

## Subdecoherent information encoding in a quantum-dot array

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A potential implementation of quantum-information schemes in semiconductor nanostructures is studied. To this end, the formal theory of quantum encoding for avoiding errors is recalled and the existence of noiseless states for model systems is discussed. Based on this theoretical framework, we analyze the possibility of designing noiseless quantum codes in realistic semiconductor structures. In the specific implementation considered, information is encoded in the lowest energy sector of charge excitations of a linear array of quantum dots. The decoherence channel considered is electron-phonon coupling. We show that besides the well-known phonon bottleneck, reducing single-qubit decoherence, suitable many-qubit initial preparation, as well as register design may enhance the decoherence time by several orders of magnitude. This behavior stems from the effective one-dimensional character of the phononic environment in the relevant region of physical parameters. [S0163-1829(99)07607-9]

### I. INTRODUCTION

Devices using unique quantum-mechanical features can perform information processing in a much more efficient—or even unattainable—way than those relying just on classical physics. This fundamental discovery has stimulated in the last few years a large amount of work and scientific debates in the newborn field of quantum computation.<sup>1</sup> From a conceptual point of view these results represent a serious challenge to the time-honored notion of *universal computational schemes* independent of an underlying physical theory: information as well as computation are *intrinsically* physical. On the other hand, physical realizations of a quantum computer (QC) would result in tremendous practical advantages.

The key ingredients which endow QC devices with computational capabilities that supersede their classical counterparts are basically: (i) the linear structure of their state space; (ii) the unitary character of their dynamical evolution; and (iii) the tensorized form of multiparticle state spaces. The first two properties allow for a parallel processing of an arbitrary number of data sets, encoded in suitable quantum states. By resorting to quantum interference, between different computational branches, one can selectively amplify desired parts of the state vector in order to optimize the probability that a final (i.e., read-out) measurement will give us the information we were looking for. Point (iii) represents another striking departure from classicality: due to *entanglement*, combining different quantum systems results in an exponential growth of the available coding space; moreover, the tensor-product structure is at the very basis of many efficient quantum manipulations.

Unfortunately, all this holds just for *closed* quantum systems. Real systems are unavoidably coupled with environmental (i.e., noncomputational) degrees of freedom. Such open character spoils points (i) and (ii), eventually turning

quantum computing to classical. Different computational branches get entangled with different (quasiorthogonal) quantum states of the environment and their interference is then no longer observable. From a mathematical point of view, the relevant state space, given by density matrices, has now a *convex* structure and the allowed quantum dynamics is described by CP maps.<sup>2</sup> Initial pure preparations are typically corrupted on extremely short timescales due to quantum-coherence loss that makes them mixed: the initial information irreversibly leaks out from the system into the huge number of uncontrollable degrees of freedom of the environment. This phenomenon—the so-called decoherence problem in QC (Ref. 3)—represents the major obstacle for the experimental realization of any quantum-computing system. Other challenging requirements are of course given by the necessity of being able to perform on a system, with a *well-defined* state space, long coherent quantum manipulations (*gating*), precise quantum-state synthesis, and detection as well.

A major theoretical achievement has been made by showing that one can, in principle, *actively* stabilize quantum states by means of quantum error correction.<sup>4</sup> The latter, built in analogy with its classical counterpart, assumes that the quantum bits (*qubits*) are coupled to independent environments. The information is then encoded in a subtle redundant way that allows, monitoring the systems and conditionally carrying on suitable quantum operations, to tolerate a certain (small) amount of decoherence and imperfect gating as well.<sup>5</sup>

It is basically the need of dealing with systems sufficiently decoupled from the external environment that, up to now, has limited the existing realizations to atomic and molecular implementations. Furthermore, the extremely advanced technological state-of-the-art in these fields allows for the manipulations required in simple QC's.<sup>6</sup> However,

any interesting QC would require a large number of quantum gates and qubits as well, and all the present approaches suffer from the problem of *scalability* to large, i.e., highly integrated, quantum processors.

