\partial^2}{\partial x^2}\delta\rho = -\frac{\partial}{\partial x}j_s \ . \quad (23)$$

λ_s is the Fermi-Thomas screening length. Supercurrent j_s varies in space on a scale of coherence lengths so $(\partial^2/\partial x^2)\delta\rho \sim (1/\xi^2)\delta\rho$, and since $\lambda_s^2/\xi^2 \ll 1$, the third term on the left-hand side of (23) can be ignored. Similarly, j_s varies in time on a scale of $\hbar\pi/8k(T_c - T) \gg 4\pi\sigma$, so the first term on the left-hand side of (23) can also be ignored. Therefore, to good approximation the charge density becomes

$$\delta\rho \cong -\frac{1}{4\pi\sigma}\frac{\partial}{\partial x}j_s \ . \quad (24)$$

Integrating Eq. (24) twice and applying the boundary condition of zero electric field away from the weak-link region, V can be written for a weak link with unit cross section as

$$V(x) = \int_0^x dx' \frac{j_{tot}(x') - j_s(x')}{\sigma} \ . \quad (25)$$

Since $\delta\rho$ varies on a scale of coherence lengths spatially, so does V:

$$V \sim \xi^2\delta\rho \ . \quad (26)$$

This implies that the third term in Eq. (21) can be ignored with respect to the second term since $\lambda_s^2/\xi^2 \ll 1$. Choosing the zero of energy such that $\mu_0 = 0$, it follows from (21) that

$$\mu \cong eV \ . \quad (27)$$

III. METHODS OF SOLUTION

Because of the nonlinearity of the Ginzburg-Landau equations and the consequent difficulty in finding solutions of them analytically, the equations were solved numerically using a PDP 15 computer. The weak link was divided up into a 20-point spatial grid, with the distance between grid points adjustable so that links of different lengths could be studied. The spatial parts of the differential equations of interest were then transformed into difference equations on the 20 discrete points. The order parameter, electric fields, current, and other parameters of interest were allowed to vary in space and time on this grid. Outside of the grid, the magnitude of the order parameter was maintained constant and kept strongly superconducting with respect to the weak region. The boundary conditions were such that away from the weak-link region the electric field went to zero. The boundary conditions imposed on the order parameter between the strongly superconducting region and the weak region were continuity of the order parameter and continuity of its first spatial derivative.

The resulting differential-difference equations are all first order in time, so that numerical solution with respect to the time variable was relatively simple. Starting from some initial time t_0, at which the values of various physical quantities were specified, the first time derivatives of these quantities were calculated from the differential equations. Then at $t_0+\Delta t$ the quantities of interest, f for example, could be written

$$f_n(t_0+\Delta t)=f_n(t_0)+\Delta t\,\frac{\partial f_n(t_0)}{\partial t}\quad , \tag{28}$$

where n refers to a given spatial point. This process was then iterated to build up the various quantities as a function of time. The time interval Δt was kept variable, and its size was adjusted depending on how rapidly the functions of interest were changing in time.

There were essentially two different natural time scales in the problem. One of these was the Josephson period τ_0, which was set by the voltage $\tau_0=2\pi(2eV/\hbar)^{-1}$, and the other is the response time of the superconductor set by τ. Here τ is the effective relaxation time characterizing the weak-link section of the superconductor. The existence of these different time scales and the basic nonlinear nature of the equations made it important to be able to adjust the time interval so that the dynamics could be followed as closely as desired.

There were a number of physical parameters that described the weak link. Many of these could be changed from one computer run to the next so that their effect on the properties of the superconducting weak link could be examined. For example, computer runs were made with various values of the total current, the ratio of transition temperatures in the weak and strong regions, the total length of the weak region, and the electrical resistivity.

