Equation of State near the Critical Point. II. Comparison with Experiment and Possible Universality with Respect to Lattice Structure and Spin Quantum Number*

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The scaling functions for the $S=\frac{1}{2}$ Ising, $S=\frac{1}{2}$ Heisenberg, and $S=\infty$ Heisenberg models are compared with experimental measurements of $h(x)$ on the insulating ferromagnets $CrBr_3$ ($S=\frac{3}{2}$) and EuO ($S=\frac{7}{2}$), with the metallic ferromagnet Ni and with the conducting alloy Pd_3Fe (in both *disordered* and *ordered* states). The data agree much better with our Heisenberg model $h(x)$ than with the Ising model $h(x)$. Comparison with experimental data is made by using plots of scaled magnetization $M_H \equiv M/H^{1/\delta}$ vs scaled temperature $\epsilon_H \equiv \epsilon/H^{1/\beta\delta}$, which has the virtue that all points fall upon a *single* curve and log-log plots need not be resorted to. We proceed to consider the extent to which the equation of state depends upon parameters appearing in the Hamiltonian (the "universality" question). We find that our calculations indicate that $h(x)$, if properly normalized, does *not* depend on lattice structure and is very likely independent of spin quantum number S. Moreover, the fact that our calculated $h(x)$ agrees with data on $CrBr_3$ (for which there exists considerable *lattice anisotropy*) and on EuO (for which there exists nonnegligible *next-nearest-neighbor interactions*) suggests the further conjecture that $h(x)$ might be independent of these features of the Hamiltonian. By contrast, the change in $h(x)$ on going from Ising coupling ($D=1$) to Heisenberg coupling ($D=3$) was quite substantial, and indeed would seem to account for the fact that earlier workers did not obtain agreement between the Ising model $h(x)$ and experimental data on magnetic systems. We conclude with the working hypothesis that the scaling function depends principally on spin dimensionality D and on lattice dimensionality d.

I. INTRODUCTION

In Paper I of this work,[1] we utilized high-temperature series expansions of $H(\epsilon, M)$ to calculate the scaling function for the $S=\frac{1}{2}$ Ising model [bcc, fcc, and simple cubic (sc) lattices], for the $S=\frac{1}{2}$ Heisenberg model (fcc and bcc lattices), and for the $S=\infty$ Heisenberg model (fcc lattice only). We anticipated that the accuracy of our expressions, limited by the number of known terms in the series, is of the order of a few percent. Accordingly, we feel that the expressions derived in I are sufficiently accurate to be worth comparing with each other and with experiment. In Sec. II, we undertake a comparative study of the scaling functions for the Ising, $S=\frac{1}{2}$ Heisenberg, and $S=\infty$ Heisenberg models, with particular emphasis being placed upon the question "upon what features of an interaction Hamiltonian does the scaling function depend?" In Sec. III we shall compare the results of our calculations with experimental measurements of the scaling function on $CrBr_3$, EuO, Ni, and Pd_3Fe (both the disordered and the ordered alloys). In Sec. IV we summarize the present work and present our conclusions.

II. COMPARATIVE STUDY OF SCALING FUNCTIONS FOR $S=\frac{1}{2}$ ISING, $S=\frac{1}{2}$ HEISENBERG, AND $S=\infty$ HEISENBERG MODELS

A. Universality of Critical Properties

There have been many "conjectures"[2-7] about the extent to which $h(x)$ depends upon parameters of the systems, such as type of lattice and magnitude of the spin quantum number, *if* in fact $h(x)$ depends at all upon these parameters! This question is related to similar questions regarding the critical-point exponents[8-12] and other singularities[13,14] pertinent to the critical region. Most of the detailed evidence germane to these questions comes from the study of certain model systems.[15]

In particular, there has been almost no detailed evidence concerning what features of a system the scaling function $h(x)$ might depend upon, since the scaling function has hitherto been calculated only for the $d=2$ and $d=3$ Ising models.[3] Gaunt and Domb observed that $h(x)$ differed drastically for the two different lattice dimensionalities, $d=2$ and $d=3$.[3] Gaunt and Domb also calculated $h(x)$ for the bcc, sc, and fcc lattices (though they do not display the results of their calculation), and they state that the scaling functions agree to within the accuracy of their calculation (about 10%).[3]

We have varied the parameters "spin-space dimensionality" D (with $D=1$ and $D=3$ corresponding, respectively, to the Ising and Heisenberg models), lattice structure, and spin quantum number S. From Fig. 1 it is clear that our calculated scaling functions depend strongly upon D, even when the normalized plots $h(x)/h(0)$ vs $(x+x_0)/x_0$ are compared. In Secs. II B and II C we examine in detail the question of whether our calculated functions $h(x)$ depend upon lattice structure and spin quantum number.

