Josephson current in a finite-size junction interrupting a superconducting ring

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We study the behavior of the Josephson current I_J flowing in a finite-size Josephson junction in a superconducting ring in the presence of an externally applied magnetic field H, taking into account the effect of the shielding currents. The set of self-consistent equations for the system can be solved explicitly for I_J in the small self-inductance coefficient limit for not negligible effective junction areas. It is found that the resulting I_J versus H curve presents a Fraunhofer-like prefactor modulating a periodic quasisinusoidal odd function. [S0163-1829(99)10413-2]

I. INTRODUCTION

Superconducting rings containing small Josephson junctions are well-known systems: Their electrodynamics and thermodynamics have been extensively studied in the past.[1–3] Indeed, a broad scientific interest lies beneath the macroscopic quantum coherence phenomena, which are present in these systems because of the validity of the Bohm-Aharonov relation[2] for the superconducting state. As a result, a large variety of applications has been realized due to the interplay between quantum mechanics and classical electrodynamics.[1,4] Moreover, after the discovery of high-T_c superconductivity,[5] circuital models containing small Josephson junctions (JJ's) (Refs. 6–10) have been used in simulating the magnetic response of superconducting cuprates possessing a marked granular structure.[11,12] In dealing with these systems, when one faces the question ''How does the local magnetic field h affect the value of the Josephson current I_J which may flow into the junction?,'' one often refers to the well-known Fraunhofer-like pattern valid for a small junction in the presence of a bias current I_B. This pattern is derived for an isolated junction, and its validity is not *a priori* evident in the context of Josephson junction networks or, in the simplest case, of a JJ in a superconducting ring.

Therefore, in the present work we shall examine in detail the current distribution in a superconducting ring interrupted by a JJ in order to derive a set of self-consistent equations, which shall allow us to study the magnetic-field dependence of I_J in the general case of a JJ of finite size. In Sec. II we shall analyze the field and current distributions in the system and in Sec. III we shall give an analytic expression for the Josephson current I_J as a function of the external magnetic field for small effective inductances of the ring. Conclusions are drawn in the final section.

II. FIELD AND CURRENT DISTRIBUTIONS

Let us consider a single rectangular Josephson junction of length L in a superconducting ring of inner and outer radii R_{in} and R_{out}, respectively, in the presence of an external magnetic field $\vec{H}$ as shown in Fig. 1(a). The vector potential $\vec{A}$ in the superconducting ring satisfies the Bohm-Aharonov relation.[2] The flux Φ linked to the closed circular path $C = C_S \cup C_J$ shown in Fig. 1(a) may be expressed as follows:

$$\Phi = \oint_C \vec{A} \cdot d\vec{l} = \int_{C_S} \vec{A} \cdot d\vec{l} + \int_{C_J} \vec{A} \cdot d\vec{l}, \tag{1}$$

where C_S is the portion of C inside the superconducting ring and C_J is that inside the Josephson junction. If we choose a path well inside the superconducting ring in such a way that

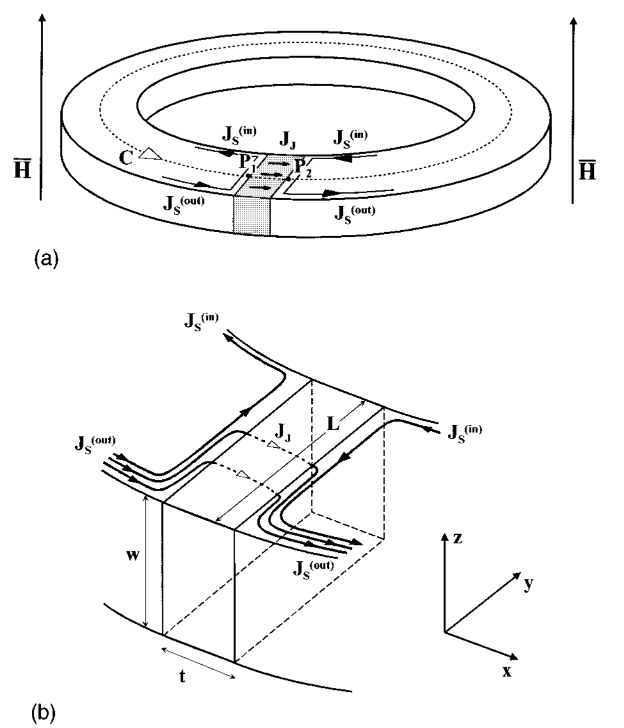

FIG. 1. (a) Schematic representation of the current distribution in the superconducting ring interrupted by a finite-size Josephson junction. (b) Detailed representation of the current distribution in the vicinity of the Josephson junction in (a).

