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Nonlinear Properties of Fluxoids in Superconductors in a High-Speed Flux-Flow State

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The order-parameter variation and the electric current about the normal core of a fluxoid in high-speed motion in clean superconductors in the low-magnetic-field and low-temperature region are discussed on the basis of a time-dependent Ginzburg-Landau equation and a classical equation of motion for the core electrons. It is found that the normal current flow along the core electric field tends to a constant when the fluxoid velocity exceeds $(\xi_0/\tau)(1+\omega_c^2\tau^2)$, where ξ_0 is the coherence length, τ is the transport relaxation time, and ω_c is the cyclotron frequency. Expressions of the strain field associated with high-speed fluxoids are derived by the use of a modified elastic-wave equation with superconductivity parameters and are used to investigate the fluxoid velocity dependence of the flux-pinning effect. It is shown that the supersonically accelerated fluxoids radiate elastic shock wave, and that flux pinning by internal strain sources like dislocations is expected to disappear. A wave radiation and lowering of the pinning are also found when fluxoids are subjected to a high-frequency vibration.

I. INTRODUCTION

There have been a number of experiments[1] associated with the motion of fluxoids in the mixed state of type-II superconductors. The flux-flow phenomena have been theoretically treated in semiphenomenological ways[2–8] or with the help of time-dependent Ginzburg-Landau (TDGL) equations.[9–11] However, the fluxoid motion in the low-temperature ($T \sim 0$ °K) and the low-field ($H \gtrsim H_{c1}$) region has only been dealt with semiphenomenologically and for slowly moving fluxoids (the distance a fluxoid moves in the electron relaxation time τ is much smaller with the core radius of fluxoids). Most of the semiphenomenological theories investigate the mixed state using a two-fluid model and hydrodynamic assumptions. The approach cannot describe consistently the properties of fluxoids in high-speed motion. In Sec. II we consider the variation of the order parameter around the core of a moving fluxoid with the help of a TDGL equation which is adequate in the low-field region and near $T = 0\,°K$, and discuss the motion of the normal excitations in the core when the fluxoid's velocity exceeds the core radius divided by the electron relaxation time.

In real type-II superconductors, the fluxoids in motion are known to suffer the flux-pinning force besides the viscous-drag force due to the power dissipation of the core electrons. Therefore, the pinning force plays an important rôle in the description of the flux-flow characteristics.[12] Because of its metallurgical structure sensitivity, the pinning effect is regarded as due primarily to the interaction of fluxoids with various lattice defects. The interaction has been discussed by many authors[13-17] who considered the interaction energy between the lattice deformation associated with fluxoids and the localized strain fields of the elastic defects. Since these theories deal with only the static interaction, it is necessary to study the dynamic aspect of the interaction in order to elucidate the pinning effect for moving fluxoids. In Sec. III expressions for the elastic fields associated with uniformly moving or harmonically vibrating fluxoids are derived on the basis of the modified elastic-wave equation with superconductivity parameters. By use of the result, the fluxoid velocity dependence of the pinning effect is discussed.

II. ORDER-PARAMETER VARIATION AND ELECTRIC CURRENT FLOW ABOUT THE NORMAL CORE OF A MOVING FLUXOID

We treat here the moving fluxoid in a pure and extreme type-II superconductor in the low-field ($H \gtrsim H_{c1}$) and the low-temperature ($T \ll T_c$) region, assuming that (a) the fluxoids are not subjected to pinning forces, (b) the distance between fluxoids is much larger than the coherence length, and (c) there are practically no normal electrons outside the core of each isolated fluxoid.

The properties of moving fluxoids have mostly been discussed either on the basis of a local model[2,6] or by the use of the linearized TDGL equation of the diffusion type.[9,11] Abrahams and Tsuneto[18] found that the TDGL equations exist near absolute zero and near the transition temperature: In the former case it has a wavelike character, and in the latter case it is of the diffusion type. We investigate the fluxoid motion with the help of a wave-type TDGL equation,[19-21] which may be appropriate within the limit of our discussion. The equation is

$$\left[\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)^2 - v_0^{-2}\left(\frac{\partial}{\partial t} + \frac{2ie}{\hbar}\varphi\right)^2 + \xi_0^{-2}\left(1 - \left|\frac{\psi}{\psi_0}\right|^2\right)\right]\psi = 0 \ , \quad (1)$$