One is then naturally led to consider the viability of solid-state implementations. In particular, by resorting to present semiconductor technology, one might benefit synergetically from the recent progress in ultrafast optoelectronics<sup>7-9</sup> and in nanostructure fabrication and characterization.<sup>10</sup>

The first drawback of such a kind of proposal is that the *typical* decoherence time  $\tau_D$  in semiconductors is of the order of picoseconds. On the other hand, the relevant parameter is the ratio between the typical time scale of gating  $\tau_G$  and  $\tau_D$ . Roughly speaking,  $\tau_D/\tau_G$  represents the number of elementary (coherent) operations that one could perform on the system before its coherence being lost.

Loss and DiVincenzo<sup>11</sup> have proposed to use nonequilibrium spin dynamics in quantum dots for quantum computation. This exploits the low decoherence of spin degrees of freedom in comparison to the one of charge excitations, being the former is much less coupled with the environment. Nevertheless, the required magnetic gating is extremely challenging from a technological point of view, and the ratio  $\tau_D/\tau_G$  does not allow for the number of gate operations within the decoherence time required by concrete QC's.

Ultrafast laser technology is now able to generate electronic excitations on a subpicosecond timescale and to perform on such states a variety of coherent-carrier-control operations.<sup>7</sup> If one can speculate to resort to such a technology for realizing gating of *charge* degrees of freedom then coherence times on nano/microsecond scales can be regarded as "long" ones.

In this paper we analyze in a detailed way the recent idea of implementing *quantum error avoiding* strategies.<sup>12</sup> The goal here is to suppress decoherence in a quantum register realized by the lowest energy *charge* excitations of a semiconductor quantum-dot array.<sup>13</sup> In this case, the noise source is given by electron-phonon scattering, which is recognized to be the most efficient decoherence channel in such a system.<sup>8,9</sup>

Despite the *a priori* complexity of the three-dimensional (3D) phononic environment, we will show that the underlying dynamical symmetry allows, by means of a proper quantum encoding, to increase the decoherence time by several orders of magnitude with respect to the bulk value. The focus of the present paper is mostly conceptual and the problem of actual preparation/manipulation of the resulting codewords will not be addressed.

The paper is organized as follows. In Sec. II the formal theory of subdecoherent quantum encoding is presented and discussed. Section III deals with the application of the proposed subdecoherence theory to realistic semiconductor-based nanostructures. More specifically, we will choose as quantum register an array of semiconductor quantum dots and for this particular system we will study the potential sources of decoherence. In Sec. IV we shall present a detailed investigation of decoherence in our quantum-dot array. In addition to a short-time analysis, we will present time-dependent simulations corresponding to a numerical solution of the master equation. They will show that by means of a proper initial many-electron state preparation it is possible to

extend the carrier-phonon decoherence time up to the  $\mu$ s scale. Finally, in Sec. V we will summarize and draw some conclusions. The Appendix is devoted to a formal analysis of the so-called circular model, which will turn out to play a major role in the semiconductor-based implementation considered.

## II. THEORY OF SUBDECOHERENT QUANTUM ENCODING

In this section we recall the basics of the theory of *noiseless coding*<sup>12</sup> in the framework of a master equation (ME) formalism, for the register subdynamics.<sup>14</sup> Generally speaking, these strategies for preserving quantum coherence rely on the possibility to design an open quantum system  $\mathcal{R}$  in such a way that (i) the environment  $\mathcal{E}$  is effectively coupled only with a subset of the degrees of freedom of  $\mathcal{R}$ . Information is then encoded in the portion  $\mathcal{C}$  of Hilbert space spanned by the remaining (decoupled) degrees of freedom, and (ii) The environment is coupled to subset of states  $\mathcal{C}$  in a state independent fashion. In both cases  $\mathcal{E}$  is not able to extract information from  $\mathcal{C}$ : the quantum coherence is then *passively* stabilized. From the above points it should be clear the first and major departure from the error correction paradigm: here one assumes the environmental noise to be correlated.  $\mathcal{E}$  is *coupled, in a strongly state-dependent way, with collective states of  $\mathcal{R}$ .*

Before embarking in a detailed analysis of subdecoherence let us shortly discuss two very simple examples, that show how this notion can come about.