In a weak link if the total current exceeds the critical current, the dynamics of the system can be explained in terms of a quantum phase slip. As discussed in the Introduction, this phase slip occurs at some point as the magnitude of the order parameter is driven very small because of a large applied current, so locally the superconductor enters a region in which the thermodynamic fluctuations become very important. This phase slippage was grafted onto the numerical analysis of the time-dependent Ginzburg-Landau equations in the following manner. The free energy F_0 of a small region in the center of the weak link was calculated as a function of time. In addition, the minimum free energy $F_{2\pi}$ this region would have if the phase were reduced by 2π across it was also calculated. When $F_{2\pi}$ became less than F_0, the phase was slipped by 2π by reducing the phase of the order parameter by 2π to the right (by convention this was the direction in which the scalar potential increased) of the center region. The length of the small center region was chosen to be one coherence length, which is roughly the length one would expect a typical thermal fluctuation to have. A more complete analysis of the nature of the phase-slip process would include the effect of all possible fluctuations weighted by the appropriate Boltzmann factor. The dynamics of the weak link near the time of phase slip proceed very rapidly compared to the Josephson period, so that the details of the phase-slip process are relatively unimportant in determining the behavior of the weak link as long as $\tau_0 \gg \tau$. In this region, alternative specifications of phase slip lead to virtually the same results.[10]

Once the phase slip has occurred, the time evolution of the weak link was again governed by the time-dependent Ginzburg-Landau equations as written down in Sec. II. The evolution of the system continued until $F_{2\pi}$ became less than F_0, and the phase slipped again. This sequence of events is shown in Fig. 1. In all of the diagrams in Fig. 1 the order parameter is plotted in cylindrical coordinates: $|f|$ is the radius, φ is the phase angle, and the spatial coordinate x of the weak link is plotted along the length of the cylinder. In addition, the phase as a function of position is plotted separately for each figure. In Fig. 1(a) the phase difference across the link is still small, and hence the supercurrent is small while the magnitude of the order parameter f is close to its equilibrium value. Figure 1(b) shows the link at a somewhat later time. The voltage developed resistively by the large current through the link has driven the phase gradient large in the weak section and consequently f has become very small in the center piece of the weak link. This is just the situation in which the phase slip can occur. Figure 1(c) is a blown-up picture of the center section of Fig. 1(b), while Fig. 1(d) is a blow-up of the center section just after the phase slip. Note that everything in the link is unchanged except that in this center section the phase bends back upon itself so as to reduce the total phase across the link by 2π. After the phase slip the dynamics proceed smoothly, and f begins to regrow and φ begins to smooth out. Figure 1(e) shows the link sometime after the phase slip as this reknitting is happening. The weak link eventually reaches the configuration in which it started, and this succession of events then continually repeats itself. Thus, Fig. 1 is a sort of movie of how the dynamics of this weak superconductor proceed.