the superconducting current density $\vec{J}_S$ is null, we can write the well-known fluxoid quantization condition:

$$\frac{2\pi}{\Phi_0}\Phi+\varphi=2\pi n, \tag{2}$$

where n is an integer and

$$\varphi=\theta(P_2)-\theta(P_1)-\frac{2\pi}{\Phi_0}\int_{P_1}^{P_2}\vec{A}\cdot d\vec{l} \tag{3}$$

with P_1 and P_2 being the extrema of the path C_J as in Fig. 1(a) and θ the superconducting phase. Notice, however, that the flux Φ can be written as a function of the variable y [see Fig. 1(b)] as follows:

$$\Phi=\Phi_J(y)+\Phi_{\rm in}, \tag{4}$$

where $\Phi_{\rm in}$ is the flux linked to the circular path $C_{\rm in}$ of radius equal to $R_{\rm in}$ and Φ_J is the flux through the area $S_J=d(L/2-y)$, where $d=2\lambda+t$, λ being the London penetration depth of the superconducting ring.

The flux $\Phi_{\rm in}$ depends on the particular magnetic state realized and, therefore, on the magnetic history of the system. Indeed, the superconducting ring may trap magnetic flux irreversibly for high enough values of the Josephson coupling energy $E_J=I_{J0}\Phi_0/2\pi$, I_{J0} being the maximum Josephson current of the JJ, and of some effective inductance coefficient $L_{\rm eff}$. Only in the very simple case of reversible behavior and of extremely small junctions can we set

$$\Phi\simeq\Phi_{\rm in}\simeq\Phi_{\rm ex}=\mu_0 HS_{\rm out}, \tag{5}$$

where $S_{\rm out}=\pi R_{\rm out}^2$, so that the Josephson current in the JJ is modulated by the field value according to the following:

$$I_J=-I_{J0}\sin\left(\frac{2\pi}{\Phi_0}\Phi_{\rm ex}\right). \tag{6}$$

However, in this simplest case one neglects (i) the flux Φ_J; (ii) the shielding currents $I_S^{(\rm in)}$ and $I_S^{(\rm out)}$, which circulate as shown in Fig. 1(b); (iii) the self-fields generated by all currents in the system.

Even though these results are valid in the extremely small junction limit and in the limit of negligible shielding current effects, a more general approach is required when these conditions are not met.

Therefore, when the size of the junction interrupting the superconducting ring is not negligible with respect to $R_{\rm out}$, the flux Φ_J should also be taken into account. In order to consider all contributions to Φ_J, let us sketch the current distribution in its surroundings as in Fig. 1(b). We notice that a superconducting shielding current density $\vec{J}_S$ flows in such a way to give $\vec{B}=0$ in the region well inside the ring for which $R_{\rm in}+\lambda\leqslant r\leqslant R_{\rm out}-\lambda$, where r is the radial distance from the center. The current density $\vec{J}_S$ is taken to be confined in the complement, with respect to the ring, of the above region. Notice also that a portion of $\vec{J}_S$ flows through the junction, giving rise to the superconducting Josephson current density $\vec{J}_J=J_J(y)\hat{x}$. As a consequence, one can identify, away from the junction and on the opposite sides of the ring, two values of the shielding current I_S, namely

$$I_S^{(\rm out)}=J_S^{(\rm out)}\lambda w, \tag{7}$$

$$I_S^{(\rm in)}=J_S^{(\rm in)}\lambda w, \tag{8}$$

where $J_S^{(\rm out)}$ and $J_S^{(\rm in)}$ are the average shielding current densities in the regions for which $R_{\rm out}-\lambda\leqslant r\leqslant R_{\rm out}$ and $R_{\rm in}\leqslant r\leqslant R_{\rm in}+\lambda$, respectively, and w is the ring thickness.