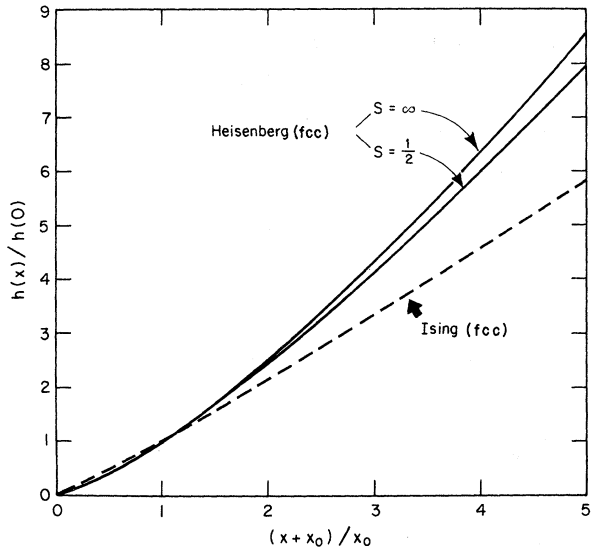

FIG. 1. Comparison of the normalized scaling functions $h(x)/h(0)$ plotted vs $(x+x_0)/x_0$, for the three ferromagnetic models—the Ising model and $S=\frac{1}{2}$ and $S=\infty$ Heisenberg models.

B. Independence of Normalized Scaling Function with Respect to Lattice Structure

1. Ising Model

The *normalized* Ising-model scaling functions $h(x)/h(0)$ are compared as functions of $(x+x_0)/x_0$ in Table I for the bcc, fcc, and sc lattices [using Eqs. (2.11a), (B1), and (B3) of I]. In obtaining the expressions for $h_1(x)$ in (B1) and (B3), we chose values of c (corresponding to points on the phase boundary) that imply corresponding values of the exponent q [cf. Eq. (2.9) of I] as close as possible to the believed[16] value $q=1$. At the same time, we could not choose c too large or we would not be in the critical region (cf. discussion in Sec. II of I). It turned out that for all three lattices $c=0.6$–0.7, implying $q\cong 1.07$. Any larger values of c would result in similar values of q, but the range of validity of the corresponding expressions for $h_1(x)$ would be too small for the present comparison. In particular, our choice $c=0.64$ for the fcc lattice (which has the smallest amplitude B in $M=B|\epsilon|^{\beta}$) might be a little too large, and this is very likely the cause for the slight discrepancy in Table I for higher values of $(x+x_0)/x_0$. Nevertheless, it is evident that the disagreement is at most 2%, and we can conclude that our calculations support the conjecture that normalized plots of $h_1(x)/h_1(0)$ vs $(x+x_0)/x_0$ are, in fact, independent of lattice structure for the Ising model.

TABLE I. Comparison of the normalized Ising-model scaling functions for the three cubic lattices.

	$h_1(x)/h_1(0)$		
$(x+x_0)/x_0$	fcc	bcc	sc
0.25	0.224	0.224	0.226
0.50	0.473	0.473	0.474
0.75	0.733	0.733	0.733
1.25	1.27	1.27	1.27
1.50	1.55	1.55	1.55
1.75	1.83	1.83	1.83
2.00	2.11	2.12	2.12
2.25	2.40	2.41	2.40
2.50	2.69	2.70	2.69
2.75	2.98	3.00	2.99
3.00	3.27	3.29	3.28
3.25	3.57	3.59	3.58
3.50	3.87	3.89	3.88
3.75	4.16	4.20	4.18
4.00	4.46	4.50	4.48
4.25	4.76	4.81	4.78
4.50	5.07	5.11	5.09

TABLE II. Comparison of the normalized scaling functions of the $S=\frac{1}{2}$ Heisenberg model for the fcc and bcc lattices.

	$h_1(x)/h_1(0)$	
$(x+x_0)/x_0$	fcc	bcc
0.25	0.167	0.164
0.50	0.409	0.405
0.75	0.690	0.688
1.25	1.33	1.34
1.50	1.69	1.69
1.75	2.06	2.06
2.00	2.44	2.45
2.25	2.84	2.85
2.50	3.26	3.26
2.75	3.68	3.68
3.00	4.12	4.11
3.25	4.57	4.55
3.50	5.02	5.00
3.75	5.46	5.46
4.00	5.96	5.92
4.25	6.45	6.38
4.50	6.94	6.86

2. Heisenberg Model

For the $S=\frac{1}{2}$ Heisenberg model, we compare in Table II normalized plots of $h_1(x)/h_1(0)$ for the fcc and bcc lattices, using Eqs. (3.7) and (B5) of I, respectively. [As discussed in Sec. IIID of I, the expansion for the sc lattices is not sufficiently well behaved to obtain a reliable expression for $h_1(x)$.] From Table II it appears that the normalized scaling function is also lattice independent for the $S=\frac{1}{2}$ Heisenberg model. This result is not too surprising, as the coincidence of the phase boundaries for the fcc and bcc lattices had been noticed previously.[17]