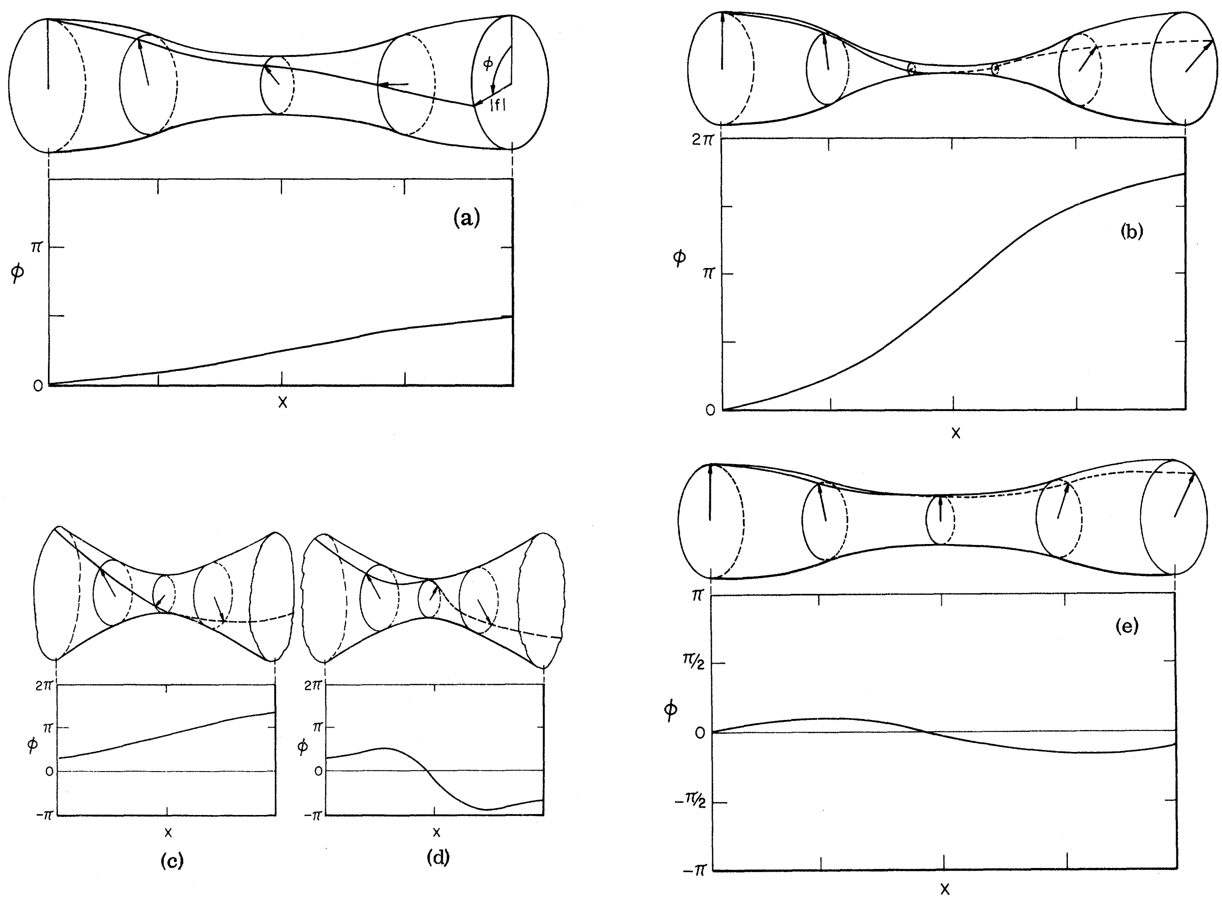

FIG. 1. The upper half of each frame shows a plot in cylindrical coordinates of the magnitude and phase of the order parameter as functions of position X along the weak link for various times during one cycle of phase slip. $|f|$ is the radius vector, φ is the phase angle, and X is plotted along the axis. The lower half of each frame shows just the phase as a function of position. (See the text for a detailed discussion of each frame.)

IV. RESULTS

The initial conditions in time were such that at $t = 0$ there was no current flowing through the weak link, the phase of the order parameter was zero everywhere, and the magnitude of the order parameter described the equilibrium configuration of the strong-superconductor–weak-superconductor–strong-superconductor junction. At $t = 0$ a current source was turned on and the t'me evolution of the weak link was calculated using the methods discussed in Sec. III.

For a total current which was less than the critical current the system rapidly approached a steady-state situation in which the total current was carried as a supercurrent everywhere. As a function of time the supercurrent in the weak link built up and approached j_{tot}, as shown in Fig. 2(a). While the supercurrent was growing there was an electric field in the weak link to drive the current, and consequently there was a voltage across the link. As the supercurrent reached j_{tot}, the electric field and the voltage vanished. Both the magnitude and the phase of the order parameter then became stationary in time. The time-dependent stable supercurrent through a weak link as a function of the total phase difference is shown in Fig. 2(b) for several weak links. The shape of the curves is roughly sinusoidal, but note that in each case there is no stable supercurrent for a phase difference larger than $\Delta\varphi_c$. Beyond $\Delta\varphi_c$ the superconductor is unstable. If the link was begun with initial conditions such that the phase difference was larger than $\Delta\varphi_c$, there was a corresponding supercurrent, but if the system was not driven by a current source, the magnitude of the order parameter rapidly died away in time until superconductivity was destroyed. The case of a weak link driven by a current larger than j_c will be discussed below, since it is the phase-slip region. Thus, only the parts of the j_s-vs-$\Delta\varphi$ curves shown in Fig. 2(b) represent a steady-state situation.

