Algebraic Bethe ansatz for the Heisenberg model with the Dzyaloshinsky-Moriya interaction and the growth model

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The Heisenberg model with Dzyaloshinsky-Moriya interaction (the constant Dzyaloshinsky vector $\mathbf{D}=D\mathbf{z}$, D is a real number) and the growth model (D is a complex number) are investigated in the framework of the quantum inverse scattering method. The Lax pair and the corresponding R matrix satisfying the Yang-Baxter relation are obtained. These models are also solved by means of the algebraic Bethe ansatz. The eigenvalues and the Bethe ansatz equations are derived. [S0163-1829(99)05013-4]

In recent years, much attention has been paid to the Dzyaloshinsky-Moriya (DM) interaction (i.e., an antisymmetric spin-spin interaction)[1,2] in the Heisenberg magnetic systems because of its important role in describing spin glasses[3] and in studying the weak ferromagnetism of the low-temperature phase of all lameblar copper oxide superconductors,[4] phase transition,[5] nonlinear spin excitations,[6] etc. The Heisenberg model with DM interaction is given by

$$H=\sum_{i=1}^{N}\left[J(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y+\Delta S_i^zS_{i+1}^z)+\mathbf{D}\cdot(\mathbf{S}_i\times\mathbf{S}_{i+1})\right], \tag{1}$$

where $S_i^{x,y,z}$ are the components of spin-$\frac{1}{2}$ operator at site i, J is the exachange integral, $\Delta(\neq 0)$ is the exchange anisotropic parameter, and $\mathbf{D}$ is the constant Dzyaloshinsky vector. Here, we use the periodic conditions: $S_{N+1}^{x,y,z}=S_1^{x,y,z}$ and assume that $\mathbf{D}=D\mathbf{z}$, D is a real number. The Hamiltonian (1) was solved by relating it to the XXZ model with a suitable boundary condition.[7] However, when D is an imaginary number, the Hamiltonian (1) is related to the growth models[8–10] of the Kardar-Parisi-Zhang universality class[11] and was solved by the coordinate Bethe ansatz.[8,10] In Ref. 12, using the algebraic Bethe ansatz,[13–15] Bogoliubov and Nassar also diagonalized the Hamiltonian (1) with $D=i$ and $\Delta=1$, which has the same R matrix with the phase-difference model. In the present paper, we prove that the Hamiltonian (1) with complex number D is completely integrable in the framework of the quantum inverse scattering method and solve it by means of the algebraic Bethe ansatz.[13–15]

A model is said to be integrable if there exists a Lax pair L_n and M_n, such that the Lax equation

$$\frac{dL_n}{dt}=M_{n+1}L_n-L_nM_n \tag{2}$$

is equivalent to the equation of motion of the model. After a direct but tedious calculation, we find that the Lax pair associated with the Hamiltonian (1) has the form

$$L_n=\begin{bmatrix}\alpha+\beta S_n^z & S_n^- \\ S_n^+ & \epsilon+\theta S_n^z\end{bmatrix},\quad M_n=\begin{bmatrix}M_n^{11} & M_n^{12}\\ M_n^{21} & M_n^{22}\end{bmatrix}, \tag{3}$$

where

$$M_n^{11}=iA_1(S_n^xS_{n-1}^x+S_n^yS_{n-1}^y)+A_2(S_n^xS_{n-1}^y-S_n^yS_{n-1}^x)+iA_3(2S_n^zS_{n-1}^z-S_n^z-S_{n-1}^z)+iA_0,$$

$$M_n^{12}=iB_1S_n^-S_{n-1}^z+iB_2S_n^zS_{n-1}^-+iB_3S_n^-+iB_4S_{n-1}^-,$$

$$M_n^{21}=-iC_1S_n^+S_{n-1}^z-iC_2S_n^zS_{n-1}^++iB_4S_n^++iB_3S_{n-1}^+,$$

$$M_n^{22}=iD_1(S_n^xS_{n-1}^x+S_n^yS_{n-1}^y)+D_2(S_n^xS_{n-1}^y-S_n^yS_{n-1}^x)+iA_3(2S_n^zS_{n-1}^z+S_n^z+S_{n-1}^z)+D_0,$$

$$A_1=-(2\epsilon+\theta)B_3+\frac{\beta(J-iD)}{2\alpha+\beta}-\frac{\theta(J+iD)}{2\epsilon-\theta},$$

$$A_2=(2\epsilon+\theta)B_3+\frac{\beta(J-iD)}{2\alpha+\beta}+\frac{\theta(J+iD)}{2\epsilon-\theta},$$

$$A_3=\frac{J\Delta}{2-(2\alpha-\beta)\left(\epsilon+\frac{1}{2}\theta\right)},$$