where ψ is the order parameter; φ is the scalar potential; $\vec{A}$ is the vector potential; v_0 is the Fermi velocity divided by $\sqrt{3}$; ξ_0 is the coherence length; and $\psi_0 = (\frac{1}{2}N)^{1/2}$, where N is the total electron density. We choose a frame such that the magnetic field is in the z direction and the vortex state moves in the x direction with velocity v_L. Then the order parameter and the electromagnetic potentials are expressible as functions of $x - v_L t$ and y, characterized by the relations

$$\frac{\partial}{\partial t}\psi = -v_L\frac{\partial}{\partial x}\psi \quad (2)$$

and so on. We assume a homogeneous "Lorentz-like" transformation for a frame X fixed to the lattice and a formally given frame X' which moves along the x axis with the velocity v_L relative to X. The transformation relates the space-time coordinates $(\vec{r}, t)$ and the electromagnetic potentials $(\vec{A}, \varphi)$ of X to the coordinates $(\vec{r}\,', t')$ and potentials $(\vec{A}\,', \varphi')$ of X' as follows:

$$x = \frac{x' + v_L t'}{g}, \quad y = y', \quad t = \frac{t' + (v_L/v_0^2)\,x'}{g}, \quad (3)$$

$$A_x = \frac{A_x' + (cv_L/v_0^2)\,\varphi'}{g}, \quad A_y = A_y', \quad \varphi = \frac{\varphi' + (v_L/c)A_x'}{g}, \quad (4)$$

where $g = (1 - v_L^2/v_0^2)^{1/2}$. This transformation leaves the form of the TDGL equation (1) invariant. From Eqs. (2) and (3) we see that ψ, $\vec{A}\,'$, and φ' are independent of t'. We look for the solution of Eq. (1) near the core of the fluxoid, where the variation of the order parameter is large. Assuming that the electric field has only a y component there or

$$E_x = -\frac{\partial}{\partial x}\left(\varphi - \frac{v_L A_x}{c}\right) = 0 \ ,$$

we get $A_x' = gA_x$ by putting

$$\varphi - v_L A_x/c = g\varphi' = 0 \, . \quad (5)$$

Then Eq. (1) reduces to

$$\left[\left(\vec{\nabla}' - \frac{2ie}{\hbar c}\vec{A}\,'\right)^2 + \xi_0^{-2}\left(1 - \left|\frac{\psi}{\psi_0}\right|^2\right)\right]\psi = 0 \, . \quad (6)$$

In the case of an extreme type-II superconductor, we can assume the magnetic field about the core region to be uniform:

$$H = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = g^{-1}\left(\frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'}\right) = H_a \, , \quad (7)$$

where H_a is a constant. Therefore a simple expression for $\vec{A}\,'$ is $A'(r')\,\vec{1}_{\theta'}$ in the cylindrical coordinates (r', θ') in the frame X', where $r' = (x'^2 + y'^2)^{1/2}$ and $\vec{1}_{\theta'}$ is the unit vector in the θ' direction. To obtain a solution of the form $\psi = f(r')\,e^{i\theta'}$, we insert this expression for ψ into Eq. (6), and, using Eq. (7), arrive at

$$\left[\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'}-\frac{a^2}{r'^2}+\xi_0^{-2}\left(1-\frac{f^2}{f_0^2}\right)\right]f=0\,, \tag{8}$$

with

$$a(r')=1-\frac{2\pi r'A'(r')}{\hbar c/2e}=1-\int\frac{dx}{g}\,dy\frac{gH_a}{\phi_0}\ ,$$

where $f_0=(\frac{1}{2}N)^{1/2}$ and ϕ_0 is the flux quantum. Although we may put $a(r')=1$ in the core region, it is difficult to find an analytical solution of Eq. (8). We recognize, however, that

$$\frac{f}{f_0}=q\left(q^2+\pi^{-2}+\frac{2}{\pi^2}\,(q^4+1)\right)^{-1/2}, \tag{9}$$

with

$$q=r'/\pi\xi_0\ , \tag{10}$$

is a good approximate solution. Equations (9) and (10) display the "Lorentz-like contraction" of the variation pattern of the order-parameter amplitude around the core by the factor g. The core electric field is given from Eqs. (5) and (7) as

$$E_y=-c^{-1}\left(-v_L\frac{\partial A_y}{\partial x}\right)-\frac{\partial\varphi}{\partial y}=\frac{v_LH_a}{c}\quad . \tag{11}$$