Baratoff, Blackburn, and Schwartz[11] (BBS) considered this steady state of a superconducting weak

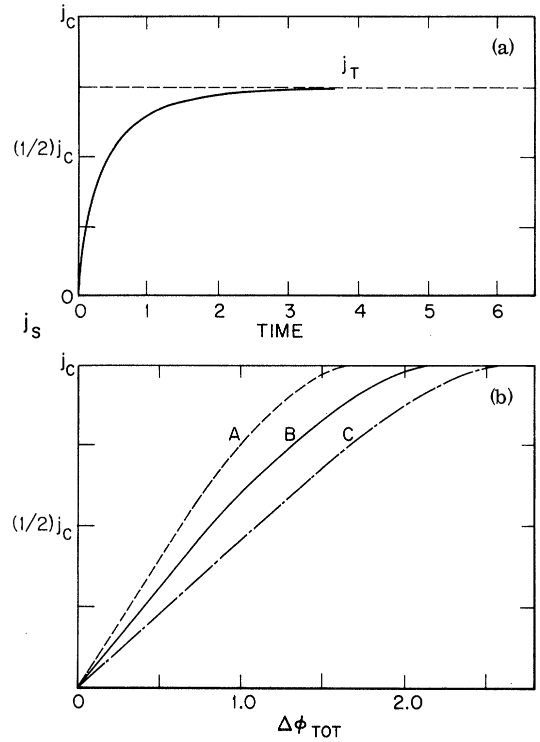

FIG. 2. (a) Supercurrent as a function of time for $j_T < j_c$. (b) Stable supercurrent as a function of phase difference across the weak link for three weak links. A is the weakest link, C is the strongest, and B is intermediate between A and C.

link with a current flowing using the time-independent Ginzburg-Landau equations. In their paper they presented a current vs phase change across the link curve, but they subtracted an amount from the phase difference equal to the phase difference the weak region would have had if it had all been strongly superconducting. This subtraction implies that as the weak link becomes less weak, $\Delta\varphi_c$ becomes smaller, since in the limit that the weak link becomes as strongly superconducting as the strong region, $\Delta\varphi_c$, as defined by BBS, becomes zero. Without this subtraction $\Delta\varphi_c$ increased as the link became stronger, as indicated in Fig. 2(b). With this difference in definition of $\Delta\varphi$ in mind, the BBS results and the results of the present work can be compared. The piece of their curve with positive slope corresponds to Fig. 2(b), so the two curves are roughly in agreement for this positive-slope piece. The dynamic equations and the boundary conditions used here, however, indicate that part of their supercurrent-phase curve with negative slope is unstable.[12] In the region with negative slope the dynamics are such that if the system were allowed to develop, superconductivity would disappear. As the link became weaker in the BBS work, $\Delta\varphi_c \rightarrow \frac{1}{2}\pi$. This may be true for a short weak section (BBS used one coherence length), but as the link becomes longer the total phase across it must inevitably become larger, since a large phase gradient can exist over a longer length.

For a total applied current which exceeds the critical current, the weak link was in the phase-slip regime. As a function of time the current began to grow, but since $j_{tot} > j_c$ the current and phase gradient eventually became sufficiently large so as to drive the magnitude of the order parameter into the phase-slip regime. As described in Sec. III, the phase then slips by 2π. The magnitude of the order parameter then grew, and the supercurrent, which had decreased as f approached zero, also grew. The growing current and phase then again drove f small so that the phase