$$A_0=\left(\alpha-\frac{1}{2}\beta\right)B_3-\left(\epsilon+\frac{1}{2}\theta\right)B_4-iD_0,$$

$$B_1=2B_3+\frac{2(J+iD)}{2\epsilon-\theta},$$

$$B_2=(2\alpha-\beta)A_3,$$

$$B_3=-\frac{(1+\beta\epsilon-\alpha\theta)(J+iD)}{\left[2-(2\alpha-\beta)\left(\epsilon+\frac{1}{2}\theta\right)\right](2\epsilon-\theta)},$$

$$B_4=-\frac{(1+\beta\epsilon-\alpha\theta)(J-iD)}{\left[2-(2\alpha-\beta)\left(\epsilon+\frac{1}{2}\theta\right)\right](2\alpha+\beta)},$$

$$C_1=(\beta-2\alpha)A_3,$$

$$C_2=(2\epsilon+\theta)A_3,$$

$$D_1=(\beta-2\alpha)B_4+\frac{\beta(J-iD)}{2\alpha+\beta}-\frac{\theta(J+iD)}{2\epsilon-\theta},$$

$$D_2=(\beta-2\alpha)B_4+\frac{\beta(J-iD)}{2\alpha+\beta}+\frac{\theta(J+iD)}{2\epsilon-\theta}. \tag{4}$$

Here, $S_n^{\pm}=S_n^x\pm iS_n^y$, D_0 is an arbitrary constant, and the constants α, β, ϵ, and θ are subject to the constraints

$$J\Delta(4\alpha^2-\beta^2)=(J-iD)(4\alpha\epsilon-\beta\theta-2),$$

$$J\Delta(4\epsilon^2-\theta^2)=(J+iD)(4\alpha\epsilon-\beta\theta-2). \tag{5}$$

Explicitly, Eq. (5) can be parametrized as follows: (i) when $D=-iJ$, then

$$\alpha=\frac{1}{2}u,\quad \beta=u,\quad \epsilon=\frac{1}{\Delta}u-\frac{1}{2u},\quad \theta=-\frac{1}{u}, \tag{6}$$

where u is an arbitrary constant that plays a role of spectral parameter.

(ii) When $D=iJ$, then

$$\alpha=\frac{1}{\Delta}u-\frac{1}{2u},\quad \beta=-\frac{1}{u},\quad \epsilon=\frac{1}{2}u,\quad \theta=u. \tag{7}$$

(iii) When $D=\pm J\sqrt{\Delta^2-1}$, then

$$\alpha=\frac{1}{2}i\sigma\theta,\quad \beta=-2i\sigma\epsilon,\quad \epsilon=\frac{1}{2}iu+\frac{i}{\sqrt{2\sigma}},\quad \theta=u, \tag{8}$$

where $\sigma=\sqrt{(J-iD)/(J+iD)}$.

(iv) When $D\neq\pm iJ$ and $\pm J\sqrt{\Delta^2-1}$, then

$$\alpha=\sigma\epsilon,\quad \beta=-\sigma\theta,\quad \epsilon=a\sin u,\quad \theta=b\cos u, \tag{9}$$

where

$$a=\frac{1}{2}\sqrt{\frac{2(J+iD)}{\sigma(J+iD)-J\Delta}}\quad \text{and}\quad b=\sqrt{\frac{2(J+iD)}{\sigma(J+iD)+J\Delta}}.$$

We have obtained the Lax pair of the Hamiltonian (1), which assures the complete integrability of the quantum system. In the following, we compute the R matrix corresponding to the L operator (3) with (i)–(iv), which satisfies the Yang-Baxter relation

$$R(u,v)L_n(u)\otimes L_n(v)=L_n(v)\otimes L_n(u)R(u,v). \tag{10}$$

Substituting the expressions of $L_n(u)$ into Eq. (10), we finally have

$$R(u,v)=\begin{bmatrix}1&0&0&0\\0&R_{22}(u,v)&R_{23}(u,v)&0\\0&R_{32}(u,v)&R_{33}(u,v)&0\\0&0&0&1\end{bmatrix}, \tag{11}$$

where (i′) when $D=-iJ$, then

$$R_{22}=R_{33}=\frac{v}{u},\quad R_{23}=\frac{u^2-v^2}{u^2},\quad R_{32}=0, \tag{12}$$

(ii′) when $D=iJ$, then

$$R_{22}=R_{33}=\frac{u}{v},\quad R_{23}=\frac{v^2-u^2}{v^2},\quad R_{32}=0, \tag{13}$$