In the vicinity of the JJ, on the other hand, we can write

$$\vec{J}_S=\pm J_S(y)\hat{y}. \tag{9}$$

By charge conservation we have

$$I_S(y+dy)-I_S(y)=J_J(y)w\,dy, \tag{10}$$

so that

$$J_J(y)=-\frac{1}{w}\frac{dI_S}{dy}=-\lambda\frac{dJ_S}{dy}. \tag{11}$$

Having schematized the current distribution in the system through Eqs. (7)–(11), we can state that the field inside the Josephson junction is given by superimposing (a) the external field $\vec{H}$; (b) the field generated by the current density $\vec{J}_S$ present on both sides of the JJ; (c) the field generated by the current density $\vec{J}_J$ flowing through the JJ.

The flux Φ_J can thus be written as follows:

$$\Phi_J=\Phi_a+\Phi_b+\Phi_c, \tag{12}$$

where the subscript refers to the three cases listed above. The fluxes Φ_a, Φ_b, and Φ_c, on their turn, may be expressed as follows:

$$\Phi_\xi(y)=\mu_0\int_{-t/2-\lambda}^{t/2+\lambda}dx\int_y^{L/2}h_\xi(x,y')dy', \tag{13}$$

where $\xi=a,b,c$ and where the $h_\xi(x,y)$'s are the corresponding field distributions in the JJ. By the assumptions set forth in Appendix A, the three fluxes can be written as

$$\Phi_a(y)=\mu_0 d(L/2-y)H, \tag{14}$$

$$\Phi_b(y)=-\frac{\mu_0}{\pi}\ln\left(1+\frac{2t}{\lambda}\right)\left[I_S^{(\rm out)}(L/2-y)-w\int_y^{L/2}dy'\int_{-L/2}^{y'}J_J(\xi)d\xi\right], \tag{15}$$

and

$$\Phi_c(y)=\mu_0(2\lambda+t)\int_y^{L/2}dy'\int_{-L/2}^{y'}J_J(\xi)d\xi. \tag{16}$$

By summing up all contributions, the flux Φ_J can be finally written as follows:

$$\Phi_J(y)=\left[\mu_0 Hd-\frac{\mu_0}{\pi}\ln\left(1+\frac{2t}{\lambda}\right)I_S^{(\rm out)}\right](L/2-y)+\mu_0\left[\frac{w}{\pi}\ln\left(1+\frac{2t}{\lambda}\right)+d\right]\int_y^{L/2}dy'\int_{-L/2}^{y'}J_J(\xi)d\xi. \tag{17}$$

III. MAGNETIC STATES AND JOSEPHSON CURRENT

As stated above, the gauge-invariant superconducting phase difference φ may be expressed in terms of the flux $\Phi=\Phi_{\rm in}+\Phi_J$ by the fluxoid quantization relation [Eq. (2)]. The flux $\Phi_{\rm in}$ depends on the particular metastable state in which the system is in, while an expression of the flux Φ_J has been given in Eq. (17). By the first Josephson equation and by the fluxoid quantization condition, the superconducting current density J_J can be written as follows:

$$J_J(y)=-J_{J0}\sin\left(\frac{2\pi}{\Phi_0}\Phi(y)\right). \tag{18}$$

The total current flowing into the junction is thus

$$I_J=-I_0\left[\frac{1}{L}\int_{-L/2}^{L/2}\sin\left(\frac{2\pi}{\Phi_0}\Phi\right)dy\right], \tag{19}$$

where $I_0=J_{J0}wL$, J_{J0} being the critical current density value of the Josephson junction.

The problem has now been completely stated and may be solved if the system's magnetic metastable state is known. We shall therefore assume that an approximate description of the magnetic state of the system can be given by the following set of equations, derived in detail in Appendix B:

$$I_S^{(\rm in)}=I_S^{(\rm out)}-I_J, \tag{20}$$

$$I_S^{(\rm out)}=\frac{\Phi_J^*-\mu_0H(S_{\rm out}-S_{\rm in})}{2L_{\rm out}}, \tag{21}$$

$$\Phi_{\rm in}=\mu_0HS_{\rm out}+L_{\rm out}I_J-\Phi_J^*, \tag{22}$$

$$\Phi^*=\Phi_{\rm in}+\Phi_J^*, \tag{23}$$

where $S_{\rm in}=\pi R_{\rm in}^2$, $\Phi^*=\Phi(-L/2)$, $\Phi_J^*=\Phi_J(-L/2)$, and $L_{\rm out}$ is the self-inductance coefficient associated to a path of radius $R_{\rm out}$.