In the following we discuss the behavior of the normal electrons in the core region assuming the existence of a sharply bounded core with radius ξ_0. It is convenient to work always in the frame X_1 which moves with the fluxoid velocity v_L. The electromagnetic fields $\vec{E}$ and $\vec{H}$ in the frame X fixed to the lattice are transformed into the fields $\vec{E}_1$ and $\vec{H}_1$ in X_1, respectively, with the help of the usual formula of uniform Lorentz transformation:

$$E_{1x}=E_x,\quad E_{1y}=\frac{E_y-(v_LH_z/c)}{(1-v_L^2/c^2)^{1/2}}\ , \tag{12}$$

$$H_{1z}=\frac{H_z-(v_LE_y/c)}{(1-v_L^2/c^2)^{1/2}}\quad . \tag{12$'$}$$

Considering the condition $v_L<v_F\ll c$, we get from Eqs. (11), (12), and (12$'$) the electromagnetic fields around the core

$$E_{1x}=E_{1y}=0\,,$$

$$H_{1z}=H_a(1-v_L^2/c^2)^{1/2}\approx H_a\ .$$

This means that in X_1 the core electrons stand still, feeling no electromagnetic force insofar as they are not scattered by the electron-lattice interaction. On the other hand, these electrons move with the fluxoid velocity in the frame X. This behavior agrees with the result of the theoretical investigation of fluxoid motion in a perfect crystal by Nozières and Vinen.[6]

Strictly speaking, however, the individual normal electron is not at rest in the frame X_1 but is regarded to be moving with velocity about v_F. Therefore the electromagnetic force acting on the core electrons in a superconductor of a perfect crystal does not vanish there. Actually the electronic properties in the normal core are observed as mean values over all electrons on the Fermi surface, and we may treat the averaged electronic motion instead of investigating the behavior of the individual core electron.

When the scattering of the core electrons is taken into consideration, the equation of motion of the electrons in the frame X_1 is given by

$$\begin{aligned}\frac{dv_{x1}}{dt}+\frac{v_{x1}+v_L}{\tau}&=\frac{e}{mc}\,v_{y1}H_a\ ,\\ \frac{dv_{y1}}{dt}+\frac{v_{y1}}{\tau}&=-\frac{e}{mc}\,v_{x1}H_a\ ,\end{aligned} \tag{13}$$

where $\vec{v}_1=(v_{x1},\,v_{y1})$ is the mean value of the electron velocity. The term $(v_{x1}+v_L)/\tau$ includes the friction force acting on the electrons by lattice scattering. Solving Eq. (13) we get

$$\begin{aligned}v_{x1}(t)&=-\frac{v_L}{1+(\omega_c\tau)^2}\\ &\quad\times\left[1+e^{-t/\tau}(\omega_c\tau\sin\omega_ct-\cos\omega_ct)\right]\,,\\ v_{y1}(t)&=\frac{v_L\omega_c\tau}{1+(\omega_c\tau)^2}\\ &\quad\times\left[\omega_c\tau(1-e^{-t/\tau}\cos\omega_ct)-e^{-t/\tau}\sin\omega_ct\right]\,,\end{aligned} \tag{14}$$

where $\omega_c=eH_a/mc$ is the cyclotron frequency. In the derivation of Eq. (14), we assumed the initial velocity of the electrons to be zero.

Since the electrons are subjected to the scattering only in the core region, it is natural to assume that the scattering characteristics vary according to the relative length of the transport relaxation time τ in comparison with the mean stay time t_s of the electrons in the core. When $t_s\gg\tau$, all core electrons are considered as having undergone several scatterings. Putting $e^{-t/\tau}\ll1$, we find from Eq. (14) an averaged velocity

$$\begin{aligned}\bar{v}_{x1}&=-v_L/(1+\omega_c^2\tau^2)\ ,\\ \bar{v}_{y1}&=v_L\,\omega_c\tau/(1+\omega_c^2\tau^2)\ .\end{aligned} \tag{15}$$

The stay time is estimated as

$$t_s\sim\xi_0/(\bar{v}_{x1}^2+\bar{v}_{y1}^2)^{1/2}=(v_c/v_L)\,\tau\ ,$$

with $v_c=\xi_0/\tau$. Thus, the condition $t_s\gg\tau$ is rewritten as $v_L/v_c\ll1$.