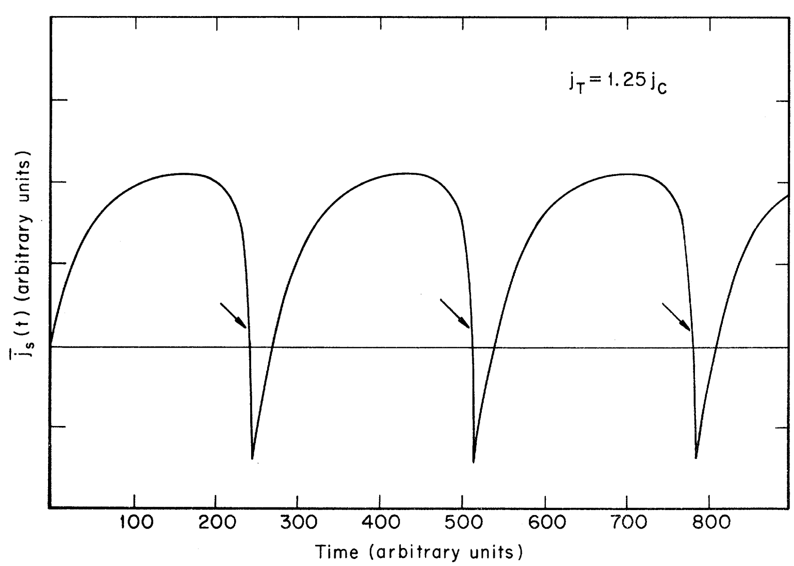

FIG. 3. Net supercurrent [$\bar{j}_s(t) = j_T - V/R$] as a function of time in the phase-slip regime for $j_T = 1.25 j_c$. The arrows indicate the times of phase slip.

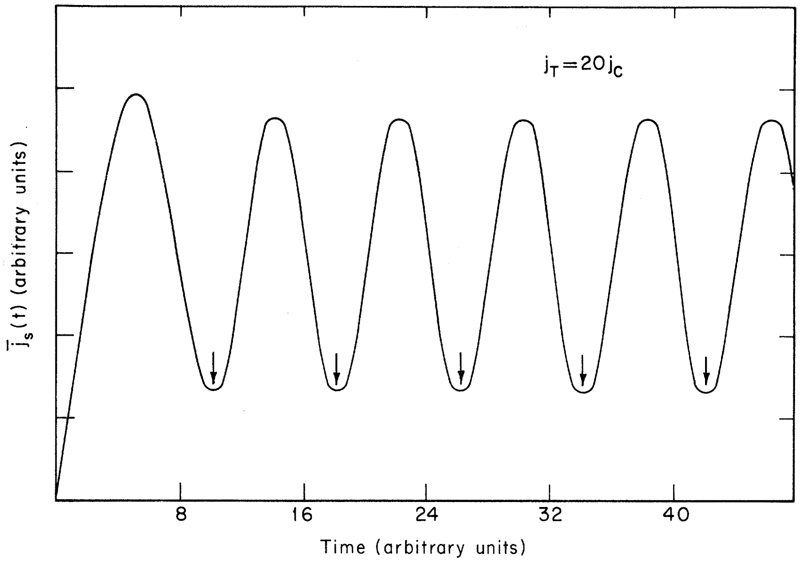

FIG. 4. Net supercurrent $[\bar{j}_s(t) = j_T - V/R]$ as a function of time in the phase-slip regime for $j_T = 20j_c$. The arrows indicate the times of phase slip.

slipped again.

For $j_{tot} > j_c$ we assume a two-fluid model in which there is both normal current and supercurrent flowing in the weak link. This implies a resistively developed voltage across the link. Setting the cross-sectional area of the weak link equal to unity, current I and current density j may be used interchangeably, and, as was noted in the Introduction,

$$V(t) = \int \frac{j_{tot} - j_s(t)}{\sigma}\, dx \tag{29}$$

$$= j_{tot}R - R\int \frac{j_s(t)}{L}\, dx \tag{30}$$

$$= j_{tot}R - \bar{j}_s(t)R\ , \tag{31}$$

so that the spatially averaged supercurrent $\bar{j}_s(t)$ defined above can be written

$$\bar{j}_s(t) = j_{tot} - V/R\ . \tag{32}$$

$\bar{j}_s(t)$ is shown as a function of time in Figs. 3 and 4 for two values of j_{tot}. $\bar{j}_s(t)$ is periodic in time and has a nonzero time-averaged value. Hence V is also periodic in time, and its time-averaged value $\langle V\rangle$ is nonzero and less than $j_{tot}R$.