(i) Let us consider  $N$  isospectral linear oscillators  $H_{\mathcal{R}} = \omega \sum_{j=1}^N b_j^\dagger b_j$  coupled with the vacuum fluctuations, i.e., zero temperature, of a bosonic field  $a_k$  by a Hamiltonian of the form  $H_{\mathcal{I}} = \sum_{jk} (g_{kj} a_k^\dagger b_j + \text{H.c.})$ . Suppose now that  $g_{kj} = g_k \mathbf{V}j$ . By introducing the Fourier transformed operators  $b_q \equiv 1/\sqrt{N} \sum_j e^{iqj} b_j$  (bosons as well) one immediately sees that only the zero modes are actually coupled:  $H_{\mathcal{I}} = b_0^\dagger (\sum_k g_k a_k) + \text{H.c.}$  and  $H_{\mathcal{R}} = \omega \sum_q b_q^\dagger b_q$ . Therefore, any state of the (infinite-dimensional) subspace

$$\mathcal{C} = |0\rangle_0 \otimes_{q>0} \mathcal{H}_q \quad (1)$$

will evolve unaffected by the environment in that  $H_{\mathcal{I}} \mathcal{C} \otimes |0\rangle_{\mathcal{E}} = 0$ .

(ii) Let the system-environment interaction Hamiltonian be of the form  $H_{\mathcal{I}} = \sum_{\mu} R_{\mu} \otimes E_{\mu}$ , where  $X_{\mu} \in \text{End} \mathcal{H}_{\mathcal{X}}$  ( $\mathcal{X} = \mathcal{R}, \mathcal{E}$ ). Moreover, let us suppose that the Hermitian  $R_{\mu}$ 's are commuting operators, i.e., they span an *abelian* algebra  $\mathcal{A}$ . Let  $\mathcal{C} \subset \mathcal{H}_{\mathcal{R}}$  a simultaneous eigenspace of  $\mathcal{A}$ . This means that

$$H_{\mathcal{I}}|_{\mathcal{C}} = \sum_{\mu} \rho_{\mu} E_{\mu} \quad (\rho_{\mu} \in \mathbf{R}). \quad (2)$$

In other words, if one restricts himself to  $\mathcal{C}$  the interaction with the environment amounts simply to a *state-independent renormalization of  $H_{\mathcal{E}}$* . It is then clear that—provided  $\mathcal{C}$  is invariant under the system self-Hamiltonian  $H_{\mathcal{R}}$ —any initial preparation in  $\mathcal{C}$  evolves in a unitary fashion regardless of the strength of the system-environment coupling and the environment initial state as well. Of course, for all this to be useful in quantum encoding one must have  $\dim \mathcal{C} > 1$ .

### A. Master-equation approach

The system under investigation  $\mathcal{R}$  is given by  $N$  identical two-level systems ( $N$ -qubits quantum register), representing our computational degrees of freedom, coupled with an external (uncontrollable) environment. The register  $\mathcal{R}$  will be described in the spin-1/2 language by means of the usual Pauli spin matrices  $\{\sigma_i^z, \sigma_i^\pm\}_{i=1}^N$  generating  $N$  local  $\mathfrak{sl}(2)$  algebras

$$[\sigma_i^+, \sigma_j^-] = 2\delta_{ij}\sigma_i^z, \quad [\sigma_i^z, \sigma_j^\pm] = \pm\delta_{ij}\sigma_i^\pm. \quad (3)$$

The collective spin operators  $S^\alpha = \sum_{i=1}^N \sigma_i^\alpha$ , ( $\alpha = \pm, z$ ) span a  $\mathfrak{sl}(2)$  algebra as well, and it will be referred to as the *global*  $\mathfrak{sl}(2)$ . The environment  $\mathcal{E}$  will be described by a set of non-interacting harmonic oscillators with bosonic field operators  $[b_k^\dagger, b_{k'}] = \delta_{kk'}$ .