(iii′) when $D=\pm J\sqrt{\Delta^2-1}$, then

$$R_{22}=R_{33}=\frac{(1+i)v+\sqrt{\frac{2}{\sigma}}}{(1+i)u+\sqrt{\frac{2}{\sigma}}+\sqrt{\frac{\sigma}{2}}(u-v)\left[(i-1)v-\sqrt{\frac{2}{\sigma}}\right]},\quad R_{23}=-R_{32}=\sqrt{\frac{\sigma}{2}}(u-v)R_{22}, \tag{14}$$

(iv′) when $D\neq\pm iJ$ and $\pm J\sqrt{\Delta^2-1}$, then

$$R_{22}=R_{33}=\frac{a\sin v-\frac{1}{2}b\cos v}{a\sin u-\frac{1}{2}b\cos u+\sigma ab\sin(v-u)\left(a\sin v+\frac{1}{2}b\cos v\right)},\quad R_{23}=R_{32}=\sigma ab\sin(v-u)R_{22}. \tag{15}$$

Define the monodromy matrix

$$T(u)=L_N(u)L_{N-1}(u)\cdots L_1(u)=\begin{pmatrix}A(u)&B(u)\\C(u)&D(u)\end{pmatrix}. \tag{16}$$

Combining Eqs. (10) and (16), we have

$$R(u,v)T(u)\otimes T(v)=T(v)\otimes T(u)R(u,v). \tag{17}$$

Here, we present some commutation relations used below:

$$[A(u),A(v)]=[B(u),B(v)]=[C(u),C(v)]=[D(u),D(v)]=[A(u),D(v)]=0,$$

$$A(u)B(v)=\left[R_{32}(u,v)-\frac{R_{22}(u,v)R_{33}(u,v)}{R_{23}(u,v)}\right]B(v)A(u)+\frac{R_{22}(u,v)}{R_{23}(u,v)}B(u)A(v),$$

$$D(u)B(v)=\frac{1}{R_{23}(u,v)}B(v)D(u)-\frac{R_{22}(u,v)}{R_{23}(u,v)}B(u)D(v). \tag{18}$$

Then the transfer matrix $\tau(u)=\mathrm{Tr}\,T(u)=A(u)+D(u)$ satisfies the commutation relation

$$[\tau(u),\tau(v)]=0. \tag{19}$$

Obviously, the transfer matrix plays a role of the generating functional of an infinite number of conserved quantities. The Hamiltonian (1) is one of these conserved quantities. From Eqs. (3), (6)–(9), and (16), we find

$$H=\begin{cases}\frac{1}{2}J\Delta^{3/2}\frac{\partial}{\partial u}\ln\tau(u)|_{u=\sqrt{\Delta}}-\frac{1}{4}NJ\Delta & \text{for } D=\pm iJ\\ \frac{iJ\Delta}{\sqrt{2\sigma}}\frac{\partial}{\partial u}\ln\tau(u)|_{u=(i-1)/\sqrt{2\sigma}}-\frac{1}{4}NJ\Delta & \text{for } D=\pm J\sqrt{\Delta^2-1}\\ \frac{2abJ\Delta}{4a^2-b^2}\frac{\partial}{\partial u}\ln\tau(u)|_{u=\arctan b/2a}-\frac{1}{4}NJ\Delta & \text{for } D\neq\pm iJ \text{ and } \pm J\sqrt{\Delta^2-1}.\end{cases} \tag{20}$$

Define the pseudovacuum $|0\rangle$ as $S_n^+|0\rangle=0$, then

$$C(u)|0\rangle=0,\quad A(u)|0\rangle=\left[\alpha(u)+\frac{1}{2}\beta(u)\right]^N|0\rangle,\quad D(u)|0\rangle=\left[\epsilon(u)+\frac{1}{2}\theta(u)\right]^N|0\rangle. \tag{21}$$

$B(u)$ is the creation operator of the particles excited. The M particle (i.e., M spins down) eigenstates of the transfer matrix $\tau(u)$ can be constructed as

$$|\Psi(u_1,u_2,\ldots,u_M)\rangle=\prod_{j=1}^{M}B(u_j)|0\rangle. \tag{22}$$

Using Eqs. (18) and (21), we get the eigenvalues of the tranfer matrix τ acting on the states $|\Psi(u_1,u_2,\ldots,u_M)\rangle$ Eq. (22) as

$$\tau(u)|\Psi(u_1,u_2,\ldots,u_M)\rangle=\Lambda(u)|\Psi(u_1,u_2,\ldots,u_M)\rangle,$$