Therefore, the problem can be solved under the assumptions made. However, we see that an explicit analytic solution to the problem, even though it has been stated in its simplest form, is not attainable. We must thus resort to more restrictions in order to obtain a qualitative answer to the question we originally put forth: ''How does the externally applied magnetic flux through the junction Φ_J^* influence the value of the Josephson current I_J?'' Let us notice that the double integral term in Eq. (17) is a bounded quantity, since it depends on the sine of the total flux Φ, given by Eq. (18). Thus, for increasing values of H, the first term grows linearly while the second addendum does not. We therefore restrict ourselves to those magnetic states where the second addendum can be neglected, i.e., to high enough values of the applied field, and write the flux Φ_J as follows:

$$\Phi_J(y)=\left[\mu_0Hd-\frac{\mu_0}{\pi}\ln\left(1+\frac{2t}{\lambda}\right)I_S^{(\rm out)}\right](L/2-y). \tag{24}$$

It is now easy to verify that, by means of Eqs. (20)–(24), Φ_J^* can be expressed in terms of H by the following elementary relation:

$$\Phi_J^*=\mu_0HlL, \tag{25}$$

where

$$l=\frac{t+2\lambda+\dfrac{\mu_0}{2\pi}\ln\left(1+\dfrac{2t}{\lambda}\right)\dfrac{(S_{\rm out}-S_{\rm in})}{L_{\rm out}}}{1+\dfrac{\mu_0L}{2\pi L_{\rm out}}\ln\left(1+\dfrac{2t}{\lambda}\right)}. \tag{26}$$

Equation (24) can now be written as follows:

$$\Phi_J(y)=\frac{\Phi_J^*}{L}[L/2-y] \tag{27}$$

and the Josephson current can be found by a straightforward integration to be

$$I_J=-I_0\frac{\sin\left(\dfrac{\pi\Phi_J^*}{\Phi_0}\right)}{\dfrac{\pi\Phi_J^*}{\Phi_0}}\sin\left(\frac{2\pi}{\Phi_0}\Phi_{\rm in}+\frac{\pi}{\Phi_0}\Phi_J^*\right). \tag{28}$$

By now introducing the normalized quantities

$$i_J=\frac{I_J}{I_0},\quad \Psi_J^*=\frac{\Phi_J^*}{\Phi_0},\quad \Psi_{\rm ex}=\frac{\mu_0HS_{\rm out}}{\Phi_0},\quad \beta_0=\frac{L_{\rm out}I_0}{\Phi_0}, \tag{29}$$

and by recalling Eqs. (20)–(23), Eq. (28) can be rewritten as follows:

$$i_J=-\frac{\sin(\pi\Psi_J^*)}{\pi\Psi_J^*}\sin(2\pi\Psi_{\rm ex}-\pi\Psi_J^*+2\pi\beta_0i_J). \tag{30}$$

Equation (30) is the general equation for the magnetic-field dependence of I_J in a finite-size JJ interrupting a superconducting ring. The solution to Eq. (30) could be found numerically by Newton's method. In the small ring inductance limit, i.e., for very small β_0 values, we can write the solution explicitly in terms of the externally applied flux $\Psi_{\rm ex}$. In this case, indeed, the system behaves reversibly and the following single-valued function i_J can be found:

$$i_J=-f(\Psi_{\rm ex})\frac{\sin\{2\pi\Psi_{\rm ex}[1-1/(2M)]\}}{1+2\pi\beta_0f(\Psi_{\rm ex})\cos\{2\pi\Psi_{\rm ex}[1-1/(2M)]\}}, \tag{31}$$

where

$$f(\Psi_{\rm ex})=\frac{\sin(\pi\Psi_{\rm ex}/M)}{(\pi\Psi_{\rm ex}/M)} \tag{32}$$

and $M=S_{\rm out}/lL$. Notice that for extremely small junctions $f(\Psi_{\rm ex})=1$ and the expression

$$i_J=-\frac{\sin(2\pi\Psi_{\rm ex})}{1+2\pi\beta_0\cos(2\pi\Psi_{\rm ex})} \tag{33}$$

reduces to Eq. (6) when we let $\beta_0\to0$.

