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PHYSICAL REVIEW B VOLUME 6, NUMBER 1 1 JULY 1972

Dynamic Structure of Vortices in Superconductors. II. $H \ll H_{c2}$ †

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(Received 22 February 1972)

The time-dependent Ginzburg-Landau equations applicable to a gapless superconductor containing a high concentration of paramagnetic impurities are solved to find the field and charge distributions around an isolated vortex moving in a transport current. The initial slope of the flux-flow resistance with respect to the average magnetic field is found to be approximately $\frac{1}{3}$ the normal resistance divided by H_{c2}. A backflow current is generated for all physical values of the parameters, vanishing only for a special case not possible for this system.

I. INTRODUCTION

In a previous paper[1] we solved a complete set of time-dependent Ginzburg-Landau (GL) equations to find the local current, charge, and field distributions when a transport current is forced through a superconductor in the mixed state near the upper critical magnetic field H_{c2}. This explicit solution was obtained by linearizing the equations in the order parameter, which becomes small as H_{c2} is approached . In the present paper we extend our work to lower magnetic fields, particularly to quite low fields where the vortices are well separated and may be studied individually. Our method of approaching this problem must be different from before owing to the nonlinearity of the GL equations, since explicit analytic solutions for the spatial dependence of the order parameter and magnetic field have not been obtained even in the static case when no transport current is applied. The equations can be linearized in the regions near and far from the vortex core, and explicit features of the solution are obtained. However, not enough information is obtained from these asymptotic solutions to determine the dissipation rate. Consequently, we, like Schmid,[2] derive energy-balance relations involving integrals of the solutions over the sample, which we can evaluate using an approximation introduced by Schmid. Unlike Schmid, we consider the backflow current which generally arises around each vortex.

A major qualitative feature of the effect of a transport current on the mixed state is that in the absence of pinning defects the vortex structure moves at right angles to the transport current. If a vortex consisted of a normal region (core) of a certain size with a sharp phase boundary into the surrounding superconducting region, the magnitude of the drift velocity v and the current distribution would be easily obtained by a Lorentz transformation of the static structure. If the shape of the core does not change along the direction of the magnetic field, it will have a uniform value B inside the core and vanish outside of it. Movement of the core then generates a uniform electric field in the core $\vec{E} = -\vec{v}\times\vec{B}$. Outside the core $E=0$, and the superfluid flow is not disturbed. If the strength of E is just the ratio of the transport current j_t to the normal-state conductivity σ, the transport current will be driven across the normal core at the same rate as it approaches from the superconducting region, and a steady-state solution is obtained with a uniform transport current and the vortex translating with $\vec{v}=\vec{j}_t\times\vec{B}/\sigma B^2$.

Actually, however, for a singly quantized vortex, the width of the superconducting-normal boundary region is the same as the radius of the core, and this simple picture is incomplete. The steady state is achieved as the result of a balance between the various screening processes which are governed by different characteristic lengths. Generally, the local electric field is not proportional to the local magnetic field, and an additional backflow current arises to compensate for the different characteristic lengths. The purpose of our paper is to determine the average dissipation rate, which involves determining the relation between v and j_t, and also some details of the local current, charge, and field distributions.

Section II is devoted to a derivation of expressions for the energy input and dissipation rate. In Sec. III these results are applied to an isolated vortex in the high-κ limit to find the effective resistivity. The backflow and local-field distributions are studied for this high-κ limit in Sec. IV. The local electric field is proportional to the local magnetic field only when their respective screening lengths are equal, and this limit is studied in Sec. V for all κ. Finally, the only special case where backflow is absent, when the two screening lengths are equal and also $\kappa=1/\sqrt{2}$, is examined in Sec. VI.

II. ENERGY BALANCE

The basic equations whose solutions we seek are the same ones we used in our previous paper, which have been derived from microscopic theory by Gor'kov and Ėliashberg[3] for a gapless superconductor containing a high concentration of paramagnetic impurities:

$$\gamma\left(\frac{\partial}{\partial t}+i\,2e\,\psi\right)\Delta+\xi^{-2}(|\Delta|^2-1)\Delta+\left(\frac{\vec{\nabla}}{i}-2e\,\vec{A}\right)^2\Delta=0\,, \quad (1)$$

$$\vec{j}=\sigma\left(-\vec{\nabla}\psi-\frac{\partial\vec{A}}{\partial t}\right)+\mathrm{Re}\left[\Delta^*\left(\frac{\vec{\nabla}}{i}-2e\,\vec{A}\right)\Delta\right]\Big/8\pi e\lambda^2\,, \quad (2)$$

$$\rho=(\psi-\varphi)/4\pi\lambda_{TF}^2\,. \quad (3)$$

γ is the inverse of the normal-state diffusion constant D, ψ is the electrochemical potential divided by the electronic charge e, and Δ is the order parameter divided by its equilibrium value in the absence of fields $\Delta_0=\pi[2(T_c^2-T^2)]^{1/2}$, where T is the temperature and T_c its critical value. Thus our reduced Δ equals unity in the absence of fields. The temperature-dependent coherence length is $\xi=(6D/\tau_s)^{1/2}/\Delta_0$, where τ_s is the spin-flip scattering time. σ is the normal-state conductivity, and the temperature-dependent magnetic field screening length is $\lambda=(8\pi\sigma\tau_s)^{-1/2}/\Delta_0$. The GL parameter κ is defined as usual as $\kappa=\lambda/\xi$, and λ_{TF} is the Thomas-Fermi static-charge screening length. (We have set $\hbar=c=k_B=1$.) This set of equations is completed by the Maxwell equations coupling the scalar and vector potentials φ and A to the charge and current densities ρ and j.

A set of equations having essentially the same form was derived earlier by Schmid. However, Ėliashberg[4] has shown that this set of equations is not valid in the case of weak pair breaking considered by Schmid. The one difference in the forms of Eqs. (1)–(3) and Schmid's is a very small additional term which Schmid included in ρ resulting from the curvature of the Fermi surface. Such corrections are of order $[(\text{Fermi wavelength})/\xi]^2$ relative to the main effects we are studying, so we ignore them. If we wanted to study these higher-order corrections, we would like to find all of them and not just the contribution to ρ. In fact, Schmid only kept this correction briefly to indicate the possibility of a small charge redistribution in a static vortex, and then dropped the correction when he went on to investigate dynamic properties. Our way of finding the rate of dissipation for a moving vortex is very similar to Schmid's work, but with some corrections and additions.

Using the Eqs. (1)–(3), Schmid has derived an energy-balance relation

$$\frac{\partial F}{\partial t} + W + \vec{\nabla}\cdot\vec{j}^E = 0 . \qquad (4)$$

This equation shows that the rate of increase of the free energy F plus the rate of dissipation W equals the inflow of energy current $\vec{j}^E$. Converting his expressions into our notation, F and $\vec{j}^E$ consist of three parts: electromagnetic (em), normal (n), and ordering [superconducting-normal (sn) difference].

$$F_{em} = (B^2 + E^2)/8\pi ,$$

$$F_n = \int (\psi - \varphi) d\rho ,$$

$$F_{sn} = [\,|(\vec{\nabla}/i - 2e\vec{A})\,\Delta|^2 + \xi^{-2}(\tfrac{1}{2}|\Delta|^4 - |\Delta|^2)\,]/32\pi e^2\lambda^2 ,$$

$$W = \sigma\left(\vec{\nabla}\psi + \frac{\partial\vec{A}}{\partial t}\right)^2 + \gamma\left|\left(\frac{\partial}{\partial t} + i\,2e\psi\right)\Delta\right|^2/16\pi e^2\lambda^2 , \qquad (5)$$

$$\vec{j}^{\,E}_{em} = (\vec{E}\times\vec{B})/4\pi ,$$

$$\vec{j}^{\,E}_{n} = (\psi - \varphi)\vec{j} ,$$

$$\vec{j}^{\,E}_{sn} = -\,\mathrm{Re}\left[\left(\frac{\partial}{\partial t} + i\,2e\psi\right)\Delta(\vec{\nabla} + i\,2e\vec{A})\,\Delta^*\right]\Big/16\pi e^2\lambda^2 .$$

To obtain the dissipation rate in low magnetic fields, Schmid introduced an approximate form for the static order parameter which enabled him to approximately solve for the field distributions and evaluate the expression for the average $\langle W\rangle$. $\langle W\rangle$ is then proportional to the square of the velocity of translation of the vortex structure v, but the relation of v to the transport current is not yet known. The ratio would be known if one could solve for the moving order parameter as has been done near H_{c2}, but this is much more difficult for low magnetic fields owing to the nonlinearity of Eq. (1). To circumvent this difficulty, he found a relation indirectly by stating without proof that $\langle W\rangle$ should equal the average current times the average effective electric field, $j_t\langle -\partial A/\partial t - \nabla\psi\rangle$. Although this relation may appear evident if there is no backflow, we now know that backflow is a general feature of moving vortices. Thus one is not sure whether the backflow current may also contribute to the dissipation .