Thus our calculations support the conjecture that the normalized scaling function is independent of detailed lattice structure for both the $S=\frac{1}{2}$ Ising and $S=\frac{1}{2}$ Heisenberg models. The $S=\infty$ Heisenberg model was studied only for the fcc lattice, since only for this case were the appropriate series available.[18]

C. Possible Independence of Normalized Scaling Function with Respect to Spin Quantum Number S

As was made clear in Secs. III and IV of I, the critical-point exponents for the $S=\frac{1}{2}$ and $S=\infty$ Heisenberg models are not yet firmly established, but it is at least *possible* that these exponents are independent of spin quantum number. Therefore we consider in this section the question of whether the corresponding scaling functions $h(x)$ are spin independent.

Figure 1 compares the normalized scaling functions $h(x)/h(0)$ for the $S=\frac{1}{2}$ and $S=\infty$ Heisenberg models; also shown for comparison is the Ising-model scaling function.[19] One observes that for all negative values of x [i.e., for all values of the abscissa $(x+x_0)/x_0<1$) corresponding to $T<T_c$], and for small positive values of x, there is almost no difference between $S=\frac{1}{2}$- and $S=\infty$-Heisenberg-model cases (cf. also Table III). Hence we are led to conclude that for $T<T_c$ any spin dependence of the scaling function is sufficiently small ($\lesssim 2\%$) that it is within the accuracy of our calculation of $h(x)$.

On the other hand, for larger positive values of x the discrepancy increases very slowly and hence we cannot infer that for $T>T_c$ the spin independence of the $h(x)$ function is confirmed by our calculations. For example, if $(x+x_0)/x_0 \cong 50$, then $h(x)/h(0)$ for the $S=\infty$ Heisenberg model is 5% larger than for the $S=\frac{1}{2}$ case, providing we use $\beta=0.35$, $\gamma=1.40$, and $\delta=5$ for both models; this selection was made because a necessary condition for $h(x)$ being independent of S is that the exponents be independent of S [cf. Eq. (3.9) of I, which implies $h(x)\sim x^{\gamma}$ for large x].

TABLE III. Comparison of the normalized scaling functions of the $S=\frac{1}{2}$ and $S=\infty$ Heisenberg models, fcc lattice.

$(x+x_0)/x_0$	$h_1(x)/h_1(0)$ $S=\frac{1}{2}$	$h(x)/h(0)$ $S=\infty$
0.125	0.07	0.06
0.250	0.17	0.16
0.375	0.28	0.27
0.500	0.41	0.40
0.625	0.55	0.53
0.750	0.69	0.68
0.875	0.84	0.84
1.125	1.16	1.17
1.250	1.33	1.35
1.375	1.51	1.53
1.500	1.69	1.72
1.625	1.87	1.92
1.750	2.06	2.12
1.875	2.25	2.32
2.00	2.44	2.53

In summary, then, we have seen that any spin dependence of $h(x)$ may be within the error of our calculation and hence may be spurious. In particular, for the region of x corresponding to $T<T_c$ $(-x_0 \leq x \leq 0)$, we find that the discrepancy between $h(x)$ for $S=\frac{1}{2}$ and $S=\infty$ Heisenberg models is within 1–2%. Of course, we know of no argument which says that if $h(x)$ is spin independent for $T<T_c$ it should also be spin independent for $T>T_c$, but it might surprise us if $h(x)$ were spin independent for $T<T_c$ but spin dependent for $T>T_c$.

III. COMPARISON OF CALCULATED SCALING FUNCTIONS WITH EXPERIMENTAL RESULTS

A. Plots of Scaled Magnetization $M/H^{1/\delta}$ vs Scaled Temperature $\epsilon/H^{1/\beta\delta}$

In Sec. II A of I we assumed that the magnetic field $H(\epsilon, M)$ is a generalized homogeneous function in the critical region [cf. Eq. (2.2) of I]. Now if one formulates the scaling hypothesis by means of the essentially equivalent hypothesis that a thermodynamic potential (e.g., the Gibbs potential G) is a generalized homogeneous function, then one can straightforwardly show[20] that all its Legendre transforms and derivatives are *also* generalized homogeneous functions. In particular, the magnetization $M(\epsilon, H)=(\partial G/\partial H)_T$ is also a generalized homogeneous function—i.e., there exist two numbers a and b such that, for all positive values of the number λ,

$$M(\lambda^a\epsilon, \lambda^b H)=\lambda M(\epsilon, H)\ . \tag{3.1}$$

The "scaling parameters" a, b in Eq. (3.1) are unspecified, but they may be related to critical-point exponents. For example, by setting $\lambda=|\epsilon|^{-1/a}$ and choosing $H=0$ in (3.1) we find that $M(\epsilon,0)\propto|\epsilon|^{1/a}$ and $a=1/\beta$; similarly by setting $\lambda=|H|^{-1/b}$ and choosing $\epsilon=0$ in (3.1) we find that $M(0,H)\propto H^{1/b}$ and $b=\delta$. Equation (3.1) thus becomes

$$M(\lambda^{1/\beta}\epsilon, \lambda^{\delta} H)=\lambda M(\epsilon, H). \tag{3.2}$$