An i_J versus $\Psi_{\rm ex}$ graph for negligible values of β_0 is shown in Figs. 2(a) and 2(b). These results are clearly different from those obtained in the case of a single current biased small junction. Indeed, in the case of a superconducting ring, we have an odd symmetry of the superconducting current with respect to the applied field, in contrast with the

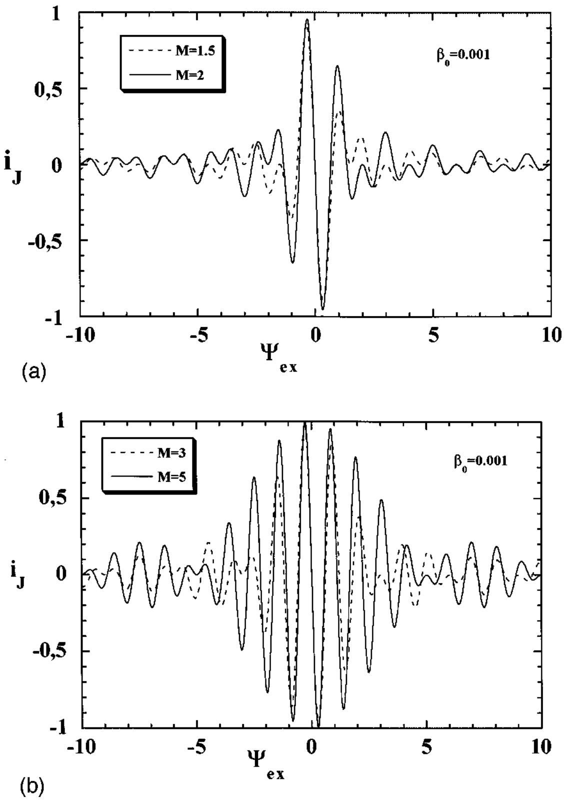

FIG. 2. Josephson current i_J vs normalized applied flux $\Psi_{\rm ex}$ for $\beta_0=0.001$ and (a) $M=1.5$ (dashed line), $M=2.0$ (full line); (b) $M=3.0$ (dashed line), $M=5.0$ (full line).

even symmetry present in the usual Fraunhofer-like pattern. Furthermore, the analytic dependence of the i_J versus $\Psi_{\rm ex}$ curves contains only a prefactor which is similar to that seen in a current biased junction, while an additional term modulates the value of the Josephson current in the loop.

IV. CONCLUSIONS

We have studied the magnetic-field dependence of the Josephson current I_J flowing in a finite-size Josephson junction interrupting a superconducting ring. The analysis has been carried out by schematically considering shielding current effects. It is found that, if the Josephson junction effective area is not negligible with respect to the geometrical area enclosed by the ring, the I_J versus H dependence acquires new structures when compared to the corresponding dependence of an infinitely small junction. It is also noted that this system is intrinsically different from a current biased Josephson junction, even though, in the case of small ring inductances, a Fraunhofer-like prefactor is seen to modulate a periodic quasisinusoidal odd function of the external magnetic field.

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APPENDIX A

We derive here the expressions for Φ_b and Φ_c given in Sec. II. In order to obtain rather immediate results for the flux Φ_b, let us assume that the effect due to the current density $\vec{J}_S$ could be schematized through two thin wires, carrying a current $I_S(y)=J_S(y)w\lambda$ placed at a distance $\lambda/2$ from the two barrier interfaces. In order to avoid divergences, we restrict the integral in Eq. (13) to the x interval $[-t/2,t/2]$. In this way, we can write

$$\vec{h}_b(x,y)=-\frac{I_S}{2\pi}\left[\frac{1}{\left(\frac{\lambda+t}{2}+x\right)}+\frac{1}{\left(\frac{\lambda+t}{2}-x\right)}\right]\hat{z},$$
$$-t/2\leqslant x\leqslant t/2, \tag{A1}$$

so that

$$\Phi_b(y)=-\frac{\mu_0}{\pi}\ln\left(1+\frac{2t}{\lambda}\right)\int_y^{L/2}I_S(y')dy'. \tag{A2}$$