We think it worthwhile to examine the basis of Schmid's alternative expression for $\langle W\rangle$. Using Maxwell's equations one finds immediately

$$\vec{\nabla}\cdot\vec{j}^{\,E}_{em} + \frac{\partial F_{em}}{\partial t} + \vec{j}\cdot\vec{E} = 0 . \qquad (6)$$

The driving potential for electrons in a metal is not the electric scalar potential φ but the electrochemical potential $e\psi$. Consequently, it is worthwhile to formally define a vector field $\vec{\mathcal{E}} = -\partial\vec{A}/\partial t - \vec{\nabla}\psi$, which is to be distinguished from the actual electric field $\vec{E} = -\partial\vec{A}/\partial t - \vec{\nabla}\varphi$. The practical difference between ψ and φ and between $\mathcal{E}$ and E is negligible since λ_{TF} is much less than the distance over which the charge density varies ($\sim\xi$). Rewriting $\vec{j}\cdot\vec{E}$ in terms of $\mathcal{E}$ we get

$$\vec{j}\cdot\vec{E} = \vec{j}\cdot\vec{\mathcal{E}} + \vec{j}\cdot\vec{\nabla}(\psi - \varphi)$$
$$= \vec{j}\cdot\vec{\mathcal{E}} + \nabla\cdot\vec{j}^{\,E}_{n} + \frac{\partial F_n}{\partial t} . \qquad (7)$$

Substituting Eqs. (6) and (7) back into (4) and (5), we get

$$\vec{j}\cdot\vec{\mathcal{E}} = W + \frac{\partial F_{sn}}{\partial t} + \vec{\nabla}\cdot\vec{j}^{\,E}_{sn} . \qquad (8)$$

This relation was also obtained by Schmid during his derivation of Eqs. (4) and (5). It is a straightforward consequence of the basic equations (1) and (2), so we will not repeat its derivation. We note that the net energy input in the steady state is fundamentally the total current times $\mathcal{E}$ and not simply the average transport current times $\mathcal{E}$, although the two quantities may be practically equal.

To investigate the difference between $\vec{j}\cdot\vec{\mathcal{E}}$ and $\vec{j}_t\cdot\vec{\mathcal{E}}$, we define $j = j_t + \mathcal{J}$, where $j_t = \langle j\rangle$ is a constant and $\langle\mathcal{J}\rangle = 0$. $\mathcal{J}$ is the screening and backflow current. We similarly define $B = B_t + \mathcal{B}$, where B_t is the magnetic field generated by j_t and $\mathcal{B}$ is the rest: $\vec{\nabla}\times\vec{\mathcal{B}} = 4\pi\vec{\mathcal{J}} + \partial\vec{E}/\partial t$. Repeating the above steps in the reverse order we obtain

$$\vec{j}_t\cdot\vec{\mathcal{E}} = W + \frac{\partial F_{sn}}{\partial t} + \vec{\nabla}\cdot\vec{j}^{\,E}_{sn} + \frac{\partial F_n}{\partial t} + \vec{\nabla}\cdot\vec{\mathcal{J}}^{\,E}_{n} + \partial\mathcal{F}_{em}/\partial t + \vec{\nabla}\cdot\vec{\mathcal{J}}^{\,E}_{em} , \qquad (9)$$

where $\vec{\mathcal{J}}^{\,E}_n = (\psi - \varphi)\vec{\mathcal{J}}$, $\mathcal{F}_{em} = (\mathcal{B}^2 + E^2)/8\pi$, and $\vec{\mathcal{J}}^{\,E}_{em} = \vec{E}\times\vec{\mathcal{B}}/4\pi$. The net energy currents enter the vortex from both ends along the lines of the external magnetic field $\vec{B}_e$. $\langle\vec{j}_t\cdot\vec{\mathcal{E}}\rangle$ gives the main energy input to the lowest interesting order $\sim v^2$, since at the sample surfaces normal to $\vec{B}_e$ the perpendicular component of $\vec{\mathcal{J}}$ vanishes and $\vec{\mathcal{B}}$ is parallel to $\vec{B}_e$ to order v.

III. DISSIPATION RATE OF AN ISOLATED VORTEX WHEN $\kappa \gg 1$

To discuss an isolated vortex it is convenient to separate Δ into its magnitude and phase $|\Delta| = f$, $\Delta = f e^{i2e\chi}$, and to define the gauge invariant quantities $\vec{Q} = \vec{A} - \vec{\nabla}\chi$ and $P = \psi + \partial\chi/\partial t$. Our basic equation are then rewritten in the new notation with Eq. (1) separated into its real and imaginary parts:

$$\gamma\frac{\partial f}{\partial t} + \xi^{-2}(f^3 - f) - \nabla^2 f + 4e^2Q^2 f = 0 , \qquad (10)$$

$$\gamma f^2 P + \vec{\nabla}\cdot(f^2\vec{Q}) = 0 , \qquad (11)$$

$$\vec{j} = \sigma\left(-\vec{\nabla}P - \frac{\partial\vec{Q}}{\partial t}\right) - \frac{f^2\vec{Q}}{4\pi\lambda^2} , \qquad (12)$$

$$\rho=(\psi-\varphi)/4\pi\lambda_{\rm TF}^2 . \tag{13}$$

Similarly, the dissipation rate becomes

$$W=\sigma\left(\vec{\nabla}P+\frac{\partial\vec{Q}}{\partial t}\right)^2+\gamma\left[\left(\frac{\partial f}{\partial t}\right)^2\Big/4e^2+P^2f^2\right]\Big/4\pi\lambda^2 . \tag{14}$$

An equation to determine Q is obtained from Eq. (12) using a Maxwell equation:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{Q})=4\pi\vec{j}+\frac{\partial\vec{E}}{\partial t}$$

$$=4\pi\sigma\left(-\vec{\nabla}P-\frac{\partial\vec{Q}}{\partial t}\right)+\left(-\vec{\nabla}\frac{\partial P}{\partial t}-\frac{\partial^2 Q}{\partial t^2}\right)-f^2\vec{Q}/\lambda^2 . \tag{15}$$

In the static case this equation reduces to the usual screening equation $\vec{\nabla}\times(\vec{\nabla}\times\vec{Q})+f^2\vec{Q}/\lambda^2=0$. In the dynamic case, the terms involving P can be eliminated by taking the curl of the entire equation.

A relation to determine P is obtained by taking the divergence of Eq. (12) and using the continuity equation $\vec{\nabla}\cdot\vec{j}+\partial\rho/\partial t=0$, which is a consequence of Maxwell's equations. Using Eq. (11) to eliminate the divergence of the last term in Eq. (12), we obtain

$$\sigma\left(\nabla^2P+\frac{\partial\vec{\nabla}\cdot\vec{Q}}{\partial t}\right)=\frac{\gamma f^2P}{4\pi\lambda^2}+\frac{\partial\rho}{\partial t} . \tag{16}$$

The last term $\partial\rho/\partial t$ is of second order in v since ρ vanishes in equilibrium to the order we are interested in. We will neglect it since we only wish to find P to first order in v. It is natural to combine the constants to form a screening length ζ for P, $\zeta^2=4\pi\lambda^2\sigma/\gamma$. Examining the microscopic expressions for λ and ξ we see that $\zeta=\xi/\sqrt{12}$ for a superconductor containing a high concentration of paramagnetic impurities. This is the physical value of ζ for the system for which the basic equations were derived. However, we may also study the system of equations for an arbitrary ζ/ξ ratio for general interest and in case the equations are later found to apply to another system with a different ratio.

We choose a system of coordinates with the z direction along the externally applied magnetic field B_e with the origin of the x-y or r-θ plane at the center of a vortex. We choose the x or $\theta=0$ axis as the direction of motion v. The direction of $\vec{j}_t$ is then along the y or $\theta=\frac{1}{2}\pi$ axis.