If we follow the argument leading up to Eq. (2.4) of I, we argue that (3.2) must be valid for all values of λ and hence, in particular, for the choice $\lambda\equiv(c/H)^{1/\delta}$, (3.2) becomes

$$M(\epsilon,H)/H^{1/\delta}=M(c^{1/\beta\delta}\epsilon/H^{1/\beta\delta}, c)/c^{\delta}. \tag{3.3}$$

Equation (3.3) says that plots of "scaled magnetization" $\tilde{M}\equiv M(\epsilon, H)/H^{1/\delta}$ vs "scaled temperature" $\tilde{\epsilon}\equiv\epsilon/H^{1/\beta\delta}$ should be described by the function $M(c^{1/\beta\delta}\tilde{\epsilon}, c)/c^{\delta}$, which is essentially the magnetization function for a constant but small (if c is small)

value of the magnetic field. In other words, Eq. (3.3) states that the scaled magnetization $M/H^{1/\delta}$ is a function of only one variable, the scaled temperature $\epsilon/H^{1/\beta\delta}$.

It turns out[20] that plots of the experimental data in the form $M/H^{1/\delta}$ vs $\epsilon/H^{1/\beta\delta}$ have some advantages over the original presentations of experimental results, namely, all experimental data can be captured within one curve for both regions, $T<T_c$ and $T>T_c$, without breaking the curve into two parts and without using a log-log plot. This advantage arises from the fact that with current equipment it is difficult to make measurements for H arbitrarily close to zero and hence the ordinate $M/H^{1/\delta}$ and the abscissa $\epsilon/H^{1/\beta\delta}$ never become extremely large. Furthermore, data do not appear clustered in a certain region,[21] nor does the behavior of the corresponding curve become dominated by the value of only one critical-point exponent.

Therefore we will compare our calculations of the scaling function[22] with the experimental results in the form of plots $M/H^{1/\delta}(\equiv 1/h^{1/\delta})$ against $\epsilon/H^{1/\beta\delta}$ $(\equiv x/h^{1/\beta\delta})$. In doing this we will deal only with the normalized quantities $h(x)/h(0)$ and x/x_0 (cf. Appendix A of I).

B. Chromium Tribromide ($CrBr_3$)

Since $CrBr_3$ was the first *insulating* magnet for which accurate measurements germane to the scaling-law equation of state were made,[23] we treat this material first. Now chromium has $S=\frac{3}{2}$,

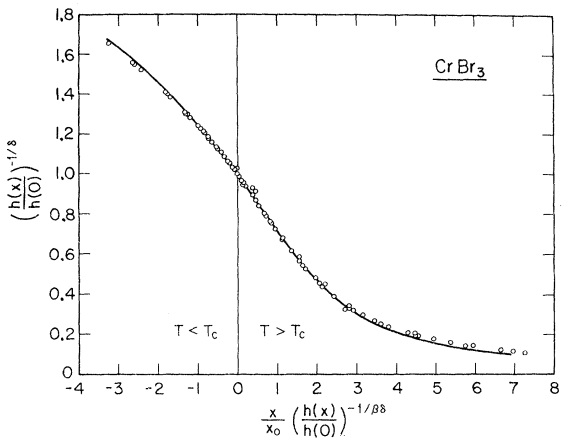

FIG. 2. Comparison of the scaling function of the $S=\frac{1}{2}$ Heisenberg model calculated in this work (solid curve) with the experimental results of Ref. 23 for $CrBr_3$ (small circles). The region between 1.2 and 1.6 of the abscissa corresponds to a range where the theoretical expressions for $h_1(x)$ and $h_2(x)$ [Eqs. (3.7a) and (3.19) of I, respectively] overlap. For smaller values of the abscissa the solid curve is calculated only according to $h_1(x)$, whereas for the larger values it was calculated by using $h_2(x)$ only.

so we compared the experimental data for the scaling function with both the $S=\frac{1}{2}$ and the $S=\infty$ Heisenberg calculations, and also with the $S=\frac{1}{2}$ Ising-model calculation. The data disagreed strongly with the Ising-model calculation, and disagreed with the $S=\infty$ Heisenberg model in those regions where the $S=\frac{1}{2}$ and $S=\infty$ Heisenberg calculations are slightly distinguishable. We show in Fig. 2 the $S=\frac{1}{2}$-Heisenberg-model calculation compared with the data.

