Ground state of uniformly frustrated Josephson-junction arrays at irrational frustration

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A conjectured outline of the structure of vortex lattices of uniformly frustrated Josephson-junction arrays at irrational frustration is presented. [S0163-1829(99)07513-X]

The structure of the energy landscape of uniformly frustrated Josephson-junction arrays is not yet well understood. Such systems are usually studied when the frustration parameter which denotes the number of flux quanta of the external field through the unit cell of the array is rational; $f=p/q$. It is believed that, at least at some large q values of f, the system may exhibit glassylike behavior.[1,2] Such rational numbers are the best representatives of the system at irrational frustration.

Uniformly frustrated XY models, embodied in arrays of Josephson junctions in constant perpendicular magnetic field, are described in the Landau gauge by

$$H=-J\Sigma_{\langle ij\rangle}\cos(\theta_i-\theta_j)-J\Sigma_{\langle ij\rangle}\cos(\theta_i-\theta_j-2\pi mf). \tag{1}$$

The first term of this Hamiltonian includes the energies of nearest-neighbor sites forming bonds that are directed along the x axis of the square array. The second term is for bonds directed along the y axis; m is an integer expressing the x coordinate of these bonds in units of the lattice constant. The Josephson coupling constant J sets the energy scale, and is taken to be positive. The Hamiltonian is invariant if f is changed to $-f$ (reversing the direction of magnetic field) or if an integer is added to f (adding and integral number of flux quanta).

For a rational f, the structure of the ground state of the above Hamiltonian forms a superlattice of vortices having (typically) a qxq unit cell. A clue to why this occurs is suggested by Eq. (1). At m equal to integer multiples of q, the frustrating phase $2\pi mf$ disappears, and the magnetic period becomes commensurate with the period of the underlying lattice. From isotropy of the system qxq periodicity follows. This is certainly not proof of the formation of the unit cell of the vortex lattice, indeed few exceptions exist,[3] yet it indicates how the periodicity of frustration is manifested in the superlattice of vortices.

Recently, it was shown that for a class of local minimum states of Hamiltonian (1), the two-dimensional problem of finding the vortex structure, i.e., phases at sites such as a,b, is reduced to a problem in one dimension.[4] The idea amounts to proving that the phase correlation between sites a,b and $a+1,b+1$, which produces the Halsey staircase state, can be generalized to the case where suitable phase correlations are established between sites a,b and $a+n,b+1$. In this way one can construct the two-dimensional lattice row by row, each row having the same one-dimensional (1D) arrangement, only shifted by n plaquettes relative to the row below [Figs. 1(a) and 1(c)]. It is shown that this results in the lattice of vortices in a given row, to sit in the potential relief of the row below, leading to a low-energy structure of the 2D array, with a qxq unit cell. The outcome is a highly ordered set of phases; due to certain symmetries, the number of independent phases is $q/2$ $[(q-1)/2$ for q odd].[5]

When f is irrational, one deals with vortex lattices incommensurate with the underlying lattice—no unit cell exists. In this case, the usual strategy for studying the spectrum of Eq. (1), and the corresponding vortex structures, is to make use of best rational approximants of f. As our prime example, we consider the case of $f=\tau$, the golden ratio.

As mentioned above, due to the symmetries of Eq. (1), $f=\tau=(\sqrt{5}+1)/2, f=\tau-1$, and $f=2-\tau$ are equivalent. The sequence, r_i, of best rational approximants of $f=\tau$ are then effectively given by

$$\frac{1}{2},\frac{1}{3},\frac{2}{5},\frac{3}{8},\frac{5}{13},\frac{8}{21},\frac{13}{34},\frac{21}{55},\dots\frac{F_N}{F_{N+2}},\dots, \tag{2}$$

where F_N denotes the Nth Fibonacci number, with $F_5=5$ and $r_5=\frac{5}{13}$.

The study of the above sequence of rationals amounts to approximating the incommensurate vortex lattice, with the best (most) commensurate lattices of increasingly larger periods. One then hopes to gain insight into the structure of the vortex lattices at $f=\tau$. Although the sequence of ground state energies, thus found, approaches the ground state energy $E(\tau)$ due to the continuity of the spectrum,[6] it is not possible to infer any property of the ultimately infinite vortex lattice from the finite vortex lattices found at each rational approximant. This is basically because of the periodic boundary conditions which stabilize a particular vortex struc-

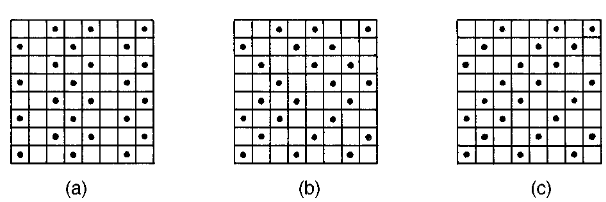

FIG. 1. Vortex lattices for $f=\frac{3}{8}$. In (a), $n=4$; the lattice is effectively 8×2, but it is not stable: the vortices move to fill the empty columns. The resulting lattice is depicted in (b); it has an energy per site of -1.2764 J. In (c), the Halsey staircase state is shown, $n=1$, and the energy per site is -1.28146 J.