In the static limit the solution for $\vec{Q}$, $\vec{Q}_0$ is known to have the form $\vec{Q}_0=q_0(r)\hat{e}_\theta$. The static screening current is thus also just a function of r times $\hat{e}_\theta$. Near the origin, the behavior of $\vec{Q}_0$ is dominated by the phase contribution $-\vec{\nabla}\chi$. Since the phase $2e\chi$ is just the angle θ there, $q_0(r)\to-1/2\,er$. This term gives no contribution to $\vec{B}_0=\vec{\nabla}\times\vec{Q}_0$. The next term for small r gives the field at the origin: $q_0(r)\to-1/2er+\frac{1}{2}B_0(0)r$.

The form of P to first order in v follows from the form of $\partial\chi/\partial t$. The time derivative in the steady state is replaced by $-\vec{v}\cdot\vec{\nabla}$. Thus $P=p(r)\times\sin\theta$. Near the origin p must have the limit $p(r)\to v/2er+[\frac{1}{2}vB_0(0)-\mathcal{E}(0)]r$ so that $\vec{\mathcal{E}}(0)$ is the limit as r goes to 0 of $(-\vec{\nabla}P-\partial\vec{Q}/\partial t)$.

Using these forms for Q and P we can obtain a simplified expression for the dissipation rate $\int W dV$ in the volume around a vortex. Beginning with Eq. (14) we find

$$\int W dV=\sigma\int\left\{\mathcal{E}^2+\left[\left(\frac{\partial f}{\partial t}\right)^2\Big/4e^2+P^2f^2\right]\Big/\zeta^2\right\}dV . \tag{17}$$

Using Eq. (16), noting that $\vec{\nabla}\cdot\vec{Q}_0=0$, the last term becomes

$$\int(P^2f^2/\zeta^2)dV=\int P\nabla^2P\,dV=-\int P\vec{\nabla}\cdot\vec{\mathcal{E}}dV . \tag{18}$$

The first term on the right-hand side of Eq. (17) gives

$$\int\mathcal{E}^2dV=\int\left(-\vec{\nabla}P-\frac{\partial\vec{Q}}{\partial t}\right)\cdot\vec{\mathcal{E}}dV . \tag{19}$$

Combining Eqs. (18) and (19) we have

$$\int\left(\mathcal{E}^2+\frac{P^2f^2}{\zeta^2}\right)dV=\int\left[\left(-\frac{\partial\vec{Q}}{\partial t}\right)\cdot\vec{\mathcal{E}}-\vec{\nabla}\cdot(P\vec{\mathcal{E}})\right]dV. \tag{20}$$

The last term of Eq. (20) is a surface integral and the first term on the right-hand side can be further rewritten

$$\int\left(-\frac{\partial\vec{Q}}{\partial t}\right)\cdot\vec{\mathcal{E}}dV=\int\left[\left(\frac{\partial Q}{\partial t}\right)^2+\left(\frac{\partial\vec{Q}}{\partial t}\right)\cdot\vec{\nabla}P\right]dV . \tag{21}$$

The last term is again a surface integral since $\vec{\nabla}\cdot\vec{Q}_0=0$. Using the explicit form for Q and performing the angular average we get

$$\int\left(\frac{\partial Q}{\partial t}\right)^2dV=\tfrac{1}{2}v^2\int\left[\left(\frac{\partial q}{\partial r}\right)^2+\left(\frac{q}{r}\right)^2\right]dV$$

$$=\tfrac{1}{2}v^2\int\left[B^2-2\left(\frac{q}{r}\right)\frac{\partial q}{\partial r}\right]dV . \tag{22}$$

The last term of Eq. (22) is also a perfect integral. Q and P vanish exponentially as $r\to\infty$ so the contributions to the surface integrals come only from the region $r\to0$. Combining all the previous steps, with L being the length of the cylindrical volume along the z direction and using the limiting expressions for Q and P as $r\to0$, we obtain our final simplified formula for the dissipation rate:

$$\int W dV=\tfrac{1}{2}\sigma v^2\int B^2dV+\sigma\int\left[\left(\frac{\partial f}{\partial t}\right)^2\Big/4e^2\zeta^2\right]dV$$

$$+L\sigma\int d\theta\left[\vec{r}\cdot\left(P\vec{\mathcal{E}}-P\frac{\partial\vec{Q}}{\partial t}\right)+\tfrac{1}{2}v^2q^2\right]\Big|_{r\to0},$$

$$\int W\,dV = \tfrac{1}{2}\sigma v^2 \int B^2 dV + \sigma \int \left[\left(\frac{\partial f}{\partial t}\right)^2 \Big/ 4e^2\zeta^2\right] dV + \frac{L\pi\sigma v[2\mathcal{E}(0) - vB(0)]}{2e} . \quad (23)$$

To proceed further and evaluate the dissipation to the leading order $\sim v^2$, we need expressions for the static magnetic field $B_0(r)$, for the static order parameter $f_0(r)$, and for the electric field $\mathcal{E}$ in the lowest order $\sim v$. Exact analytic expressions for the static quantities are not known. Consequently, we use an approximation introduced by Schmid (with an erroneous factor of 2 deleted): $f_0(r) \approx (1 + \xi^2/r^2)^{-1/2}$, for the region $r \ll \lambda$. This approximation has the virtue of being correct in the large region $\xi \ll r \ll \lambda$ where $f_0 = 1 - \xi^2/2r^2$. It also has the correct form at the origin $f \sim r$, although the slope is most probably wrong. One may easily verify that this approximation for f_0 is not an exact solution of Eq. (10).

With this approximation for f_0, Eqs. (15) and (16) become soluable for $\vec{Q}_0$ and P as found by Schmid:

$$\vec{Q}_0 = -\frac{(r^2/\xi^2+1)^{1/2}}{2er} \frac{K_1((r^2+\xi^2)^{1/2}/\lambda)}{K_1(\xi/\lambda)}\hat{e}_\theta ,$$
$$P = \frac{v(r^2/\xi^2+1)^{1/2}}{2er} \frac{K_1((r^2+\xi^2)^{1/2}/\zeta)}{K_1(\xi/\zeta)} \sin\theta . \quad (24)$$

The functions K_n are the modified Bessel functions of imaginary arguments which vanish exponentially at infinite arguments. Taking the curl of $\vec{Q}_0$, we find $\vec{B}_0$, where $2eH_{c2} = \xi^{-2}$:

$$\vec{B}_0(r) = \kappa^{-1} H_{c2} K_0((r^2+\xi^2)^{1/2}/\lambda)/K_1(\kappa^{-1})\,\hat{e}_z ,$$
$$\vec{B}_0(0) = \kappa^{-2} H_{c2} (\ln\kappa + 0.1)\,\hat{e}_z . \quad (25)$$

Similarly, $\mathcal{E}$ can be found from $\vec{\mathcal{E}} = -\vec{\nabla}P + \vec{v}\cdot\vec{\nabla}\vec{Q}$:

$$2e\mathcal{E}_r = v\left[(r^2/\xi^2+1)^{1/2} r^{-2}\left(\frac{K_1((r^2+\xi^2)^{1/2}/\zeta)}{K_1(\xi/\zeta)} - \frac{K_1((r^2+\xi^2)^{1/2}/\lambda)}{K_1(\xi/\lambda)}\right) + (\xi\zeta)^{-1} K_0((r^2+\xi^2)^{1/2}/\zeta)/K_1(\xi/\zeta)\right]\sin\theta ,$$

$$2e\mathcal{E}_\theta = v\left[(r^2/\xi^2+1)^{1/2} r^{-2}\left(-\frac{K_1((r^2+\xi^2)^{1/2}/\zeta)}{K_1(\xi/\zeta)} + \frac{K_1((r^2+\xi^2)^{1/2}/\lambda)}{K_1(\xi/\lambda)}\right) + (\xi\lambda)^{-1} K_0((r^2+\xi^2)^{1/2}/\lambda)/K_1(\xi/\lambda)\right]\cos\theta .$$

Notice that only if the two screening lengths are equal $\lambda = \zeta$ is the simple relation $\vec{\mathcal{E}} = -\vec{v}\times\vec{B}$ obtained, as for a rigid Lorentz transformation. At the origin

$$\vec{\mathcal{E}}(0) = \tfrac{1}{2} vH_{c2}\left(\frac{\xi K_0(\xi/\zeta)}{\zeta K_1(\xi/\zeta)} + \frac{\xi K_0(\xi/\lambda)}{\lambda K_1(\xi/\lambda)}\right)\hat{e}_y$$
$$= \tfrac{1}{2} vH_{c2}\left(\kappa^{-2}(\ln\kappa + 0.1) + \frac{\xi K_0(\xi/\zeta)}{\zeta K_1(\xi/\zeta)}\right)\hat{e}_y . \quad (26)$$