When $t_s\ll\tau$ and $\omega_ct_s\ll1$, we have another averaged velocity

$$\begin{aligned}\bar{v}_{x1}&\sim\left(\frac{\partial v_x}{\partial t}\right)_{t=t_s}t_s\approx\frac{-v_Lt_s\cos\omega_ct_s}{\tau}\ ,\\ \bar{v}_{y1}&\sim\left(\frac{\partial v_{y1}}{\partial t}\right)_{t=t_s}t_s\approx\frac{v_Lt_s\sin\omega_ct_s}{\tau}\quad .\end{aligned} \tag{16}$$

The stay time is estimated as $t_s\sim(v_c/v_L)^{1/2}\tau$, and the conditions $t_s\ll\tau$ and $\omega_ct_s\ll1$ are rewritten $v_L/v_c\gg1+(\omega_c\tau)^2$.

If $t_s \ll \tau$ and $\omega_c t_s \gg 1$, the time-averaged velocity may be obtained by putting $\sin\omega_c t = \cos\omega_c t = 0$ in Eq. (14), because the cyclotron motion which the electrons are supposed to experience may be neglected for the time-averaged process. Then we have the same expression for the velocity as Eq. (15), and the conditions $t_s \ll \tau$ and $\omega_c t_s \gg 1$ give $1 \ll v_L/v_c \ll \omega_c\tau$.

In the lattice frame X, the mean velocity of the core electrons $\vec{v} = (v_x, v_y)$ is found as $\vec{v} = \vec{v}_1 + \vec{v}_L$. Therefore from Eqs. (15) and (16) the electric current that flows inside the core is obtained as follows:

$$J_\parallel = \frac{Nev_L\omega_c\tau}{(1+\omega_c^2\tau^2)} \quad \text{when } \frac{v_L}{v_c} \ll 1 \text{ and } 1 \ll \frac{v_L}{v_c} \ll \omega_c\tau$$
$$= Nev_c\omega_c\tau \quad \text{when } \frac{v_L}{v_c} \gg 1+\omega_c^2\tau^2, \tag{17}$$

$$J_\perp = \frac{Nev_L\omega_c^2\tau_c}{1+\omega_c^2\tau^2} \quad \text{when } \frac{v_L}{v_c} \ll 1 \text{ and } 1 \ll v_L/v_c \ll \omega_c\tau$$
$$= Nev_L\left[1-\left(\frac{v_c}{v_L}\right)^{1/2}\right] \quad \text{when } \frac{v_L}{v_c} \gg 1+\omega_c^2\tau^2\,, \tag{18}$$

where $J_\parallel$ and $J_\perp$ are the normal-current components parallel and perpendicular to the core electric field, respectively, and N is the electron density. Figures 1 and 2 represent the fluxoid velocity dependence of the core current densities given by Eqs. (17) and (18).

It must be emphasized that, when $v_L/v_c \ll 1 + \omega_c^2\tau^2$, $J_\parallel$ reduces markedly as $\omega_c\tau$ becomes larger than unity, and that $J_\parallel$ tends to a constant $Nev_c\omega_c\tau$ when $v_L/v_c \gg 1+\omega_c^2\tau^2$. These characteristics may be important in view of the realization of the high-speed acceleration of fluxoids because they show the possibility of the reduction of the power dissipation $J_\parallel E$ in pure specimens. A numerical example of the threshold velocity v_c is 2×10^4 cm/sec for the values $\xi_0 = 20$ Å and $\tau = 10^{-11}$ sec. The velocity is less than that of sound in metals.