The total Hamiltonian is assumed to be  $H = H_{\mathcal{R}} + H_{\mathcal{E}} + H_{\mathcal{I}}$ , where  $H_{\mathcal{R}} = ES^z$  and  $H_{\mathcal{E}} = \sum_k \omega_k b_k^\dagger b_k$  are, respectively, the register and the environment self-Hamiltonians. Here,  $E$  represents the energy spacing between states  $|0\rangle_i$  and  $|1\rangle_i$  in each qubit. The  $\mathcal{R}$ - $\mathcal{E}$  interaction is given by

$$H_{\mathcal{I}} = \sum_{ki} (g_{ki} b_k^\dagger \sigma_i^- + \text{H.c.}) \quad (4)$$

Let us now briefly recall the standard Born-Markov scheme for tracing out the  $\mathcal{E}$  degrees of freedom and obtaining a master equation for the register subdynamics. The Liouville-von Neumann equation for the *total* density matrix of  $\mathcal{R} \otimes \mathcal{E}$  in the interaction picture reads  $i\partial_t \tilde{\rho} = [H_{\mathcal{I}}, \tilde{\rho}]$ . One assumes a factorized initial condition  $\tilde{\rho}(0) = \rho \otimes \Omega$ . After a formal time integration one obtains

$$\begin{aligned} \tilde{\rho}(t) &= \tilde{\rho}(0) + \int_0^t d\tau [H_{\mathcal{I}}(\tau), \tilde{\rho}(\tau)] \\ &= \tilde{\rho}(0) - i \int_0^t d\tau [H_{\mathcal{I}}(\tau), \tilde{\rho}(0)] + (-i)^2 \\ &\quad \times \int_0^t d\tau \int_0^\tau d\tau' [H_{\mathcal{I}}(\tau), [H_{\mathcal{I}}(\tau'), \tilde{\rho}(\tau')]]. \end{aligned} \quad (5)$$

Now we set  $\tilde{\rho}(\tau') = \rho(\tau') \otimes \Omega$  ( $\Omega \sim e^{-\beta H_{\mathcal{E}}}$ ) and we perform a partial trace over  $\mathcal{E}$  in order to get an equation for the reduced density matrix of  $\mathcal{R}$ :  $\dot{\rho}(t) = \text{tr}_{\mathcal{E}} \tilde{\rho}(t)$ . The resulting ME is of the form  $\dot{\rho} = \mathcal{L}(\rho)$ . The Liouvillian *superoperator*  $\mathcal{L}$  is given by the sum of two contributions:  $\mathcal{L}_u$  representing the unitary component of the dynamics ruled by the (renormalized) register self-Hamiltonian;  $\mathcal{L}_d$  describing the irreversible decoherence/dissipation processes induced by the coupling with the external bath. By denoting with  $H_{\mathcal{R}}$  the unperturbed register self-Hamiltonian, one has that  $\mathcal{L}_u(\rho) = i/\hbar [\rho, H_{\mathcal{R}} + \delta H_{\mathcal{R}}]$  where the environment-induced  $\delta H_{\mathcal{R}}$  is given by

$$\delta H_{\mathcal{R}} = \sum_{\eta=\pm} \sum_{ij=1}^N \Delta_{ij}^{(\eta)} \sigma_i^- \sigma_j^\eta. \quad (6)$$

These contributions—usually referred to as the Lamb-shift terms—describe a sort of qubit-qubit effective interaction

mediated by the external environment. The dissipative Liouvillian is given by  $\mathcal{L}_d = \sum_{\eta=\pm} \mathcal{L}_d^\eta$ , where

$$\mathcal{L}_d^\eta(\rho) = \frac{1}{2\hbar} \sum_{ij=1}^N \Gamma_{ij}^{(\eta)} ([\sigma_i^\eta \rho, \sigma_j^{-\eta}] + [\sigma_i^\eta, \rho \sigma_j^{-\eta}]). \quad (7)$$

Here, the term  $\eta = -$  ( $\eta = +$ ) is associated to deexcitation (excitation) processes of the qubits by emission (absorption) of bosonic quanta. The Hermitian matrices  $\Gamma$  and  $\Delta$  are the input data defining our ME, and their actual form depends on the details of the physical constants ( $E, \{\omega_k\}, \{g_{ki}\}$ , etc.) and will be given later.