The two integrals on the right-hand side of Eq. (23) may also be evaluated with the approximate f_0:

$$\int B_0^2 dV = (L\pi H_{c2}/2e\kappa^2)[1 - K_0(\kappa^{-1})^2/K_1(\kappa^{-1})^2]$$
$$= L\pi H_{c2}/2e\kappa^2 , \quad (27)$$
$$\int \left(\frac{\partial f}{\partial t}\right)^2 dV = \int (\vec{v}\cdot\vec{\nabla}f_0)^2 dV$$
$$= \tfrac{1}{4} L\pi v^2 .$$

Combining all these results together we get

$$\int \frac{W\,dV}{L} = \frac{\pi\sigma v^2 H_{c2}}{2e}\left(\frac{1}{2\kappa^2} + \frac{\xi^2}{4\zeta^2} + \frac{\xi K_0(\xi/\zeta)}{\zeta K_1(\xi/\zeta)}\right) . \quad (28)$$

In the physical limit where $\xi^2/\zeta^2 = 12$, the first term in the parentheses is negligible since $\kappa \gg 1$. The second term which came from $\int (\partial f/\partial t)^2 dV$ gives 3, while the third term which came from $\mathcal{E}(0)$ gives 3.05.

We have thus found an expression for the dissipation rate per unit length of an isolated vortex in terms of its velocity of translation. However, to find the flux-flow resistance we need to know it as a function of the transport current j_t. To find a relation between j_t and v we set the dissipation rate equal to $\int \vec{j}_t \cdot \vec{\mathcal{E}}\,dV = j_t \int \mathcal{E}\,dV$. Although we noted that locally $\vec{\mathcal{E}} \neq -\vec{v}\times\vec{B}$, the averages are equal $\langle\vec{\mathcal{E}}\rangle = -\vec{v}\times\langle\vec{B}\rangle$. This result can be verified from our explicit solutions for $\mathcal{E}$ and B or more directly from Maxwell's equations $\int \vec{E}\cdot d\vec{y} = -d(\int B_z\,dx\,dy)/dt = -v\int B_z\,dy$. The average over x gives the above relation. The integral of B is known by flux quantization. Consequently, we have

$$\int \frac{W\,dV}{L} = j_t v\pi/e . \quad (29)$$

A relation between j_t and v is obtained by equating Eqs. (28) and (29):

$$j_t = \tfrac{1}{2}\sigma v H_{c2}\left(\frac{1}{2\kappa^2} + \frac{\xi^2}{4\zeta^2} + \frac{\xi K_0(\xi/\zeta)}{\zeta K_1(\xi/\zeta)}\right) . \quad (30)$$

The flux-flow resistivity may be defined as $\rho_f = \langle W\rangle/j_t^2 = \langle\mathcal{E}\rangle/j_t$. The ratio of ρ_f to the normal-state resistivity is R:

$$R = \sigma\langle\mathcal{E}\rangle/j_t = \sigma v\langle B\rangle/j_t$$
$$= \left(\frac{1}{4\kappa^2} + \frac{\xi^2}{8\zeta^2} + \frac{\xi K_0(\xi/\zeta)}{2\zeta K_1(\xi/\zeta)}\right)^{-1} \frac{\langle B\rangle}{H_{c2}} . \quad (31)$$

We find that the slope

$$H_{c2}\left.\frac{dR}{d\langle B\rangle}\right|_{\langle B\rangle\to 0} = \left(\frac{1}{4\kappa^2} + \frac{\xi^2}{8\zeta^2} + \frac{\xi K_0(\xi/\zeta)}{2\zeta K_1(\xi/\zeta)}\right)^{-1} . \quad (32)$$

In the physical case when $\zeta = \xi/\sqrt{12}$, the slope is evaluated to be

$$H_{c2} \frac{dR}{d\langle B\rangle}\bigg|_{\langle B\rangle\to 0} = 0.33 \,. \qquad (33)$$

Comparison with our previous result near H_{c2}, where the slope equals $\xi^2\kappa^2/\zeta^2[1.16(2\kappa^2-1)+1]$ (which for the physical case with κ large is 5.2), shows that the R-vs-$\langle B\rangle$ curve has positive curvature $d^2R/d\langle B\rangle^2 > 0$ in the physical case. If $\zeta \approx \xi/\sqrt{2}$, the curve would be approximately flat, and if $\zeta > \xi$, the curvature would be negative. Figure 1 illustrates the prediction for the physical case.

After the completion of our work, another independent work was published by Gor'kov and Kopnin[5] which also attempts to solve the same set of basic equations to find the dissipation rate in the low-field, high-κ limit. They found a relation between j_t and v similar to our Eq. (30) but neglected the electric field everywhere. Only the $\int(\partial f/\partial t)^2 r\,dr$ term contributes to the right-hand side of their equation. They have evaluated this integral numerically and found 0.247 where we got $\frac{1}{4}$ using Schmid's approximation. This indicates that Schmid's approximation is rather good as a leading approximation. However, our result shows that Gor'kov and Kopnin's approximation of neglecting the electric field everywhere is not valid, since the $\mathcal{E}(0)$ term makes an equally large contribution to the relation between j_t and v as the term they evaluated. Proceeding to find R as in our Eq. (31), they therefore found a value twice as large as ours.

Gor'kov and Kopnin further emphasized important three-dimensional structures of the moving vortices including their necessary bending, which we have not yet studied. In Sec. IV we show the backflow structure in the plane perpendicular to a vortex, which was not studied by them. Thus the two works complement each other in bringing out new features of the dynamics of vortices.

IV. BACKFLOW AROUND AN ISOLATED VORTEX WHEN $\kappa \gg 1$

In our previous work we found that near H_{c2} the local field relation $\vec{\mathcal{E}}(r) = -\vec{v}\times\vec{B}_0(r)$ and $\rho = \vec{v}\cdot\vec{j}_0$ characteristic of a rigid low-velocity Lorentz transformation of the static vortex field B_0 and current $\vec{j}_0 = \vec{\nabla}\times\vec{B}_0/4\pi$ occurred when the two screening lengths were equal, $\lambda = \zeta$. Furthermore, the total current was just the translating j_0 plus a uniform transport current j_t only in this case, independent of κ, except that the physical case $\zeta = \xi/\sqrt{12}$ requires that $\kappa = 1/\sqrt{12}$. In this section, we investigate whether these relations are still true for the low-field limit.

In Sec. III we found that indeed $\vec{\mathcal{E}}(r) = -\vec{v}\times\vec{B}_0(r)$ if only $\lambda = \zeta$. As mentioned earlier, the difference between $\mathcal{E}$ and E is very small. We can calculate ρ from $\vec{\nabla}\cdot\vec{\mathcal{E}}/4\pi$ and verify that the corrections are small. Since $\vec{\nabla}\cdot\vec{Q}_0 = 0$, we obtain

$$4\pi\rho = -\nabla^2 P = -f^2 P/\zeta^2 \qquad (34)$$

using Eq. (16) to obtain the last equality. The difference between ψ and φ follows from Eq. (3):

$$\psi - \varphi = -\left(\frac{\lambda_{TF}}{\zeta}\right)^2 f^2 P = -\left(\frac{\lambda_{TF}}{\zeta}\right)^2 f^2\left(\psi + \frac{\partial\chi}{\partial t}\right) \,. \qquad (35)$$

Since λ_{TF} is only a few angstroms whereas $\zeta = \xi/\sqrt{12} \sim 100$ Å, the difference is indeed practically negligible. Comparing Eq. (24) for P and Eq. (25) for B_0, we find that $\rho = \vec{v}\cdot\vec{j}_0 = \vec{v}\cdot\vec{\nabla}\times\vec{B}_0/4\pi$ when only $\lambda = \zeta$.