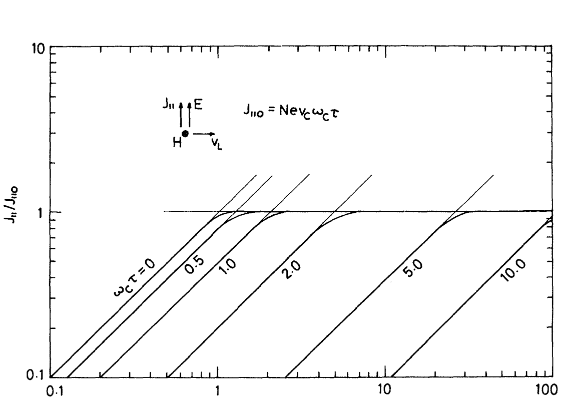

FIG. 1. Fluxoid velocity dependence of the normal-current density $J_\parallel$ along the electric field ($v_c = \xi_0/\tau$, $J_{\parallel 0} = Nev_c\omega_c\tau$).

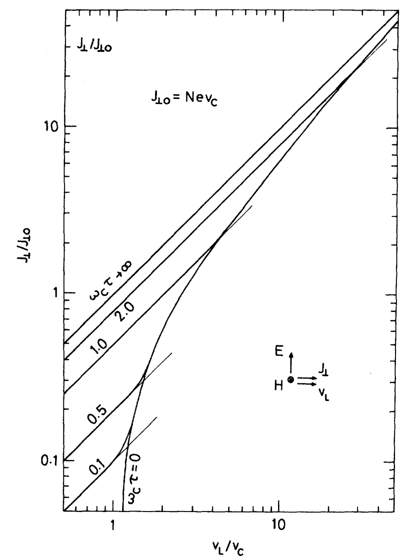

FIG. 2. Fluxoid velocity dependence of the normal-current density $J_\perp$ perpendicular to the electric field ($J_{\perp 0} = Nev_c$).

III. ELASTIC FIELD ABOUT A HIGH-SPEED FLUXOID

In this section we investigate the elastic-field distribution around a fluxoid in motion by solving a modified elastic-wave equation with superconductivity parameters, and we discuss the fluxoid velocity dependence of the flux-pinning effect. The wave equation is a generalization of the modified elastic equation[16,17] which is obtained by a variational method from the Ginzburg-Landau free energy with appropriate phenomenological parameters. We treat our problem with the following assumptions. (a) The crystal is treated as an isotropic elastic continuum. (b) All the surface effects are negligible. (c) The elastic modulus and the lattice density are constant.[22] (d) Space and time variations of lattice displacements occur only over distances greater than the coherence length ξ_0 and over times greater than $h/k_B T_c$, where $k_B T_c$ is the critical temperature. (e) We investigate extreme type-II superconductor in low magnetic field ($H \gtrsim H_{c1}$). The modified wave equation is given by

$$\mu[(1-2\nu)^{-1}\,\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \nabla^2\vec{u}] - \rho\,\frac{\partial^2\vec{u}}{\partial t^2} = \frac{H_c^2}{8\pi}\,\vec{\nabla}\left(a\left|\frac{\psi}{\psi_0}\right|^2 - \frac{b}{2}\left|\frac{\psi}{\psi_0}\right|^4\right), \tag{19}$$

with $a = \partial \ln H_{c2}/\partial\epsilon$ and $b = \partial \ln\kappa^2/\partial\epsilon$, where $\vec{u}$ is the displacement vector, μ is the modulus of rigidity, ν is Poisson's ratio, ρ is the mass density of the lattice, H_c is the critical field at zero strain, ϵ is the strain, κ is the Ginzburg-Landau parameter, and H_{c2} is the upper critical field.

As was discussed in Sec. II, in a pure superconductor in which $\omega_c\tau \gg 1$ is satisfied, the power dissipation p in the normal core of a moving fluxoid may be small. The viscous-drag force f_v, which nearly amounts to p/v_L, is also small there. We treat this pure case neglecting the lattice deformation by f_v, though the effect of the deformation is easily taken into consideration by including in Eq. (19) the term of the drag force.

We solve Eq. (19) when the fluxoid velocity is in the neighborhood of or less than the sound velocity v_s, where the Lorentz-like contraction of the variation pattern of $|\psi|$ near the core is negligible, as seen from Eqs. (9) and (10). We may consider our problem in two dimensions, since the strain field about the fluxoid lacks an axial component.[17] We choose the frame X with the magnetic field in the z direction.