From our numerical calculations, the frequency of the periodic function $v_s(t)$ [or $\bar{j}_s(t)$] was found to be the Josephson frequency $\omega_0 = 2e\langle V\rangle/\hbar$, to within about 10%. This 10% variation reflects computational errors introduced by the 20-point spatial grid and finite-sized time intervals, rather than any physical effect. If one examines Eq. (19) for $\dot{\varphi}$, it can be seen that the frequency should be exactly ω_0. The total phase across the link is the value of φ at the last grid point where the $j_{tot} = j_s$ = const. The second and third terms on the right-hand side of Eq. (19) are proportional to $(\vec{\nabla}\cdot\vec{j}_s)/f^2$, which is zero outside the weak-link region, so

$$\dot{\varphi}_{tot} = \frac{2eV_{tot}}{\hbar}\ . \tag{33}$$

In one period τ_0, φ_{tot} goes through 2π, so that

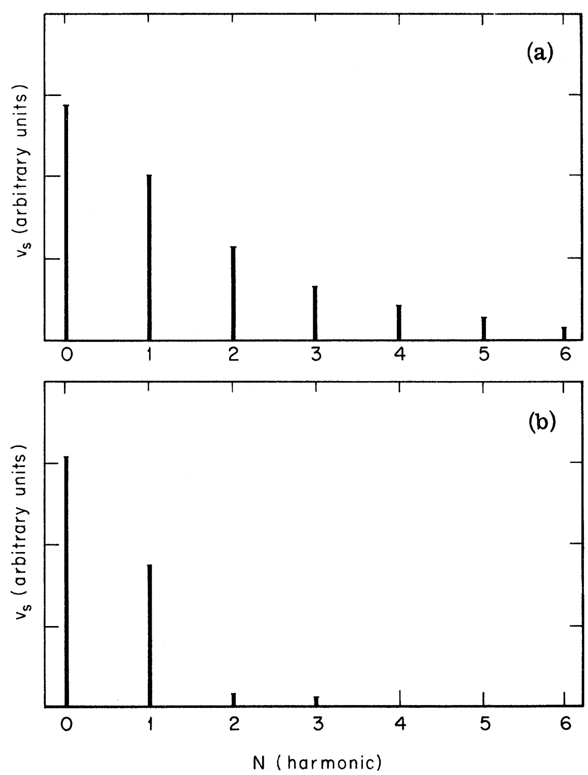

FIG. 5. Frequency decomposition of the voltage in the phase-slip regime in terms of the fundamental ($N=1$) Josephson frequency. (a) $j_T = 1.25\ j_c$, (b) $j_T = 20j_c$.