However, the equality of the electric field and charge distributions to those of a rigid Lorentz transformation is not sufficient to eliminate the backflow current in the low-magnetic-field limit, as was true near H_{c2}. Since $f = 0$ at the vortex core $j(0)$ equals $\sigma\mathcal{E}(0)$. Therefore, combining Eqs. (23), (29), and (30), we see that

$$j_t - j(0) = \tfrac{1}{2}\sigma v e\int \frac{B^2\,dV}{\pi L} + \sigma\int\left(\frac{\partial f}{\partial t}\right)^2 \frac{dV}{4\pi e\zeta^2 v} - \tfrac{1}{2}\sigma v B(0)$$
$$= \tfrac{1}{2}\sigma v H_{c2}\left[\tfrac{1}{2}\kappa^{-2} + \xi^2/4\zeta^2 - \kappa^{-2}(\ln\kappa + 0.1)\right] \,. \qquad (36)$$

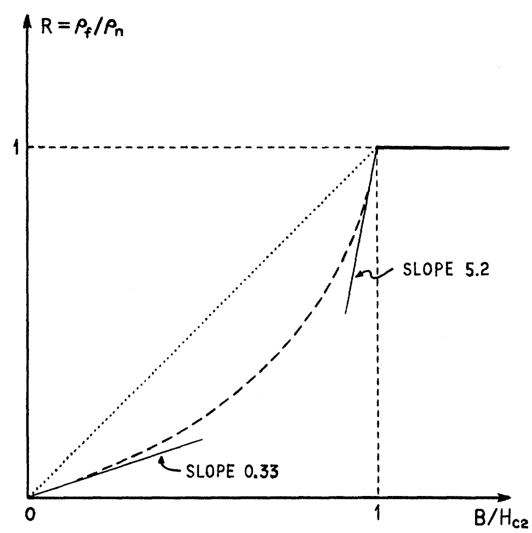

FIG. 1. Ratio R of the flux-flow resistivity ρ_f to the normal-state resistivity ρ_n for a high-κ superconductor containing a large concentration of paramagnetic impurities. Only the limiting slopes have been calculated for the low-field $B\to 0$ and high-field $B \approx H_{c2}$ limits. The dashed curve extrapolates between the limits using only a positive curvature. We suggest that the optimum experimental arrangement for verifying the prediction is a flat sample perpendicular to the externally applied magnetic field H_e (demagnetization coefficient ≈ 1), so that the average local field $B \approx H_e$. Some unphysical cases considered in the text can give the dotted straight line or curves lying above it with negative curvature.

In the physical case $\zeta = \xi/\sqrt{12}$ this equation gives

$$j_t - j(0) = \tfrac{3}{2}\sigma v H_{c2} = 0.98\, j(0)\ ,$$
$$j(0) = 0.5\, j_t\ . \tag{37}$$

However, in the unphysical case for large κ when $\lambda = \zeta$, where j_t would equal $j(0)$ if backflow were absent, we get

$$j_t - j(0) = \tfrac{1}{2}\sigma v H_{c2}\kappa^{-2}[\tfrac{3}{4} - (\ln\kappa + 0.1)]\ , \tag{38}$$

which does not vanish but is negative, i.e., $j(0) > j_t$. From Eq. (36), we see that $j(0) = j_t$ when $\zeta = \frac{1}{2}\lambda/(\ln\kappa - 0.4)^{1/2}$. However, the current being equal to j_t at the origin for this value of ζ does not mean that the backflow vanishes elsewhere, as may be expected since the field distributions do not correspond to a Lorentz transformation here.

We can find the backflow pattern to first order in v outside the core $r \gg \xi$ by using asymptotic expressions valid in this region. To obtain a scalar equation to solve we take the curl of Eq. (12):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -4\pi\sigma\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\times(f^2\vec{Q})/\lambda^2\ . \tag{39}$$

We will concentrate on the two-dimensional character of the solution, ignoring the response to the magnetic field generated by j_t which will include bending of the flux lines and additional screening currents flowing along the z direction parallel to the external magnetic field. We thus assume that $\vec{B} = \vec{B}_0 + \vec{B}_t + \vec{B}_b$, where $\vec{B}_0$ is the translating static field given in Eq. (25), $\vec{\nabla}\times\vec{B}_t = 4\pi\vec{j}_t$, and significantly the $\vec{B}_b$ generated by the backflow current $\vec{j}_b$ is only in the z direction.

$\vec{Q}$ can similarly be broken down into three parts: $\vec{Q} = \vec{Q}_0 + \vec{Q}_t + \vec{Q}_b$. From Eq. (24), $\vec{Q}_0 = -K_1(r/\lambda)\,\hat{e}_\theta/2e\lambda$. $\vec{Q}_t$ is a constant vector whose magnitude follows from $\vec{j}_t = -\vec{Q}_t/4\pi\lambda^2$ in the superconducting region and $\vec{B}_b = \vec{\nabla}\times\vec{Q}_b$. For $r \gg \xi$, the leading contribution to f follows from Eq. (10):

$$f^2 = 1 - 4e^2\xi^2Q^2$$
$$= 1 - 4e^2\xi^2(Q_0^2 + 2\vec{Q}_0\cdot\vec{Q}_t)\ . \tag{40}$$

Inserting this into Eq. (39) and keeping only the contributions to $\vec{B}_b$ we find

$$\nabla^2\vec{B}_b = 4\pi\sigma\frac{\partial\vec{B}_0}{\partial t} + \frac{\vec{B}_b}{\lambda^2} - \frac{4e^2\xi^2}{\lambda^2}\vec{\nabla}\times(Q_0^2\vec{Q}_t + 2\vec{Q}_0\cdot\vec{Q}_t\vec{Q}_0)\ ,$$
$$(\nabla^2 - \lambda^{-2})\vec{B}_b = \frac{2\pi\sigma v}{e\lambda^3}K_1\left(\frac{r}{\lambda}\right)\cos\theta - \frac{4\pi j_t\xi^2}{\lambda^2}\left[\frac{4}{r}K_1\left(\frac{r}{\lambda}\right)^2 + \frac{6}{\lambda}K_1\left(\frac{r}{\lambda}\right)K_0\left(\frac{r}{\lambda}\right)\right]\cos\theta\ . \tag{41}$$

The homogeneous solution for $\vec{B}_b$ which is proportional to $\cos\theta$ is $K_1(r/\lambda)$, which can be added in an arbitrary amount. An exact solution has been found which satisfies Eq. (41):

$$\vec{B}_b = \left(\frac{2\pi\sigma v}{e\lambda}\right)\left\{-\frac{1}{2}\frac{r}{\lambda}K_0\left(\frac{r}{\lambda}\right) + \frac{2j_t}{\sigma v H_{c2}}\left[-K_0\left(\frac{r}{\lambda}\right) + C\right]K_1\left(\frac{r}{\lambda}\right)\right\}\cos\theta\,\hat{e}_z\ . \tag{42}$$

The first two terms of $\vec{B}_b$ give the two source terms on the right-hand side of Eq. (41), whereas the last term is the homogeneous solution whose magnitude C is not yet determined. In order for $\vec{B}_b$ to stay finite as $r \to \xi$, the magnitude of C should be $\sim \ln\kappa$. The backflow current is found immediately by differentiation $\vec{\nabla}\times\vec{B}_b = 4\pi\vec{j}_b$.

An interesting feature of $\vec{B}_b$ is that for any fixed θ it changes sign as a function of r at some r_0. For a two-dimensional j the lines of constant B are the streamlines of $\vec{j}$ $(\vec{\nabla}\times\vec{B}\hat{e}_z = -\hat{e}_z\times\vec{\nabla}\vec{B})$. Consequently, on either side of the y axis there are closed loops of $\vec{j}_b$ circulating in one sense in the region $r < r_0$ and another set of closed loops of $\vec{j}_b$ for $r > r_0$ circulating in the opposite way. On the side nearer the core, the direction of the current flow of the inner loops is opposite to j_t, whereas that of the outer loops is the same as j_t. An illustration of this behavior is shown in Fig. 2.

The location of r_0 depends on the ratio j_t/v, which is obtained from Eq. (30). In the physical case, where $\zeta = \xi/\sqrt{12}$, $2j_t/\sigma v H_{c2} = 6.05$, so r_0 is rather large in the asymptotic region where $K_0 \approx K_1$, giving $r_0 = 12.1C\lambda$. For this large value of r_0, the K functions are exponentially small, so the outer loops are moved out so far that they are negligibly small, and the backflow pattern is dominated by the inner loops, giving j_b opposite to j_t near the origin in agreement with our result $j(0) < j_t$.