As far as the analysis of this section is concerned it is sufficient to know that  $\Gamma \geq 0$ . One can go on with general considerations by diagonalizing  $\Gamma^{(\eta)}$  in order to obtain the canonical form for the dissipative part of the Liouvillian<sup>15</sup>

$$\mathcal{L}_d(\rho) = \frac{1}{2\hbar} \sum_{\eta=\pm, \mu=1}^N \lambda_\mu^\eta ([L_\mu^\eta \rho, L_\mu^{-\eta}] + [L_\mu^\eta, \rho L_\mu^{-\eta}]), \quad (8)$$

where  $\{\lambda_\mu^\eta\}$  are the (non-negative) eigenvalues of  $\Gamma^{(\eta)}$ . Moreover,  $L_\mu^\eta = \sum_i u_i^\mu \sigma_i^\eta$ ,  $u_i^\mu$  denoting the components of the eigenvectors of  $\Gamma^{(\sigma)}$ . The  $L_\mu^\eta$ 's will be referred to as the Lindblad operators. The operator (Lie) algebra  $\mathcal{A}$  spanned by the Lindblad operators contains the information about the existence of coding spaces stable at least on a short time-scale. The finite-time stability depends on the interplay between the dissipative and the unitary components of the Liouvillian in  $\mathcal{H}_{\mathcal{R}}$ .

In order to quantify the efficiency of the environment in destroying quantum coherence it is useful to define a (*first-order*) *decoherence time (rate)*  $\tau_1$  ( $\tau_1^{-1}$ ) by means of the short-time expansion of the *fidelity*<sup>14</sup> for *pure* initial state preparations  $|\psi\rangle$

$$F(t) \equiv \langle \psi | \rho(t) | \psi \rangle = 1 - \frac{t}{\tau_1} + o(t^2). \quad (9)$$

From Eq. (8) one obtains

$$\tau_1^{-1} [|\psi\rangle] = \sum_{\eta=\pm, \mu=1}^N \lambda_\mu^\eta (\|L_\mu^\eta |\psi\rangle\|^2 - |\langle \psi | L_\mu^\eta |\psi \rangle|^2). \quad (10)$$

This expression is nothing but a sort of fluctuation-dissipation relation connecting the dispersion of the Lindblad operators  $L_\mu^\eta$  in the initial register state with the rate at which quantum coherence is destroyed. It is important to point out that the unitary component of the Liouvillian does not contribute to the first-order decoherence time. If  $\tau_1^{-1} [|\psi\rangle] = 0$  then the state  $|\psi\rangle$  will be called *subdecoherent* and a linear subspace  $\mathcal{C} \subset \mathcal{H}_{\mathcal{R}}$  will be referred to as a *subdecoherent code*.

In general, the register Hilbert space splits in  $\mathcal{A}$ -invariant subspaces,

$$\mathcal{H}_{\mathcal{R}} = \bigoplus_r \bigoplus_{j=1}^{n_j} \mathcal{H}_r^{(j)}, \quad (11)$$

where  $J$  labels the irreducible representations (irrep) of  $\mathcal{A}$ , and the integers  $n_j$  are the associated multiplicities  $[\mathcal{H}_r^{(j)} \cong \mathcal{H}_r^{(j)}]$ . The *singlet sector*  $\mathcal{C}$  of  $\mathcal{A}$  is the direct sum (possibly empty) of the one-dimensional irreps. In Ref. 14 it has