At first we suppose a uniform motion of fluxoid in the x direction with a velocity v_L. From Eq. (19) the Fourier transform of the displacement vector $\vec{u}$ becomes

$$\vec{u}_k = -iK\vec{k}\, e^{-ik_x v_L t}/(k_x^2 g_-^2 + k_y^2) , \tag{20}$$

with

$$K = \frac{(a-\frac{1}{3}b)(H_c^2/8\pi)\,\xi_0^2}{8\rho v_s^2} , \quad g_- = \left(1 - \frac{v_L^2}{v_s^2}\right)^{1/2} ,$$

$$v_s = \left(\frac{2(1-\nu)\mu}{(1-2\nu)\rho}\right)^{1/2} ,$$

and wave vector $\vec{k} = k_x\vec{1}_x + k_y\vec{1}_y$, where $\vec{1}_x$ and $\vec{1}_y$ are the unit vectors in the x and y direction, respectively. To derive Eq. (20), we use the assumption $|k| < \xi_0^{-1}$ and a simplified expression of the variation of the order-parameter amplitude[19]

$$\begin{aligned} |\psi/\psi_0| &= r_1/\pi^{1/2}\xi_0 \quad &&\text{when} \quad 0 < r_1 < \pi^{1/2}\xi_0 \\ &= 1 &&\text{when} \quad \pi^{1/2}\xi_0 < r_1 , \end{aligned}$$

where $r_1 = [(x - v_L t)^2 + y^2]^{1/2}$. The reverse transform of Eq. (20) obtained by excecuting the integration in the complex k_x and k_y plane becomes

$$\vec{u} = \frac{2\pi K}{g_-} \frac{(x - v_L t)\,\vec{1}_x + g_-^2 y\,\vec{1}_y}{(x - v_L t)^2 + g_-^2 y^2} , \tag{21}$$

when $0 < v_L < v_s$; and for $v_L > v_s$

$$\vec{u} = (2\pi^2 K)\,\delta(x - v_L t + g_+|y|)\,[g_+^{-1}\vec{1}_x + (y/|y|)\,\vec{1}_y] , \tag{22}$$

with $g_+ = (v_L^2/v_s^2 - 1)^{1/2}$ and $x - v_L t < 0$. If we keep in mind the assumption $|k| < \xi_0^{-1}$, the δ function in Eq. (22) may be better expressed by

$$\frac{\sin[(x - v_L t + g_+|y|)/\xi_0]}{\pi(x - v_L t + g_+|y|)} .$$

We see from Eq. (21) that the displacement field associated with a subsonic fluxoid is localized around it. Since the flux pinning is caused by the interaction betwen this localized strain field and those of lattice defects, a subsonic fluxoid must experience essentially the same kind of pinning effects as a static fluxoid does. A supersonically accelerated fluxoid, however, does not carry a localized field but radiates an elastic shock wave, as shown in Eq. (22). This phenomenon is analogous to the Čerenkov radiation,[23] which is observed when a high-energy charged particle moves through a medium at velocities exceeding the local speed of light in the medium.

It is well known that the fluxoids moving in an actual superconducting material are always subjected to the drag force by the pinning effect.[1] The disappearance of the pinning effect for supersonic fluxoids may result in a jump of the fluxoid velocity or macroscopic electric field, when we accelerate the fluxoids over the sound velocity by increasing the driving force with applied conduction current.

Now we discuss the elastic field associated with a fluxoid vibrating in a superconductor. It is well known that circularly polarized vibration modes can propagate along fluxoids.[24] Since a circularly polarized vibration is decomposed into two linearly polarized modes, we can deal with one of the decomposed modes instead of investigating the circular mode directly. Assuming a linear harmonic oscillation of a fluxoid along the x axis with a condition $\int v_L dt = -d\cos\omega t$, we get from Eq. (19) the following Fourier-transformed expression of the displacement vector:

$$\vec{u}_k = -iK\vec{k}\left(\frac{J_0(k_x d)}{k^2} + 2\sum_{l=1}^{\infty} \frac{i^l J_l(k_x d)\cos l\omega t}{k^2 - l^2\omega^2/v_s^2}\right) ,$$

where $J_l(z)$ is the lth-order Bessel function. By assuming $\omega d/v_s \ll 1$ or the subsonic fluxoid velocity, the reverse transform is obtained as