As the ratio ζ/ξ increases from its physical value, the value of r_0 decreases. The backflow vanishes at the origin and $j(0)$ equals j_t when ζ/ξ has increased just enough to make $r_0 = 0$. Further increase of ζ/ξ gives an enhancement of j at the origin, $j(0) > j_t$, since only the outer loops remain. Our formula, Eq. (42), for $\vec{B}_b$ is not correct in the core and cannot be used to verify that indeed $r_0 = 0$ exactly when $\zeta = \frac{1}{2}\lambda/(\ln\kappa - 0.4)^{1/2}$, where we earlier found $j(0) = j_t$. However, Eq. (42) is correct well outside of the core and shows that $\vec{B}_b$ and $\vec{j}_b$ do not vanish everywhere for any ζ including this particular ζ where $\vec{j}_b$ vanishes at the origin.

Although the condition $\lambda = \zeta$ is enough to give the correct electric-field charge distribution for a rigid Lorentz transformation of a vortex, it is not sufficient to eliminate backflow, which we see exists for all ζ/λ ratios for high κ. In Secs. V and VI we will investigate further the case $\zeta = \lambda$ for all κ values and will find that there is a special κ value $\kappa_c = 1/\sqrt{2}$ where the backflow does vanish, requiring simultaneously $\zeta = \lambda = \xi/\sqrt{2}$.

V. RESPONSE OF AN ISOLATED VORTEX WHEN $\zeta=\lambda$ FOR ALL κ

In Sec. IV we found that $\vec{\mathcal{E}}(r)=-\vec{v}\times\vec{B}_0(r)$ and $\rho(r)=\vec{v}\cdot\vec{j}_0(r)$ were obtained as in a rigid Lorentz transformation if and only if $\xi=\lambda$ for $\kappa\gg 1$. We first will show in this section that this result is true for all κ if only $\zeta=\lambda$. Then we will proceed to find the dissipation rate in terms of H_{c1}, the static lower critical field for vortex entry when the demagnetization coefficient is zero. Backflow is also analyzed.

Our equation for P to first order in v is obtained from Eq. (16):

$$\zeta^2[\nabla^2 P+\vec{v}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{Q}_0)]=f^2P\ . \tag{43}$$

From Eq. (15) the equilibrium $\vec{Q}_0$ satisfies

$$\begin{aligned}&-\lambda^2\vec{\nabla}\times(\vec{\nabla}\times\vec{Q}_0)=f^2\vec{Q}_0\ ,\\&\lambda^2[\nabla^2\vec{Q}_0-\vec{\nabla}(\vec{\nabla}\cdot\vec{Q}_0)]=f^2\vec{Q}_0\ .\end{aligned} \tag{44}$$

The boundary conditions at the vortex core follow from the space and time derivatives of the phase $2e\chi=\theta$:

$$\begin{aligned}&\vec{Q}_0\to-(2er)^{-1}\hat{e}_\theta\ ,\\&P\to-(2er)^{-1}\vec{v}\cdot\hat{e}_\theta\ .\end{aligned} \tag{45}$$

Comparing these equations we see that the solution when $\zeta=\lambda$ is $P=\vec{v}\cdot\vec{Q}_0$. From this solution we find

$$\begin{aligned}\vec{\mathcal{E}}&=-\vec{\nabla}(\vec{v}\cdot\vec{Q}_0)+(\vec{v}\cdot\vec{\nabla})\vec{Q}_0\\&=-\vec{v}\times\vec{\nabla}\times\vec{Q}_0=-\vec{v}\times\vec{B}_0\ .\end{aligned} \tag{46}$$

Taking the divergence of this equation it follows that $\rho=\vec{v}\cdot\vec{j}_0$.

To discuss the dissipation rate we wish to rewrite our expression Eq. (23) in terms of H_{c1}. For this purpose we use some identities proved using the static GL equations for the equilibrium solution in a book on type-II superconductivity[6]:

$$\begin{aligned}4e\xi^2\lambda^2H_{c1}&=\int_0^\infty(1-f_0^2)r\,dr\\&=\int_0^\infty[(2e\xi\lambda B_0)^2+\tfrac{1}{2}(1-f_0^4)]r\,dr\ .\end{aligned} \tag{47}$$

Subtracting the two right-hand sides it follows that

$$\int_0^\infty(2e\xi\lambda B_0)^2r\,dr=\tfrac{1}{2}\int_0^\infty(1-f_0^2)^2r\,dr\ . \tag{48}$$

We need an additional similar identity in order to evaluate the quantity $\int(\partial f/\partial t)^2dV/L=\pi v^2\int(\nabla f_0)^2r\,dr$:

$$\begin{aligned}\int(\nabla f_0)^2r\,dr&=-\int f_0\nabla^2f_0r\,dr\\&=\int f_0[\xi^{-2}f_0(1-f_0^2)-4e^2Q_0^2f_0]r\,dr\end{aligned}$$

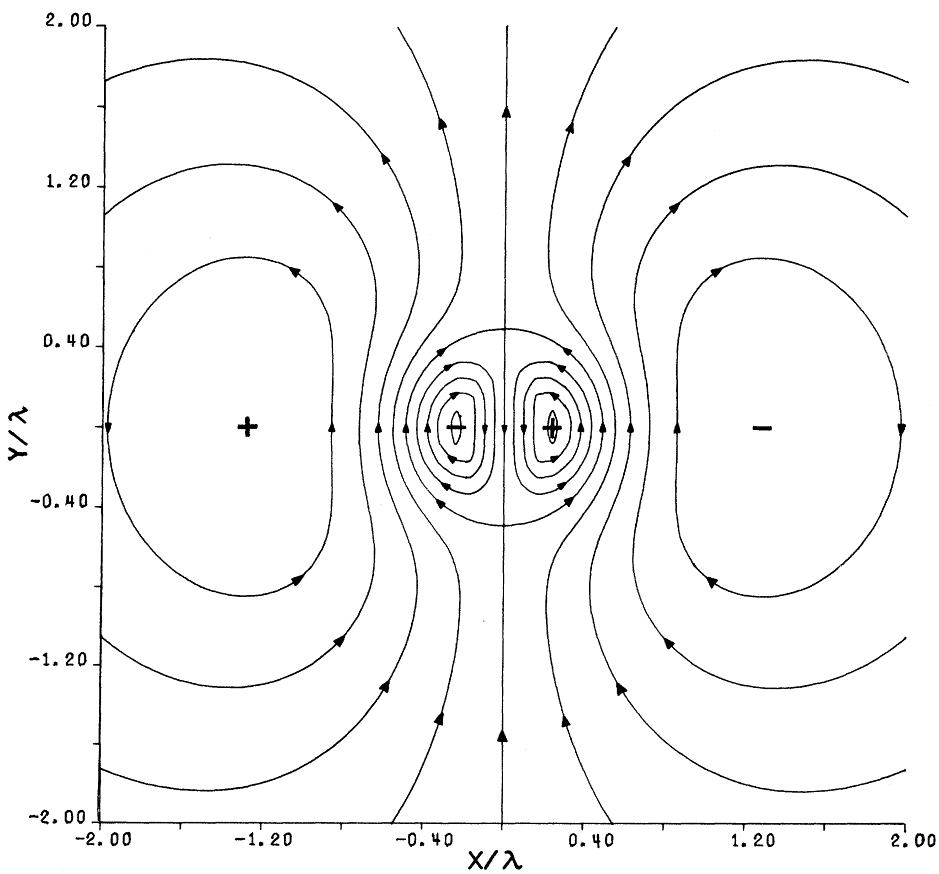

FIG. 2. Equal-interval contours of the value of the magnetic field B_b generated in the $\hat{e}_z$ direction by the backflow current j_b are drawn. The + and − signs indicate the local maxima and minima of B_b. These contours are also streamlines of j_b, which flows in the directions indicated by the arrows. The vortex centered at the origin is moving in the $\hat{e}_x$ direction as forced by a uniform transport current j_t flowing in the $\hat{e}_y$ direction. B_b vanishes on the line $x=0$ and on the perfect circle $r=r_0$ intersecting this line. The values of B_b are calculated from Eq. (42) for $r\geq 2\xi$ and linearly extrapolated to vanish at the origin for smaller r where Eq. (42) is invalid. The parameter values $\kappa=10$ and $2j_t/\sigma vH_{c2}=0.1$ are used.

$$= \int [\xi^{-2}(f_0^2 - f_0^4) + 4e^2\lambda^2 \vec{Q}_0 \cdot \vec{\nabla}\times(\vec{\nabla}\times\vec{Q}_0)] r\,dr$$

$$= \int [\xi^{-2}(f_0^2 - f_0^4) + 4e^2\lambda^2 B_0^2] r\,dr - 2e\lambda^2 B_0(0) \; . \qquad (49)$$