It is important to emphasize that there are *no adjustable parameters* whatsoever used in the Heisenberg-model calculation, so we found the agreement to be rather gratifying and even perhaps somewhat surprising when one remembers that the critical-point exponents of $CrBr_3$ ($\beta=0.368$, $\gamma=1.215$, and $\delta=4.28$) are not at all close to the values used in our calculations for the $S=\frac{1}{2}$ Heisenberg model ($\beta=0.385$, $\gamma=1.43$, and $\delta=4.71$). This difference in exponents is likely to be the reason for the slight discrepancy between the "tails" of the experimental and theoretical curves in Fig. 2. For example, at very large x we have $h(x)\sim x^{\gamma}$ [cf. Eq. (3.9) of I] and hence $[h(x)]^{-1/\delta}\sim x^{-\gamma/\delta}$ for the behavior of the ordinate in Fig. 2. Using the values of the exponents quoted above, we expect experimental and theoretical curves to behave as $x^{-0.284}$ and $x^{-0.304}$, respectively, and one can notice that the experimental points do indeed lie somewhat *above* the theoretical curve in Fig. 2.

Figure 3 shows an enlarged region of Fig. 2 centered about $x=0$ (corresponding to $T=T_c$). The curve labeled 1 in Fig. 3 is the $S=\frac{1}{2}$ Heisenberg calculation from Fig. 2, while the curve labeled 2 in Fig. 3 is the same functional form, Eq. (3.7a) of I, as curve 1 except that we have used $\beta=0.35$ and $\delta=5$ rather than $\beta=0.385$ and $\delta=4.71$ (cf. the discussion in Sec. III B of I). We note that the two curves are barely distinguishable in this region, though the discrepancy between them is somewhat larger in the region $T<T_c$ $(x<0)$ than in the region $T>T_c$ $(x>0)$. Also shown in Fig. 3 is the scaling function calculated for the Ising model and one sees rather dramatically how much more closely the data are fit by the Heisenberg model than by the Ising model.

According to our understanding of the universality hypothesis, the fact that $CrBr_3$ has a rather large amount of "lattice anisotropy"[24] [different coupling strengths in different directions, the coupling in the z direction (J_z) being about 17 times weaker than in the xy plane (J_{xy})] does not mean that the scaling function should be any different from the "isotropic" case $(J_z=J_{xy})$. Since we cannot easily calculate the scaling function for the case of arbitrary lattice anisotropy, we cannot test this hypothesis *theoretically*, but the fact that our isotropic calculation agrees as well as it does with the ex-

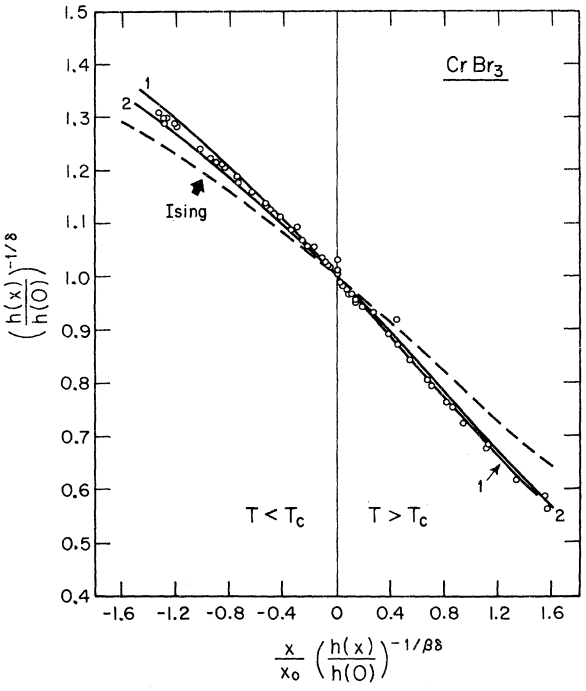

FIG. 3. Enlarged portion of Fig. 2. The curve labeled 1 is the same curve as that in the previous figure, i.e., it corresponds to the $S=\frac{1}{2}$ Heisenberg model with $\beta=0.385$ and $\delta=4.71$; the curve numbered 2 corresponds to the same model but the estimates $\beta=0.35$ and $\delta=5$ were used in its construction. The broken curve corresponds to the Ising-model scaling function.

perimental data for $CrBr_3$ (for which $J_z/J_{xy}=\frac{1}{17}$) suggests that *if* there is a dependence of the scaling function upon lattice anisotropy, this dependence is very weak.