$$\vec{u} = 2\pi K\left[\frac{\vec{r}}{r^2} - 2\pi\vec{\nabla}\sum_{l+1}^{\infty}\left(\frac{l\omega d}{v_s}\right)^l \cos l\omega t/l\,! \times \left(\frac{\partial}{\partial X}\right)^l J_0\left((X^2+Y^2)^{1/2}\right)\right] ,$$

with $\vec{r} = x\vec{1}_x + y\vec{1}_y$, $X = l\omega x/v_s$, and $Y = l\omega y/v_s$. Neglecting the terms $l \geqq 2$ and using formulas

$$2\,\frac{\partial}{\partial z} J_n(z) = J_{n-1}(z) - J_{n+1}(z) ,$$

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z) ,$$

we get

$$\vec{u} = 2\pi K\left\{\frac{\vec{r}}{r^2} + \frac{\pi\omega^2 d}{v_s^2}\left[\left(J_0\left(\frac{\omega r}{v_s}\right) - \frac{x^2-y^2}{r^2} J_2\right.\right.\right.$$

$$\times\left(\frac{\omega r}{v_s}\right)\bigg)\vec{1}_x-\frac{2xy}{r^2}J_2\left(\frac{\omega r}{v_s}\right)\vec{1}_y\bigg]\cos\omega t\bigg\} \ . \quad (23)$$

In order to obtain the displacement field associated with the circularly polarized vibration, we combine Eq. (23) with the elastic field due to a fluxoid which is linearly vibrating along the y axis with $\frac{1}{2}\pi$ phase difference. The resultant expression is

$$\vec{u}=2\pi K\left\{\frac{\vec{r}}{r^2}+\frac{\pi\omega^2 d}{2^{1/2}v_s^2}\left[J_0\left(\frac{\omega r}{v_s}\right)(\vec{1}_x\cos\omega t+\vec{1}_y\sin\omega t)\right.\right.$$
$$\left.\left.-J_2\left(\frac{\omega r}{v_s}\right)[\vec{1}_x\cos(2\theta-\omega t)+\vec{1}_y\sin(2\theta-\omega t)]\right]\right\} \ , \quad (24)$$

with $\theta=\tan^{-1}(y/x)$. The static term of Eq. (24) corresponds to Eq. (21) with $v_L=0$ and shows the localized strain field around the fluxoid. The elastic energy density brought about the static strain decreases by r^{-2}. For large r, Eq. (24) reduces to

$$u=2\pi K\frac{\pi^{1/2}\omega^2 d}{v_s^2}\left(\frac{\omega r}{v_s}\right)^{-1/2}\cos\left(\frac{\omega r}{v_s}-\frac{1}{4}\pi\right)$$
$$\times\cos(\omega t-\theta)(\vec{1}_x\cos\theta+\vec{1}_y\sin\theta) \ .$$

This expression denotes that the total energy density due to the oscillatory terms in Eq. (24) is proportional to r^{-1} for large r. These terms express an elastic-field radiation since they contribute to the net-energy flow through a cyclindrical surface of infinite radius.

Comparing the static term and the oscillatory ones in Eq. (24), we see that the static localized elastic field near the fluxoid core suffers a considerable disturbance from the oscillatory field when

$$\pi\omega^2 d/v_s^2 \gtrsim 1/r_{\text{core}}\approx 1/\xi_0 \ , \quad (25)$$

where r_{core} is the core radius. Then we may say that the vibrating fluxoid which happens to be pinned by a localized elastic source easily finds the opportunity to get over them insofar as the vibration satisfies the condition (25). Thus the flux pinning is expected to be ineffective for fluxoids moving at high frequency with subsonic velocity. If we choose numerical values $v_s=3\times10^5$ cm/sec, $\omega d=0.2v_s$, and $\xi_0=20\sim100$ Å, the threshold frequency is found to be 8×10^{10}–4×10^{11} Hz.

On the other hand, the localized strain field may vanish for a fluxoid vibrating with supersonic velocity, which is easily deduced from the properties of the elastic field for a supersonic fluxoid in uniform motion. Therefore the disappearance of flux pinning is also expected when

$$\omega d>v_s \ . \quad (26)$$

The threshold frequency becomes about 5×10^8 Hz in this case for the vibration amplitude $d=1$ μm.