FIG. 2. The spacing between successive vortices is in the self-similar quasiperiodic order of the Fibonacci sequence $f=\frac{8}{21}$.

ture at a particular member of the sequence, while for another member, a totally different ground state lattice structure forms.

No doubt the structure of the infinite vortex lattice exists, is well defined, and in principle can be formulated. To this end, one must observe a kind of continuity in the finite lattice structures, which result from the sequence of rationals. This constitutes the main objective of the present article. In what follows, we shall argue for a possible way of fulfilling this requirement.

The one-dimensional problem of finding a generalized Halsey state is that of distributing fq vortices over q plaquettes of a ladder of Josephson junctions in order to minimize the interaction energy. Recalling the Coulomb gas version of the XY model, one is not surprised to find that the exact solution by Hubbard and also by Pokrovsky and Uimin (HPU), applies to the present case.[7] Empirically, we have verified that the arrangement of vortices along the ladder, which minimizes the energy, is given by

$$x_i=[i/f], \quad i=1,\ldots,p. \tag{3}$$

In Eq. (3), x_i denotes the location of the plaquette occupied by a vortex; the square brackets indicate the integer part. This equation states that the unit cell of the structure is q plaquettes long, and contains p vortices; it also observes the symmetries of Hamiltonian (1). Equation (3) is equivalent to the HPU solution.[8] For the arrangement of vortices at $f=\tau$, this equation gives the celebrated one-dimensional quasiperiodic self-similar Fibonacci lattice (Fig. 2).

In the language of our problem, the solution indicates that the one-dimensional minimum energy structure is independent of the shift parameter n. Knowing the 1D vortex lattice, the choices for possible values of n are immediately determined: one requires no nearest-neighbor vortices, once the 2D lattice is developed. However, not all possible n's give viable results [Fig. 1(b)].

Using the methods previously described, we studied the generalized Halsey staircase local minima of the first few members of the sequence (2) aiming at finding the particular shift parameter leading to the lowest-energy 2D vortex lattice. The results are summarized in Table I. For $q=21$, the generator of the 2D vortex lattice is given in Fig. 2. For other entries except for those indicated by an asterisk, the generator is similarly found. It is customary to indicate the order in the Fibonnaci lattice in terms of the spacing between the neighboring vortices, e.g., $LSLLSLSL$ for $q=21$, L denoting long (two plaquette spacing) and S, short (single plaquette spacing). The entries with asterisks do not obey such order (and do not compose the lowest-energy states). For instance, $n=4$, $q=21$ is a $LSLSLSLL$ state. At present, it is not known why the 1D lattice for some n values is not stable and converts to a different state when the 2D lattice is found. In addition to the generalized staircase states not obeying Eq. (3) (and denoted by an asterisk in Table I), there are nonstaircase states as well. These are easier to understand. Again consider Fig. 1(a). Here, we have $f=\frac{3}{8}, n=4$. The lattice contains empty columns with no vortex resident; these may be called striped lattices. At high densities, a more uniform distribution of vortices exists giving lower energy; neighboring vortices migrate to occupy the empty columns. The stable state found [Fig. 1(b)] is not a local minimum characterized by a shift parameter. We have indicated such unstable striped states (or nearly striped, in the case of $f=\frac{21}{55}$, $n=17$) with the letter u, in Table I. It should be noted that the stable lattice resulting from such unstable patterns may actually turn out to be a low-energy state (though none of those mentioned are), but we shall argue below that such lattices need not concern us.

The $n=1$ state is the ordinary Halsey staircase state and is seen to possess the lowest-energy configuration for each member of the rational sequence of $f=\tau$ studied. The entries of Table I give five significant digits; at times the 2D lattice had to be cooled to 0.00001 J/k_B. We conjecture that this result holds for all members of the sequence.

The importance of this result becomes apparent once we recall that the best rational approximants come from the continued fraction expansion. The continued fraction expansion builds upon itself and the 1D vortex lattices inherit this property, i.e., they are assembled sequentially.[9] It is also important to note the implicit property of best rational approxi-

TABLE I. Energies for the sequence of best rational approximants of $f=\tau$ as a function of possible values of n. $n=1$ gives the lowest energy. See text for the meanings of u's and asterisks. Blank entries could be filled by symmetry arguments, for instance, $n=4,9,17,22$ are all the same state when $f=\frac{5}{13}$.