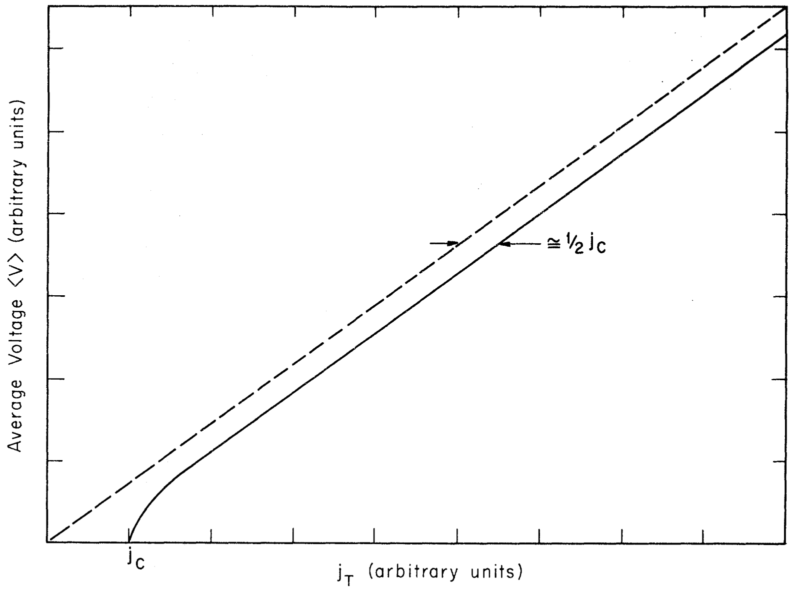

FIG. 6. Current vs time-averaged voltage for a superconducting weak link showing the excess supercurrent of $\frac{1}{2}j_c$.

$$2\pi = \int_0^{\tau_0} \dot{\varphi}_{\rm tot}\, dt = \int_0^{\tau_0} \frac{2eV_{\rm tot}}{\hbar}\, dt = \frac{2e}{\hbar}\, \tau_0 \langle V \rangle\,, \qquad (34)$$
$$\tau_0 = 2\pi (2e\langle V\rangle/\hbar)^{-1}\,.$$

As the total current was increased, the shape of the $v_s(t)$ curve became more and more of a perfect sinusoid, which can be seen in Fig. 4. At lower currents the shape is considerably less sinusoidal, as can be seen in Fig. 3. Also shown in Figs. 3 and 4 are the times at which phase slip occurred; each arrow indicates a quantum phase slip of 2π. Figure 5 shows the Fourier decompositions of v_s, i.e., v_s vs ω is plotted. Note that at currents near j_c the fundamental harmonic ($N = 1$) is dominant, but a few of the higher harmonics ($N = 2$, 3, or so) are present to an extent. For $j_{\rm tot} \gg j_c$, on the other hand, really only the fundamental harmonic is present to any degree, which indicates that $v_s(t)$ is very sinusoidal for large currents. This means that if the frequency spectrum of the voltage across a weak link is examined for total currents less than about $3j_c$ or so, it should be possible to see a peak not only at the Josephson frequency but also at twice and perhaps three times the Josephson frequency.

These analytic results are consistent with experimental observations[1] on weakly superconducting structures. In that case it was found that the function

$$v_s = \tfrac{1}{2} RI_c [1 + \cos(2e/\hbar) \textstyle\int V dt]\,, \qquad (35)$$

where $V = RI - v_s$, accurately described both the time-averaged and the time-dependent voltages developed across a weak superconductor carrying a total current I. This approximation also predicts the harmonic generation described above and seems to be a generally useful analytic form with which to describe weakly superconducting links of this type.

Another interesting property of weak-link superconductors is their j-vs-V curve. The time-averaged voltage $\langle V\rangle$ vs $j_{\rm tot}$ is shown in Fig. 6. For $j_{\rm tot} > j_c$ there is an excess supercurrent and a nonzero time-averaged voltage. Asymptotically for $j_{\rm tot} \gg j_c$, this excess supercurrent was always found to be $\frac{1}{2}j_c$ for a wide variety of physical properties of the weak link, e.g., electrical conductivity, total length, ratio of transition temperatures in the weak and strong regions. There was a spread of about 10% in the values of the excess supercurrent around $\frac{1}{2}j_c$. We believe this spread is again due to numerical errors rather than a physical effect. As was pointed out in the Introduction, this excess current reflects dissipation attendant to phase slip and comes out of this application of Ginzburg-Landau theory because of the insertion of this requirement. It does not reflect a dissipation intrinsic to the Ginzburg-Landau model.

For a weak link shorter than approximately five coherence lengths the phase slip always occurs in the center of the link, and the phase slips by 2π rather than a multiple of 2π. A multiple phase slip is in principle possible, but for a short weak link they do not occur, since $F_{4\pi}$, the free energy for the 4π phase-slip state, is much larger than $F_{2\pi}$. A multiple phase slip in a short weak link is highly improbable.