It has been observed[25] that at microwave frequencies, type-II specimens exhibit full flux-flow resistivity even though the microwave current density induced in the specimen is several orders of magnitude lower than the critical current density needed for the dc and low-frequency flux-flow state. This fluxoid-depinning phenomenon can be explainable on the basis of the ideas discussed above. Cape *et al.*[26] observed the depinning of fluxoids by superimposing a small longitudinal oscillatory field $b(t)$ on the transverse dc magnetic field B_0. They found that the depinning threshold of the oscillatory field $b_c(t)$ follows a relation

$$b_c\propto\omega^{-1/2} \ . \quad (27)$$

This proportionality seems to be explainable if we assume that the vibration velocity of fluxoids reaches the sound velocity in the threshold state, and that the fluxoid velocity is proportional to the oscillatory component of magnetic pressure gradient. Then we have

$$|\vec{\nabla}b_c^2(t)|=\text{const}\times v_L, \quad v_L=v_s \ ,$$

and with the conservation equation of the fluxoid number $\vec{\nabla}\cdot(\vec{v}_L b_c)=-\partial b_c/\partial t$, we arrive at Eq. (27).

IV. CONCLUSION

We have seen that in type-II superconductors at low temperature in low-magnetic-field region, several nonlinear effects are expected for the moving fluxoids in the high-speed flow state. When the fluxoid velocity v_L exceeds the coherence length ξ_0 divided by the electron relaxation time τ, the characteristics of the normal-excitation current inside the fluxoid core show a considerable velocity dependence. The saturation of the current component along the core electric field is found when $v_L>(\xi_0/\tau)\times(1+\omega_c^2\tau^2)$. A remarkable velocity dependence is also expected about the flux-pinning effect which is caused by the interaction between the localized elastic field associated with the fluxoids and that of lattice defects. Though a subsonic fluxoid is accompanied with a localized elastic field, a supersonically accelerated fluxoid radiates elastic shock wave. So that the flux pinning is expected to disappear when the fluxoid velocity exceeds the sound velocity. The elastic-wave radiation and lowering of the pinning force are also expected for a vibrating fluxoid. In this case the phenomena are found even for the fluxoid moving with a subsonic velocity.

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Electron-Tunneling Studies of Dilute Pb-Based In Alloys*

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The technique of electron tunneling has been used to obtain the effective phonon spectra, $\alpha^2(w)F(w)$, and the parameters defining the superconductivity for dilute Pb-In alloys by solving the Eliashberg equations. These quantities were extracted from the measured energy gap of the alloy films and the measured normalized conductance of the thin-film tunnel junctions of the form $Al-Al_2O_3-Pb-In$. All the measurements were carried out around 1 K, at which temperature both the aluminum and the alloy films were in the superconducting state. The normal-state data were taken by raising the temperature above the transition temperature of both the films. A band of frequencies, the so-called impurity band, appeared beyond the high-frequency cutoff of the phonon spectrum of pure Pb. For all the alloys studied the impurity band was found to consist of at least two peaks which are attributed to the vibrations of isolated indium atoms and the vibrations of the pairs of indium atoms, both surrounded by the host lead atoms. The position of the first peak has been found to be constant (9.57 ± 0.03 meV), and its width has been found to vary linearly with concentration of indium. The position of the second peak remains constant up to 2-at.% indium but increases with the further addition of indium. The width of the second peak also varies linearly with the indium concentration. The fraction of modes in the impurity band has been found to be a factor 1.5–2.0 less than the concentration of indium. A determination of the ratio of the energy gap to the superconducting transition temperature for the alloy films shows that the electron-phonon interaction remains nearly constant for all the alloys studied.

I. INTRODUCTION

The problem of the disordered lattice has been of considerable interest for some time. This is due to the large effects of imperfections on the properties of the solids and to the fact that imperfections are always present. Although a number of theoretical methods have been developed for the study of the disordered lattice, the problem still remains, at least in regard to the phonon spectrum, not solved rigorously. The most recent extensive work on such phonon spectra has been done by Dean[1] on one- and two-dimensional disordered lattices and by Payton and Visscher[2] on one-, two-, and three-dimensional lattices. All these calculations are for limited-size lattice only. The central result of these calculations is that when light impurities are added to the heavy host lattice, a number of peaks appear beyond the high-frequency cutoff of the phonon spectrum of the host lattice. Dean has attributed these peaks to the vibrational modes of impurity clusters of different