C. Europium Oxide (EuO)

A second ferromagnet for which accurate experimental data have very recently been obtained is EuO.[25] Europium oxide has spin quantum number $S=\frac{7}{2}$, and hence in Fig. 4(a) we compare the data with both the $S=\frac{1}{2}$ and the $S=\infty$ scaling functions. We were somewhat surprised to note that the data agree slightly better with the $S=\frac{1}{2}$ scaling function than with the $S=\infty$ scaling function, since many authors have found that $S=\infty$ is a better approximation to EuO than is $S=\frac{1}{2}$.

One reason for this might be that the scaling function $h(x)$ is, in fact, spin independent, in which case the small disagreement between the calculated functions for $S=\frac{1}{2}$ and $S=\infty$ is spurious. Another possible reason for the data agreeing better with $S=\frac{1}{2}$ than with $S=\infty$ is that the normalization amplitude $h(0)$ might not have been properly chosen for the data, and, indeed, the reader can observe by careful inspection of Fig. 4(a) that the experimental data do appear to extrapolate to a value of the ordinate that is slightly larger than unity when the abscissa value is zero (recall that for $T=T_c$, $H=h(0)M^6$). However, the value of $h(0)$ used was $h(0)=7.6\times10^{-6}$ (cgs units), which is the value quoted in Ref. 25. Nevertheless, when we undertook to independently determine (from the actual data of Ref. 25) a value for $h(0)$ using plotting techniques of the sort described in Sec. III A, we were led to estimate $h(0)=6.99\times10^{-6}$, a value about 8% smaller than the value stated in Ref. 25. Using our smaller value of $h(0)$, the experimental data were replotted as is shown in Fig. 4(b). The reader will note that the data now extrapolate to the point $h(x)/h(0)=1$ for $x=0$, and that, in fact, the data lie somewhat closer to the $S=\infty$ $h(x)$ than they do to the $S=\frac{1}{2}$ $h(x)$. Of course, the differences may well be within the computational or experimental error bars.

It might be somewhat surprising that the agreement between measured and calculated scaling functions is as good as it is for EuO, since the measured exponents were $\beta=0.368$ and $\delta=4.46$, while the exponents used in the calculation were

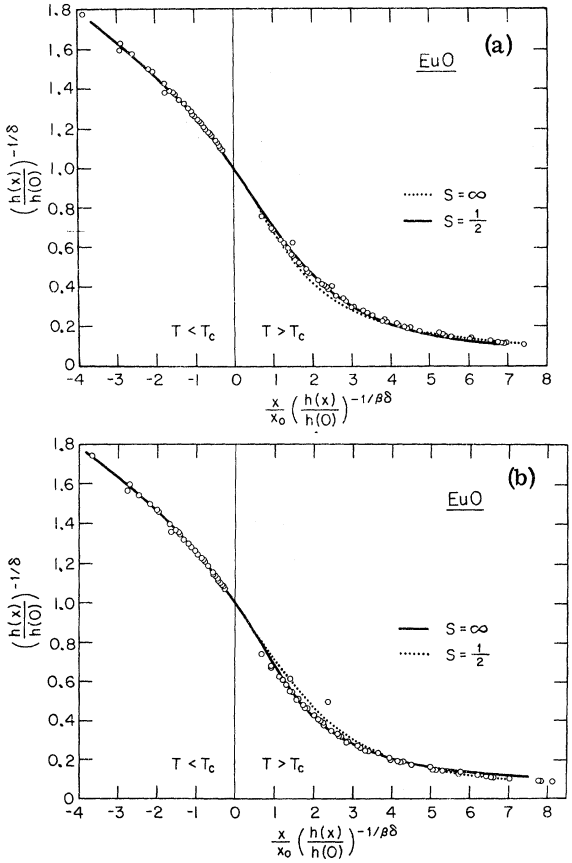

FIG. 4. Comparison of the scaling functions calculated in this work for the $S=\frac{1}{2}$ and $S=\infty$ Heisenberg models with the experimental results of Ref. 25 for EuO. Parts (a) and (b) of the figure differ in the normalization constant $h(0)$ accepted for the experimental data (cf. text).

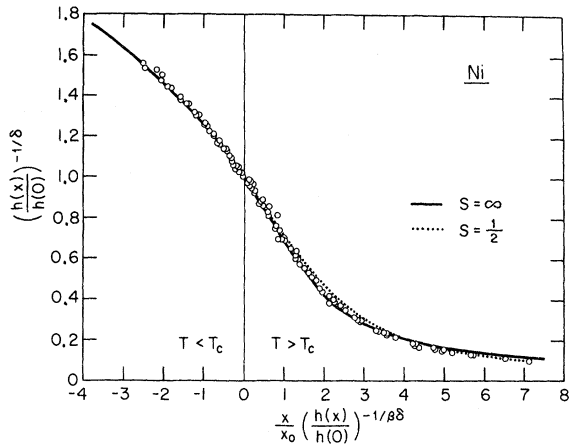

FIG. 5. Comparison of the scaling functions of the $S=\frac{1}{2}$ (dotted curve) and $S=\infty$ (solid curve) Heisenberg models with the experimental results (Ref. 26) for Ni.