A long weak link, e.g., 10 or more coherence lengths, can have a large phase difference across it. A phase slip of more than 2π may be necessary in order to leave the link in a state with a small net phase difference so that the order parameter can have a chance to regrow. A multiple phase slip at a given spot is still highly improbable, however, since $F_{4\pi}$ is much larger than $F_{2\pi}$. Instead, the phase slips by 2π across some small region, and the order parameter in the immediate vicinity of the phase slip begins to grow. If the 2π phase slip were not enough, the phase slips again by 2π at some other spot. The order parameter in the region of the first phase slip begins to grow, so if another 2π phase slip is needed, it cannot happen at the first place since locally the order parameter has grown out of the region in which the thermal fluctuations dominate the behavior. In a long link there can be a fairly large region in which the order parameter is very small. The exact position of the phase slip is determined by the detailed nature of the fluctuations and in this large depressed region is more or less random. All this implies that phase slip is a noisy process in a long weak link.

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[1]H. A. Notarys and J. E. Mercereau, in *Proceedings of the* 1969 *International Conference on Superconductivity*, edited by Frank Chilton (North-Holland, Amsterdam, 1971), p. 424; R. K. Kirschman, H. A. Notarys, and J. E. Mercereau, Phys. Letters 34A, 209 (1971).

[2]P. W. Anderson and A. H. Dayem, Phys. Rev. Letters 13, 195 (1964); P. W. Anderson, Rev. Mod. Phys. 38, 298 (1966).

[3]D. E. McCumber, J. Appl. Phys. 39, 3113 (1968); J. R. Waldram, A. B. Pippard, and J. C. Clarke, Phil. Trans. Roy. Soc. London 268, 265 (1970).

[4]V. L. Ginzburg and L. Landau, Zh. Eksperim. i Teor. Fiz. 20, 1064 (1950).

[5]L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959) [Sov. Phys. JETP 9, 7364 (1959)].

[6]L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 37, 1407 (1959) [Sov. Phys. JETP 10, 998 (1960)].

[7]E. Abrahams and T. Tsuneto, Phys. Rev. 152, 416 (1966).

[8]L. P. Gor'kov and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. 54, 612 (1968) [Sov. Phys. JETP 27, 328 (1968)].

[9]T. R. Rieger, D. J. Scalapino, and J. E. Mercereau, Phys. Rev. Letters 27, 1787 (1971).

[10]It may be possible to examine the dynamics of phase slip by studying weak links in the vicinity where $\tau_0 \sim \tau$.

[11]A. Baratoff, J. A. Blackburn, and B. B. Schwartz, Phys. Rev. Letters 25, 1096 (1970).

[12]It may be that alternative boundary conditions corresponding to different external circuits could stabilize this region of the curve, but we have not investigated this aspect of the problem.

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Localized Modes of Excitation of an Electron Gas in the Vicinity of an Impurity in Metals

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Modes of vibration of an electron gas in a uniform background of positive charge containing a fixed point charge are investigated. A hydrodynamic model is used in which the electrons are treated as a Fermi fluid. It is found that a localized excitation of electron gas may exist near a positive impurity. Such a local excitation can be detected in characteristic energy losses of fast electrons.

INTRODUCTION

Excitations of a homogeneous electron gas embedded in a uniform background of positive charge have been studied extensively.[1] The excitations of a nonhomogeneous system are not so well understood. We consider the collective oscillations of an electron gas in the presence of a fixed point impurity. Layzer[2] has shown that in the vicinity of a positive impurity localized single-particle excitations may exist. Sziklas[3] and Sham[4] have found that a plasmon-type excitation with frequency $\sim \omega_0/\sqrt{2}$ exists in the vicinity of a negative impurity under appropriate conditions. Here ω_0 is the plasma frequency of the homogeneous medium. The previous authors have used a quantum approach to the problem. However, in using a quantum approach one does not have a physical picture of the processes involved, as one has in the hydrodynamic approach. Hence Sziklas[3] and Sham[4] have also