	$f=\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{5}{13}$	$\frac{8}{21}$	$\frac{13}{34}$	$\frac{21}{55}$
$n=1$	$-\sqrt{2}$	$-4/3$	-1.2944	-1.2814	-1.2763	-1.2744	-1.2736	-1.2734
4				u	-1.2534	-1.2690**	-1.2608*	-1.2647*
9						-1.2737[a]	-1.2539	1.2684
12							-1.2637[d]	-1.2717
17							u	u
22								-1.2674[b]
25								-1.2580[c]

[a]The lattice is 21×7.
[b]The lattice is 55×5.
[c]The lattice is 55×11.
[d]The lattice is 34×17.

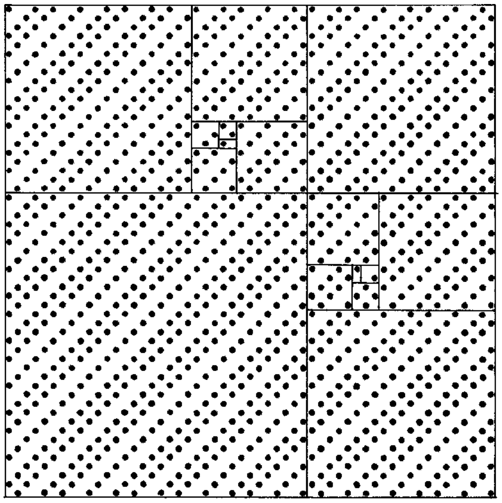

FIG. 3. The Fibonacci state at the $f=\frac{21}{55}$ stage; vortices occupy diagonals that are spaced in the Fibonacci sequence. The self-similar nesting cells are shown by nearly unfrustrated vertical boundaries. The figure shows the claimed structure for the ground state when f is the golden ratio.

mants, namely, the successive q's are such that τF_N gets closer and closer to an integer as N grows. This so-called nesting property is summarized in Fig. 3, which depicts what could be referred to as the Fibonacci state at the $N=8$ stage. The Fibonacci state is unique in that at each stage its generator is obtained from the assembly of generators of two previous stages, each repeated only *once*. For instance, $f=\frac{5}{13}$ with $LSLLS$ is obtained from LSL which belongs to $f=\frac{2}{5}$, and LS which belongs to $f=\frac{1}{3}$. This is the result of the continued fraction expansion having only 1's. Furthermore, the Fibonacci state has the property that the vortices appear in diagonals that are spaced according to the quasiperiodic ordering of the Fibonacci lattice. We believe that the Fibonacci state is the ground state of the uniformly frustrated Josephson-junction array at $f=\tau$.

Figure 3 shows the 55×55 Fibonacci state. The vertical lines at $m=0$ and $m=55$ show the unfrustrated bonds. The other vertical lines show the nearly unfrustrated bonds; the approximation becoming worse for the (vertical) boundaries of smaller cells. The nesting property obtained is, of course, gauge invariant, but manifest in the Landau gauge. Using Fig. 2 we could expand the total energy of the lattice in terms of the total energies of its constituents

$$E(r_8)=E(r_7)+3E(r_6)+2[E(r_5)+E(r_4)+E(r_3)+E(r_2)+E(r_1)], \tag{4}$$

which misses one vortex-residing plaquette. Using Halsey's formula for the constituents, the energy per site of the 55×55 lattice obtained from Eq. (4) is -1.2756 J, and is to be compared to the Halsey energy of the 55×55 lattice, which is -1.2734 J. The expansion overestimates the nesting fitness by treating the constituent cells as independent lattices, giving a lower than actual energy. The energy of the Fibonacci state is -4 J/π, which is the infinite q limit of Halsey's formula.

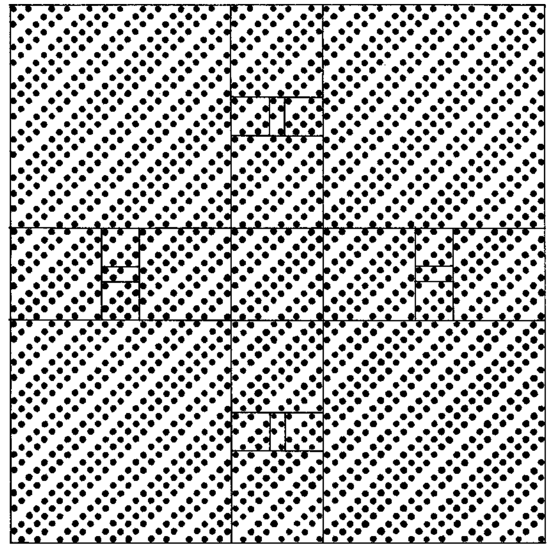

FIG. 4. The structure of the ground state for $f=\sqrt{2}$ is claimed to be a Halsey state. Here, we see the structure for one of its rational approximants, $\frac{29}{70}$, $n=1$. The spacing between vortex diagonals and nesting are governed by the continued fraction of $\sqrt{2}$. The structure in the infinite q limit is degenerate with the Fibonacci state.