By Eq. (11) we now have

$$I_S(y)=I_S^{(\rm out)}-w\int_{-L/2}^{y}J_J(\xi)d\xi. \tag{A3}$$

Furthermore, by Josephson equations we may write

$$J_J(y)=J_{J0}\sin\varphi(y). \tag{A4}$$

Notice also that the superconducting phase difference φ depends on the flux Φ through the fluxoid quantization condition. We can now make use of the expression of I_S in Eq. (A3) to rewrite Eq. (A2) as follows:

$$\Phi_b(y)=-\frac{\mu_0}{\pi}\ln\left(1+\frac{2t}{\lambda}\right)\left[I_S^{(\rm out)}(L/2-y)\right.$$
$$\left.-w\int_y^{L/2}dy'\int_{-L/2}^{y'}J_J(\xi)d\xi\right]. \tag{A5}$$

Let us now evaluate $\Phi_c(y)$. By Maxwell equations, taking $\vec{h}_c=h_c(y)\hat{z}$, we have

$$h_c(y)=\int_{-L/2}^{y}J_J(y')dy'. \tag{A6}$$

By substituting the above expression for h_c into Eq. (13), we can finally write

$$\Phi_c(y)=\mu_0(2\lambda+t)\int_y^{L/2}dy'\int_{-L/2}^{y'}J_J(\xi)d\xi. \tag{A7}$$

APPENDIX B

In order to derive the magnetic state of a superconducting ring, let us take this system as electromagnetically equivalent to two concentric superconducting loops. By neglecting the superconducting current I_J, the fluxes linked to the two loops can be written as follows:

$$\Phi^*=L_{\rm out}I_S^{(\rm out)}-MI_S^{(\rm in)}+\mu_0HS_{(\rm out)}, \tag{B1}$$

$$\Phi_{\rm in}=-L_{\rm in}I_S^{(\rm in)}-MI_S^{(\rm out)}+\mu_0HS_{(\rm in)}, \tag{B2}$$

where the quantities $L_{\rm in},L_{\rm out}$ are the self-inductance coefficients relative to the inner and outer loop, respectively, and

M is the mutual inductance between the two loops, and where the S's are the areas enclosed by the two current paths. The flux Φ^* is related to the total flux through the junction Φ_J^* as follows:

$$\Phi^* = \Phi_{\text{in}} + \Phi_J^*, \tag{B3}$$

while the current I_J flowing through the junction is given by

$$I_S^{(\text{in})} = I_S^{(\text{out})} - I_J. \tag{B4}$$

We can express the quantities Φ_{in}, Φ^*, $I_S^{(\text{in})}$, and $I_S^{(\text{out})}$ in terms of Φ_J^*, I_J, and H as follows:

$$I_S^{(\text{in})} = \frac{\Phi_J^* - \mu_0 H (S_{\text{out}} - S_{\text{in}}) - (M + L_{\text{out}}) I_J}{L_{\text{out}} + L_{\text{in}}}, \tag{B5}$$

$$I_S^{(\text{out})} = I_S^{(\text{in})} + I_J, \tag{B6}$$

$$\Phi_{\text{in}} = \mu_0 H S_{\text{eff}} + L_{\text{eff}} I_J - \kappa \Phi_J^*, \tag{B7}$$

$$\Phi^* = \Phi_{\text{in}} + \Phi_J^*, \tag{B8}$$

where

$$S_{\text{eff}} = \frac{(L_{\text{in}} + M) S_{\text{out}} + (L_{\text{out}} - M) S_{\text{in}}}{L_{\text{out}} + L_{\text{in}}} \tag{B9}$$

and

$$\kappa = \frac{(L_{\text{in}} + M)}{L_{\text{out}} + L_{\text{in}}}. \tag{B10}$$

In the case of almost identical loops, we can set

$$L_{\text{in}} \simeq L_{\text{out}} = L_{\text{eff}}; \quad M \simeq L_{\text{in}} \rightarrow S_{\text{eff}} \simeq S_{\text{out}}; \quad \kappa \simeq 1, \tag{B11}$$

so that Eqs. (B5)–(B8) can be written as in Eqs. (20)–(23).

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