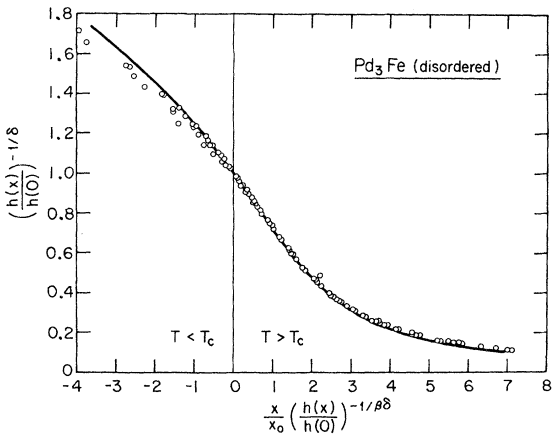

FIG. 6. Comparison of the scaling function of the $S=\frac{1}{2}$ Heisenberg model (solid curve) with the experimental results (Ref. 28) for the Pd_3Fe alloy in the *disordered* state.

$\beta=0.385$ and $\delta=4.71$ for the $S=\frac{1}{2}$ case and $\beta=0.38$, $\delta=4.63$ for the $S=\infty$ case. Besides, EuO has *next-nearest-neighbor* interactions that are of magnitude almost comparable to the nearest-neighbor interactions,[25] while our calculations were for the nearest-neighbor Hamiltonians (3.1) and (4.1). Concerning this point, perhaps we should remark that it is at least possible that the scaling function $h(x)$ is independent of the strength of second-neighbor interactions since a necessary condition for this to be so, the invariance of the critical-point exponents, has been established.[15]

D. Nickel (Ni)

We will next compare our calculated scaling functions with experimental results[26] for Ni, the first ferromagnet for which the scaling-law equation of state was tested.[27] This comparison appears here near the end since our calculations have been done for models which presumably describe *insulating* ferromagnets, whereas Ni is a *metal*, and hence it might seem unrealistic to expect agreement between the theoretical and experimental results. However, Fig. 5 reveals indeed a good agreement of the Ni data with calculated scaling functions for both the spin $=\frac{1}{2}$ and spin $=\infty$ Heisenberg models.

The agreement is somewhat better with the classical Heisenberg model, which may be due to the fact that the critical exponents[26] of Ni ($\beta=0.378$ and $\delta=4.58$) are closer to the values for $S=\infty$ ($\beta=0.38$ and $\delta=4.63$) than to those for $S=\frac{1}{2}$ ($\beta=0.385$ and $\delta=4.71$).

We find that comparison with experimental results for Ni suggests very strongly that insulating or conducting properties of a material are not important near the critical point as regards the normalized scaling function $h(x)$.

E. Palladium-Iron Alloy (Pd_3Fe)

Figure 6 shows rather good agreement between experimental data[28] for the conducting alloy Pd_3Fe in its *disordered* state with our calculated scaling function for the $S=\frac{1}{2}$ Heisenberg model.[29] In contrast, the data for the *ordered* alloy[28] shown in Fig. 7 are in rather poor agreement with the Heisenberg-model calculation.

What is the interpretation of these results? One possible interpretation is that the interactions in the *random* alloy may be rather short range,[28] and indeed the measured critical-point exponents $\beta=0.364$ and $\delta=4.61$ are rather close to those for the nearest-neighbor Heisenberg model, Eq. (3.1) of I. On the other hand, the interactions in the *ordered* alloy might be quite long range,[28] and in

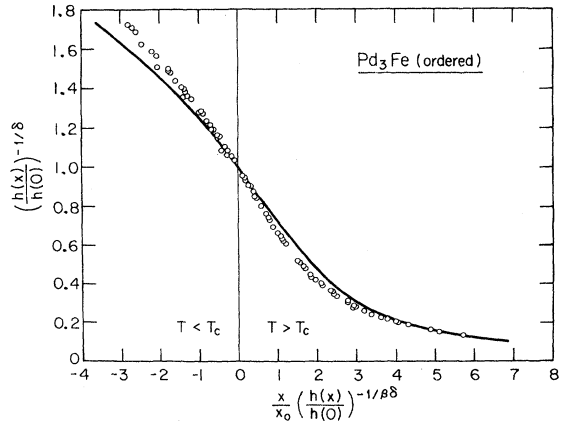

FIG. 7. Comparison of the scaling function of the $S=\frac{1}{2}$ Heisenberg model (solid curve) with the experimental results (Ref. 28) for the Pd_3Fe alloy in the *ordered* state.