Next consider $f=\sqrt{2}$. The sequence is now $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{12}{29}$, $\frac{29}{70}$, ... The possible values of n for the first five members of the sequence mentioned are in the set (1,6,11,18,23,30). The study of $\frac{29}{70}$ is cumbersome, but from our studies we place a lower bound energy of -1.26 J per site, for $n>1$. For such values of n, the best energy found was that of $f=\frac{12}{29}$, $n=11$, with -1.257 J. Again, the $n=1$ state with an energy per site of -1.2733 J, provides the lowest-energy state. Figure 4 shows the outcome; here, the continued fraction has the property that $q_{N+2}=2q_{N+1}+q_N$, in contrast with Fig. 2, where we have $q_{N+2}=q_{N+1}+q_N$.

The implicit requirement in the evolution of the states, as the sequence of rationals approaches the irrational limit, i.e., the continuity that we mentioned in the beginning, is that the structure of the lattices obtained be essentially the same. For $n=1$ this condition is naturally satisfied, as seen in Figs. 1 and 2. However, if the structure is not shared by the lowest-energy members of the sequence, the scheme fails—a different type of nesting should be found, and the simple shift map states will not do. This becomes more clear in the study of $f=\sqrt{3}$.

The sequence for $f=\sqrt{3}$, is $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{11}$, $\frac{4}{15}$, $\frac{11}{41}$, ... The continued fraction now has a mixed nature, that of $f=\tau$ and that of $f=\sqrt{2}$. The study of the first four members indicates that $n=2$ gives the lowest energy. The lattice structures, however, vary. In Fig. 5, we show the $n=2$ state for $\frac{11}{41}$, with an energy per site of -1.3410 J. The cells having correct $\frac{4}{15}$ and $\frac{3}{11}$ structures are marked by I and II, respectively. The differences between them can be traced back to the differing structure of $\frac{1}{3}$ and $\frac{1}{4}$, especially, since $f=\frac{1}{4}$ has a 4×8 unit cell.[10] As seen, the component cells cannot be fitted as well as the other two cases studied. The important point is that the structure of the lowest q members of the sequence plays the determining role.

Transcendental numbers do not have periodic continued fractions, yet the sequential property holds. For $f=\pi$, the

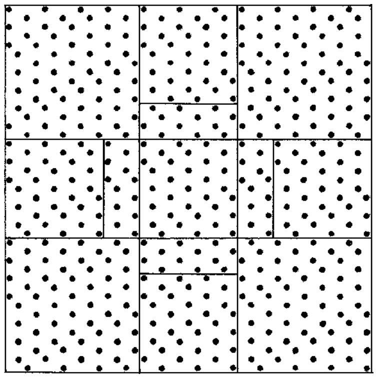

FIG. 5. The $n=2$ vortex lattice for $f=\frac{11}{41}$ in the best approximant sequence of $\sqrt{3}$. The nesting is not complete. The correct $\frac{4}{15}$ and $\frac{3}{11}$ structures are indicated by I and II, respectively.

sequence starts with $\frac{1}{7}$, has ground state energy -1.5166 J, and is obtained with $n=2$ or equivalently $n=3$. The sequence grows rapidly in q and becomes difficult to study; $\frac{15}{106}$, $\frac{16}{113}$, $\frac{4703}{33215}$, ... The case of $f=e$ is similar to that of $f=\sqrt{3}$, having the sequence $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{7}$, $\frac{9}{32}$, ...

In summary, we have argued that to obtain the structure of an infinitely large array of Josephson junctions at an irrational value of the frustration parameter, a particular limiting form of successively larger arrays needs to be considered. This is in accord with the uniform continuity of the spectrum, and it further demands a continuity of the ''eigenfunctions.'' An immediate possibility is suggested by studying the generalized Halsey staircase states of varying length n as a way of generating this particular limiting form. Without such a procedure, the low-lying-energy lattices found at each member of the sequence, should be considered as artifacts of the boundary conditions. In general, the basic one-dimensional lattice may very well be a kind of backbone for the 2D lattices, where different permutations of vortex spacings could be used to build up the 2D lattice. For the particular case of the limiting form discussed, the structures have a nesting property and hierarchical nature, resting upon the structure of the low q members of the sequence.

The study of the spectrum at irrational f sheds light on the differentiability properties of $E(f)$. Due to the uniform continuity of the spectrum, the incommensurate vortex lattices cannot all be of the same ground state energy, i.e., a measure of commensuration remains even at the infinite q limit. However, uniform continuity does not necessitate smoothness at any point of the domain. Finally, we note that for superconducting wire grids, the equivalent of what we have named the Fibonacci state has been reported to be the ground state.[11]

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