$$\Lambda(u)=\left[\alpha(u)+\frac{1}{2}\beta(u)\right]^N\prod_{j=1}^{M}\left[R_{32}(u,u_j)-\frac{R_{22}(u,u_j)R_{33}(u,u_j)}{R_{23}(u,u_j)}\right]+\left[\epsilon(u)+\frac{1}{2}\theta(u)\right]^N\prod_{j=1}^{M}\frac{1}{R_{23}(u,u_j)}, \tag{23}$$

where the spectral parameters u_j satisfy the Bethe ansatz equation

$$\left[\frac{\alpha(u_l)+\frac{1}{2}\beta(u_l)}{\epsilon(u_l)+\frac{1}{2}\theta(u_l)}\right]^N=\prod_{j\neq l}\frac{1}{R_{23}(u_l,u_j)R_{32}(u_l,u_j)-R_{22}(u_l,u_j)R_{33}(u_l,u_j)}. \tag{24}$$

From Eqs. (19) and (20), we have

$$[H,\tau(u)]=0. \tag{25}$$

Therefore, H can be also diagonalized in the states $|\Psi(u_1,u_2,\ldots,u_M)\rangle$. Its eigenvalues E_M are determined by using Eqs. (20) and (23). Here, we present explicitly the energy eigenvalues and the Bethe ansatz equations

$$E_M=\frac{1}{2}J\Delta^{3/2}\frac{\partial}{\partial u}\ln\Lambda(u)|_{u=\sqrt{\Delta}}-\frac{1}{4}NJ\Delta=\sum_{j=1}^{M}\frac{J\Delta^2}{u_j^2-\Delta}+\frac{1}{4}NJ\Delta,$$

$$\left[\frac{1}{\Delta}-\frac{1}{u_l^2}\right]^N=(-1)^{M-1}\prod_{j=1}^{M}\frac{u_j^2}{u_l^2}\tag{26}$$

for $D=\pm iJ$;

$$E_M=\frac{iJ\Delta}{\sqrt{2\sigma}}\frac{\partial}{\partial u}\ln\Lambda(u)|_{u=(i-1)/\sqrt{2\sigma}}-\frac{1}{4}NJ\Delta=-\sum_{j=1}^{M}\frac{2J\Delta}{(\sqrt{2\sigma}u_j+1)^2+1}+\frac{1}{4}NJ\Delta,$$

$$\left[\frac{(1-i)u_l-i\sqrt{\frac{2}{\sigma}}}{(1+i)u_l+i\sqrt{\frac{2}{\sigma}}}\right]^N=(-1)^{M-1}(i\sigma)^{-N}\prod_{j=1}^{M}\frac{\left\{(1+i)u_l+\sqrt{\frac{2}{\sigma}}+\sqrt{\frac{\sigma}{2}}(u_l-u_j)\left[(i-1)u_j-\sqrt{\frac{2}{\sigma}}\right]\right\}^2}{\left[\frac{1}{2}\sigma(u_l-u_j)^2+1\right]\left[(1+i)u_j+\sqrt{\frac{2}{\sigma}}\right]^2}\tag{27}$$

for $D=\pm J\sqrt{\Delta^2-1}$ and

$$E_M=\frac{2abJ\Delta}{4a^2-b^2}\frac{\partial}{\partial u}\ln\Lambda(u)|_{u=\arctan b/2a}-\frac{1}{4}NJ\Delta=\sum_{j=1}^{M}\frac{J^2+D^2-J^2\Delta^2}{J\Delta-\sigma(J+iD)\cos(2u_j)}+\frac{1}{4}NJ\Delta,$$

$$\left[\frac{a\sin u_l-\frac{1}{2}b\cos u_l}{a\sin u_l+\frac{1}{2}b\cos u_l}\right]^N=-\sigma^{-N}\prod_{j=1}^{M}\frac{\left[a\sin u_l-\frac{1}{2}b\cos u_l+\sigma ab\sin(u_j-u_l)\left(a\sin u_j+\frac{1}{2}b\cos u_j\right)\right]^2}{[\sigma^2a^2b^2\sin^2(u_j-u_l)-1]\left[a\sin u_j-\frac{1}{2}b\cos u_j\right]^2}\tag{28}$$

for $D\neq\pm iJ$ and $\pm J\sqrt{\Delta^2-1}$.

In summary, we have proved the integrability of the Heisenberg model with DM interaction and the growth model described by the Hamiltonian (1) with the real number D and the complex number D, respectively, and have solved these models by means of the algebraic Bethe ansatz. We note that when $\Delta=1$ and $D=-iJ$, the results of the present paper are nothing but the ones given in Ref. 12. When $D=\pm iJ$, the R matrices (12) and (13) are independent of the exchange anisotropic parameter Δ, and the energy eigenvalues and the Bethe ansatz equations are the same at the two points.

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