fact the measured critical-point exponents $\beta = 0.444$ and $\delta = 3.64$ are considerably closer in magnitude to the predictions $\beta = \frac{1}{2}$ and $\delta = 3$ of the Curie-Weiss or "mean-field theory" which corresponds to each spin interacting with every other spin with a force of equal magnitude. It is precisely this case of "infinite-range interactions" that, according to our understanding, the universality hypothesis[6,11] would predict different behavior. Thus the agreement of experiment and Heisenberg model in Fig. 6 and the disagreement in Fig. 7 might, in fact, be consistent with the universality prediction that systems with short-range interactions have different critical properties from systems whose interactions are infinite in range.[6,11]

In summary, then, we have seen that the nearest-neighbor isotropic Heisenberg-model equation of state [Eqs. (3.7a) and (3.19) of I for $S = \frac{1}{2}$ and Eq. (4.8) of I for $S = \infty$] appears to be adequate for a wide range of physical systems: (i) $CrBr_3$ (a rhombohedral two-sublattice ferromagnet[24] with $S = \frac{3}{2}$ and $J_z/J_{xy} = \frac{1}{17}$), (ii) EuO (a $S = \frac{7}{2}$ semiconducting magnet with next-nearest-neighbor interactions), (iii) Ni (a metallic ferromagnet), and (iv) disordered Pd_3Fe alloy.

Thus we see that the scaled equation of state, when properly normalized, appears to be independent of many specific physical parameters.[6] The one case of disagreement, ordered Pd_3Fe alloy, is thought to correspond to an example of infinite-range interactions for which case the universality hypothesis would indeed predict that the nearest-neighbor models considered in this work would be inadequate.

IV. SUMMARY AND CONCLUSIONS

In this work we have presented a method for calculating, directly from high-temperature series expansions, the scaling function $h(x) = H/M^{\delta}$, where $x \equiv \epsilon/M^{1/\beta}$. Previous calculations of $h(x)$ required the use of *low-temperature* series as well, and hence were restricted to the Ising model; we found that our calculation agreed with this previous work.[3] Our method is made possible by directly utilizing the assumed property of the magnetic field $H(\epsilon, M)$ being a generalized homogeneous function.[20]

Most magnetic materials are thought to be better described by the Heisenberg model, so we proceeded to calculate $h(x)$ for the $S = \frac{1}{2}$ and $S = \infty$ Heisenberg models. In order to obtain numerical results we had to infer the magnitude of the exponent δ from estimates of other exponents and "scaling relations" such as $\beta + \delta = \Delta$ and $\beta(\delta + 1) = 2\beta + \gamma$. Comparison of experimental results for $CrBr_3$, EuO, Ni, and disordered Pd_3Fe with our calculated $h(x)$ functions reveals discrepancies never larger than 10% and in most cases much better. For the comparison we followed previous workers[3] and normalized $h(x)$ by $h(0)$ and x by x_0 [where x_0 is defined by $h(-x_0) = 0$].

Our calculations also support the conjecture that the scaling function might not depend upon the lattice structure and spin quantum number S, at least to within the accuracy of our calculational procedure. To get more definite support concerning this "universality hypothesis," one would have to make more accurate calculations, and this would require using longer series than are currently available. The fact that our calculated $h(x)$ agree with data on $CrBr_3$ (for which there exists considerable *lattice anisotropy*) and on EuO (for which there exist nonnegligible *next-nearest-neighbor interactions*) suggests the further conjecture that $h(x)$ might be independent of these features of the Hamiltonian. By contrast, the change in $h(x)$ on going from Ising ($D = 1$) to Heisenberg ($D = 3$) coupling was quite substantial, as also is the change on going from $d = 2$ to $d = 3$ (observed in Ref. 3). Accordingly, we put forth the working hypothesis that the scaling function depends principally on spin dimensionality (or "symmetry number") D and on lattice dimensionality d.

We hope that our calculated expressions for $h(x)$ will prove useful to those wishing to compare experimental data with model systems. We also hope that the method presented in I will prove applicable to other model systems (e.g., the planar-spin model of helium), and work along these lines is in progress.[30]

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[22]One may wonder why we did not calculate the equation of state as it is given by Eq. (3.3) since we actually compare our calculations with given experimental results in the way that represents Eq. (3.3) (with suitable normalization). The reason is that the corresponding high-temperature series expansions for $M(\epsilon, H)$ contain singularities which would not allow one to perform the kind of calculation we did following (2.5) of I and using, e.g., expansion (2.6) of I. The obstacles are the same as those encountered in determination of phase boundary from the expansions for M (Refs. 16–18).

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