Putting these results into Eq. (23), noting Eq. (29), we get

$$j_t = (e/v\pi)\int W \frac{dV}{L}$$

$$= \sigma\mathcal{E}(0) - \tfrac{1}{2}\sigma v\left(1 + \frac{\lambda^2}{\zeta^2}\right)[B_0(0) - H_{c1}]$$

$$+ (\sigma v/8e\xi^2\lambda^2)(1 - \lambda^2/\zeta^2)\int_0^\infty (f_0^2 - f_0^4) r\,dr \; . \qquad (50)$$

When $\lambda = \zeta$ we can use $\mathcal{E}(0) = vB_0(0)$ and therefore we have

$$j_t = \sigma v H_{c1} \; . \qquad (51)$$

Since the flux quantum is e/π, the ratio of the flux-flow resistance to the normal-state value is just

$$R = \langle B \rangle / H_{c1} \; . \qquad (52)$$

The initial slope

$$H_{c2}\frac{dR}{d\langle B\rangle}\bigg|_{\langle B\rangle = 0} = \frac{H_{c2}}{H_{c1}} \qquad (53)$$

is (<, >, or =) to unity when κ is (<, >, or =) to $\kappa_c = 1/\sqrt{2}$. Thus, as in the high-κ case, this slope and the sign of the curvature necessary for R to reach unity smoothly at H_{c2} depend on the ratio of ζ to $\xi/\sqrt{2}$. For the physical value $\zeta/\xi = 1/\sqrt{12}$ or $\kappa = 1/\sqrt{12}$ if $\lambda = \zeta$, the slope is less than one in both this low-κ and the previous high-κ limit. An isolated vortex with a single-flux quantum will be the thermodynamically preferred solution for this low-κ value only for thin films perpendicular to B.

The backflow current at the origin is obtained directly from Eq. (50) using $j(0) = \sigma\mathcal{E}(0)$:

$$j_b(0) = j(0) - j_t$$

$$= \tfrac{1}{2}\sigma v(1 + \lambda^2/\zeta^2)[B_0(0) - H_{c1}]$$

$$+ (\sigma v/8e\xi^2\lambda^2)(1 - \lambda^2/\zeta^2)\int_0^\infty f_0^2(1 - f_0^2) r\,dr \; . \qquad (54)$$

Since $B_0(0) - H_{c1}$ has the same sign as $\kappa - \kappa_c$, it follows that $j_b(0)$ has the same sign as $\kappa - \kappa_c$ if $\zeta = \lambda$ and has the same sign as $\zeta - \lambda$ if $\kappa = \kappa_c$. This result is consistent with our previous result that $j_b(0)$ is positive, in the same direction as j_t, when $\lambda = \zeta$ for $\kappa \gg 1$. $j_b(0)$ vanishes at $\kappa = \kappa_c$, $\lambda = \zeta$. We will show in Sec. VI that, unlike the high-κ case, the backflow vanishes everywhere for this special case where $j_b(0)$ vanishes.

For the remainder of this section we wish to apply our new identity Eq. (49) to the high-κ limit, where we find that the widely quoted numerical work of Abrikosov[7] must be wrong. In the high-κ limit Abrikosov found

$$B_0(0) = \kappa^{-2}(\ln\kappa + C_0)H_{c2} \; ,$$
$$H_{c1} = \tfrac{1}{2}\kappa^{-2}(\ln\kappa + C_1)H_{c2} \; . \qquad (55)$$

He estimated numerically that $C_0 = -0.18$ and $C_1 = +0.08$. Combining our identity equation (49) with Eq. (47) we get

$$\int_0^\infty (\nabla f_0)^2 r\,dr = 2e\lambda^2[2H_{c1} - B_0(0)] - 4e^2\lambda^2\int_0^\infty B_0^2 r\,dr. \qquad (56)$$

Our result for the last integral above, calculated in Eq. (27), is correct for large κ independent of the approximation used for f_0. Consequently, we find

$$\int_0^\infty (\nabla f_0)^2 r\,dr = C_1 - C_0 - \tfrac{1}{2} \; . \qquad (57)$$

The integral on the left-hand side is positive definite. Using Schmid's approximation $f_0 = r/(r^2 + \xi^2)^{1/2}$ its value is $\frac{1}{4}$, whereas, according to the numerical work quoted by Gor'kov and Kopnin, it equals 0.247. Using Abrikosov's values for C_0 and C_1, the right-hand side of Eq. (57) is -0.24, not even positive. At least one of Abrikosov's numbers must be seriously in error.

Schmid's approximation actually gives a worse result for the identity. From Eq. (25) we find $C_0 = +0.1$. To evaluate H_{c1} from the first identity of Eq. (27), we use $f_0 = r/(r^2 + \xi^2)^{1/2}$ in the region $r < (\lambda\xi)^{1/2}$ and the asymptotic result from Eq. (40), $f_0^2 = 1 - \kappa^{-2}K_1(r/\lambda)^2$, in the region $r > (\lambda\xi)^{1/2}$. We then obtain $C_1 = -0.4$. The right-hand side of Eq. (57) then gives -1.0, worse than Abrikosov's value. Evaluation of our result for the slope $dR/d\langle B\rangle$ when $\lambda = \zeta$ and comparison with Eq. (53) give another different value $C_1 = +0.9$. Thus Schmid's approximation is not enough to obtain the corrections to the leading $\ln\kappa$ behaviors. One of us (C. R. H.) has devised a program for calculating C_0 and C_1 numerically as well as $\int(\nabla f_0)^2 r\,dr$ and hopes to report consistent results later.

VI. FLUX FLOW WITHOUT BACKFLOW WHEN $\lambda = \zeta = \xi/\sqrt{2}$

Finally, we examine the one special case where, as we will show, the backflow current j_b vanishes not just at the origin but everywhere in the plane perpendicular to the external magnetic field. Although this special case is not physical for our system since $\zeta \neq \xi/\sqrt{12}$, it is especially interesting since we manage to get a solution for all fields, not just in the low- and high-field limits. (Of course, the demagnetization coefficient of the sample must be greater than zero for the field to penetrate below H_{c2}, since $H_{c1} = H_{cB} = H_{c2}$ when $\kappa = 1/\sqrt{2}$.)

For the static case with $\kappa = 1/\sqrt{2}$ it is already known[6] that the GL equations for the field and order parameter become the same if the solution for B_0

$= H_{c2}(1-f_0^2)$ is used, leaving a single equation to determine f_0:

$$f_0\nabla^2 f_0 + \xi^{-2} f_0^2(1-f_0^2) - (\nabla f_0)^2 = 0 \ . \qquad (58)$$

We will show that this same solution is also valid for the translating vortex to order v and that f and B are not changed so the vortex does translate rigidly without backflow.

Our basic set of equations is still Eqs. (10)–(12). We should point out that since we are ignoring the effects of the magnetic field generated by the transport current j_t in order to get a two-dimensional solution, Eq. (12) for the current is connected with the magnetic field for our derivations by $\vec{\nabla}\times\vec{B} = 4\pi \times(\vec{j}-\vec{j}_t)$. Eliminating Q from Eqs. (10) and (12) in terms of f and B and neglecting terms of order v^2, we get

$$\nabla^2 f + \xi^{-2} f(1-f^2) + \gamma(\vec{v}\cdot\vec{\nabla})f$$
$$= 4e^2\lambda^4 f^{-3}[\vec{\nabla}\times\vec{B} - 4\pi(\sigma\vec{\mathcal{E}} - \vec{j}_t)]^2$$
$$= 4e^2\lambda^4 f^{-3}(\vec{\nabla}\times\vec{B})\cdot[\vec{\nabla}\times\vec{B} - 8\pi(\sigma\vec{\mathcal{E}} - \vec{j}_t)] \ , \qquad (59)$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) + \lambda^{-2} f^2\vec{B} - \gamma\xi^2\lambda^{-2}(\vec{v}\cdot\vec{\nabla})\vec{B}$$
$$= 2f^{-1}\vec{\nabla}f\times[\vec{\nabla}\times\vec{B} - 4\pi(\sigma\vec{\mathcal{E}} - j_t)] \ .$$

For a two-dimensional solution with $\vec{B} = \hat{e}_z b$ the equations can be simplified using various vector identities:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\hat{e}_z\nabla^2 b \ ,$$
$$(\vec{\nabla}\times\vec{B})^2 = (\vec{\nabla}b)^2 \ ,$$
$$(\vec{\nabla}f)\times(\vec{\nabla}\times\vec{B}) = -\hat{e}_z(\vec{\nabla}f)\cdot(\vec{\nabla}b) \ , \qquad (60)$$
$$(\vec{\nabla}\times\vec{B})\cdot(\vec{\mathcal{E}} \text{ or } \vec{j}_t) = -\hat{e}_z\cdot(\vec{\nabla}b)\times(\vec{\mathcal{E}} \text{ or } \vec{j}_t) \ .$$

If we now put in the trial solution $b = H_{c2}(1-f^2)$, Eqs. (59) become (after multiplying the first by f and dividing the second by $2H_{c2}$):

$$f\nabla^2 f + \xi^{-2} f^2(1-f^2) + \gamma f(\vec{v}\cdot\vec{\nabla})f$$
$$= 2\kappa^2(\nabla f)^2 + 16\pi e\sqrt{2}\,\kappa\lambda^2 f^{-1}\,\hat{e}_z\cdot(\vec{\nabla}f)\times(-\sigma\vec{\mathcal{E}} + \vec{j}_t) \ , \qquad (61)$$
$$f\nabla^2 f + \tfrac{1}{2}\lambda^{-2} f^2(1-f^2) + \gamma\xi^2\lambda^{-2} f(\vec{v}\cdot\vec{\nabla})f$$
$$= (\nabla f)^2 + 8\pi e\sqrt{2}\,\kappa^{-1}\lambda^2 f^{-1}\hat{e}_z\cdot(\vec{\nabla}f)\times(-\sigma\vec{\mathcal{E}} + \vec{j}_t) \ .$$

These two equations become identical and the trial solution for b is valid only if both $\kappa = 1/\sqrt{2}$ and $\lambda = \xi$. This result is incidentally a proof of the result quoted earlier in the static case for arbitrary two-dimensional solutions.

The task remaining is to show that $f = f_0$. For this purpose we let $f = f_0 + \delta f$, where δf is a possible correction of order v:

$$f_0\nabla^2\delta f + (\nabla^2 f_0)\delta f - 2(\vec{\nabla}f_0)\cdot\vec{\nabla}\delta f + 2\xi^{-2} f_0(1-2f_0^2)\delta f$$
$$= 8\pi e\xi^2 f_0^{-1}\,\hat{e}_z\cdot(\vec{\nabla}f)\times(\vec{j}_t - \sigma\vec{\mathcal{E}}) - \gamma f_0(\vec{v}\cdot\vec{\nabla})f_0 \ . \qquad (62)$$

The right-hand side of this equation is the source which determines the magnitude of δf. It can be simplified by remembering that since $\lambda = \xi$, $\vec{\mathcal{E}} = -\vec{v}\times\vec{B}_0$, and by using the solution for $\vec{B}_0 = \hat{e}_z H_{c2} \times(1-f_0^2)$, since $\kappa = 1/\sqrt{2}$:

$$-8\pi e\xi^2 f_0^{-1}\hat{e}_z\cdot(\vec{\nabla}f_0)\times\sigma\vec{\mathcal{E}}$$
$$= 8\pi e\xi^2 f_0^{-1}\hat{e}_z\cdot(\vec{\nabla}f_0)\times(\sigma\vec{v}\times\hat{e}_z H_{c2})$$
$$-\gamma f_0\hat{e}_z\cdot(\vec{\nabla}f_0)\times(\vec{v}\times\hat{e}_z) \ . \qquad (63)$$

We used the definition of $\xi^2 = 4\pi\lambda^2\sigma/\gamma$ to replace $8\pi e\xi^2\sigma H_{c2}$ by γ since $\xi/\sqrt{2} = \lambda = \xi$. Using a vector identity, the last term in Eq. (63) exactly cancels the last term in Eq. (62). Therefore, the right-hand side of Eq. (62) becomes simply

$$8\pi e\xi^2 f_0^{-1}\hat{e}_z\cdot(\vec{\nabla}f_0)\times[\vec{j}_t + \sigma\vec{v}\times\hat{e}_z H_{c2}] \ . \qquad (64)$$

The term in brackets is a constant independent of r. Its value can be determined by examining the equation for δf near a vortex core at the origin of the coordinate system. As $r\to 0$, f must vanish. Since f_0 vanishes, so must δf. This is only possible if the constant vanishes. Therefore, we find

$$\vec{j}_t = -\sigma\vec{v}\times\hat{e}_z H_{c2} = -\sigma\vec{v}\times\vec{B}_0(0) = \sigma\mathcal{E}(0) \ . \qquad (65)$$

The source term therefore vanishes everywhere, and $\delta f = 0$. There is no distortion of f and no backflow ($B_b = 0$).

Using Eq. (65) we can find R, the ratio of the flux-flow resistance in the superconducting state to the normal-state resistance:

$$R = \sigma\langle\mathcal{E}\rangle/j_t = \langle B\rangle/H_{c2} \ ,$$
$$H_{c2}dR/d\langle B\rangle = 1 \ . \qquad (66)$$

These relations are valid for all fields between 0 and H_{c2} and are consistent with both our high- and low-field limiting results.

VII. CONCLUSIONS

We have found the dissipation rate and backflow character of a vortex line moving in the plane perpendicular to the magnetic field for various ratios of the screening lengths and κ values, illustrating the backflow in the low-field region for the first time. More work remains to determine the complete three-dimensional structure of moving vortices including their bending, the screening of the transport current between vortices, and the opening up of the vortex lines at the sample surfaces, which will allow an explicit verification of the assertion made at the end of Sec. II. The thin-film limit also deserves further study owing to the importance of the boundary conditions on the fields at the sample surfaces.

†Supported by the National Science Foundation.
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Thermal Effects at Superconducting Point Contacts*

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(Received 12 October 1971)

In this paper we have calculated *I-V* curves of superconducting weak-link constriction junctions by assuming that there is a region of normal material which tends to spread with increasing power levels. The causes for the spreading of the normal resistance are twofold. One is the increase of the current-density distribution and the other is the increase of localized Joule heating at the contact as the total current is increased. The resultant rise in temperature of the link above the bath temperature, over the range of the *I-V* characteristic, is found to be significant. Using material constants that are representative of bulk Nb, we found that the calculated *I-V* characteristic is very similar to several experimentally observed Nb point-contact curves. The spreading normal-resistance analysis has suggested a model to explain the *I-V* characteristic of a superconductor–normal-metal (*S–N*) point-contact system. A calculation has indicated that large excess temperatures are also present at the contacts when biased in the millivolt region. These findings have prompted us to review several published experiments with *S–N* contacts.

I. INTRODUCTION

Current-voltage (*I-V*) characteristics of superconducting weak-link point-contact or thin-film constriction junctions usually exhibit a zero-voltage current followed by either a continuous or discontinuous transition into a nonlinear resistive region.[1–4] The types (Ohmic, flux flow, radiation) of dissipative mechanisms active in this region and the proportions which they contribute to the local resistance are difficult to distinguish in experimental situations.[3,4] It has been well documented in the literature[5] that superconducting phenomena can be present in the resistive state.

In the theory of resistive yet superconducting point contacts at a nonzero voltage the total transport current is generally assumed to consist of a superconducting and a normal component.[6] The superconducting component consists of Cooper pairs and is dissipative because of the emission of photons. The normal component consists of quasiparticles and is also dissipative because of Joule heating. In the simplest approximation such a point contact may be represented[7,8] by an ideal Josephson element, in parallel with a shunt conductance G that exhibits Ohmic behavior.

Several authors[7–9] have calculated the *I-V* characteristics that result from this model when it is modified by the circuit capacitance and inductance. Scott[10] has found satisfactory agreement between the experimental observation of hysteresis in the *I-V* characteristics of thin-film Pb-PbO-Pb sandwich junctions and the theoretical predictions of Stewart[7] and McCumber.[8] Although this theory gives good agreement with experiment in the case of some types of weak-link junctions,[11,12] in the case of point-contact junctions poor agreement[13] with experiment is obtained, especially at large biases. This inconsistency may be attributed to an oversimplified picture of the contact model.

To explain their *I-V* curves with weak-link junctions (both thin-film constrictions and pressure contacts) several authors[1,3] have suggested that the superconducting region adjacent to the contact interface is driven normal by the large current densities localized there, while the surrounding material with lower current density remains superconducting. This process is accompanied by Joule heating at the contact. As current through the contact is increased, the Joule heating and current density in the contact region are increased, resulting in a spread of the region of normal material and hence an increase of the Ohmic resistance. In the models of Stewart and McCumber the shunt con-