been shown that, for non-Abelian  $\mathcal{A}$ , the subdecoherent codes coincide with  $\mathcal{C}$ . In an equivalent group-theoretic language one can say that the code  $\mathcal{C}$  is the subspace of vectors *invariant* under the action of group  $\mathcal{G}=\exp(\mathcal{A})$  generated (infinitesimally) by the Lindblad operators  $L_\mu$ . ( $\mathcal{C}$  is the trivial  $\mathcal{G}$ -representation space.) This group acts, of course, on the general mixed states:  $\rho \mapsto X\rho X^\dagger$ , ( $X \in \mathcal{G}$ ). The same argument holds for the subdecoherent (pure) states. When  $\mathcal{C}$  is invariant under the action of  $H'_R$ ; then the contribution to the dynamics of  $\mathcal{L}_d$  vanishes

$$\rho = |\psi\rangle\langle\psi| \mapsto e^{-itH'_R} \rho e^{itH'_R} \quad (\forall |\psi\rangle \in \mathcal{C}, t \geq 0). \quad (12)$$

The finite-time evolution is unitary, and in this case  $\mathcal{C}$  will be referred to as noiseless code: quantum coherence is preserved—in principle—for an arbitrarily long time. When  $\mathcal{C}$  is not  $H'_R$  invariant the initial preparation  $|\psi\rangle$  on a greater time scale leaks out from the code and its quantum coherence will be eventually washed out. For instance, the condition  $[H'_R, \mathcal{A}] = 0$  suffices to have such a noiseless coding or even that  $H'_R$  belonging to the *associative* operator algebra  $\mathcal{A}_a$  generated by the  $L_\mu$ 's and the identity operator. Notice that if  $\mathcal{C}$  is subdecoherent for the  $L_\mu$ 's it is subdecoherent for any set of Lindblad operators included in  $\mathcal{A}_a$ .

From a physical point of view, the algebra  $\mathcal{A}$  of Lindblad operators represents the set of the register modes that are incoherently excited by the environment; looking for states that are annihilated by as many Lindblad operators as possible is then like looking for states that are “vacua” for the largest number of such excitations and, therefore, maximally decoupled with environment.<sup>16</sup> It is important to emphasize that such a decoupling can be achieved thanks to the algebraic-dynamical structure of the model without any assumptions about the (weakness) of the register-environment interaction. Loosely speaking, one can say that for generic  $\Gamma$ 's, the Liouvillian is such that, given *any* register preparation, the environment forces the coding system to explore the totality of its Hilbert space so that there is no safe place for storing quantum information. Except for some “magic”  $\Gamma$  the Lindblad algebra gets smaller allowing just for a limited probing of the register space of states by the environment strongly dependent on the initial register data: free room is left for “hiding” quantum information.

Rather interestingly, the problem of analyzing state stability against decoherence can be cast in a Hamiltonian form by observing that, for an initial condition  $|\psi\rangle$  that is a  $S^z$  eigenstate one has  $\tau_1^{-1} = \langle\psi|\tilde{H}|\psi\rangle$  where

$$\begin{aligned} \tilde{H} &= \sum_{\eta=\pm, \mu=1}^N \lambda_\mu^\eta L_\mu^- \eta L_\mu^\eta \\ &= \sum_{ij=1}^N (\Gamma_{ij}^{(-)} \sigma_i^+ \sigma_j^- + \Gamma_{ij}^{(+)} \sigma_i^- \sigma_j^+). \end{aligned} \quad (13)$$

In other words: the problem of finding decoherence rates is mapped onto the spectral problem for the (positive) operator (13). In particular, “robust” states (i.e., the ones with minimal decoherence rates) are *ground states* of  $\tilde{H}$ . Let  $E_N$  denote the lowest eigenvalue of  $\tilde{H}$ .  $E_N = 0$  means that there

exist subdecoherent states, and in this case  $\mathcal{C} \equiv \text{Ker}\tilde{H}$  and  $d_N \equiv \dim \text{Ker}\tilde{H}$  gives the dimension of the code. The subdecoherence property is stable against small perturbations of the state. Indeed, if  $|\psi\rangle \in \text{Ker}\tilde{H} \mapsto |\psi\rangle + |\delta\psi\rangle$  then  $\delta\tau_1^{-1} = \langle\delta\psi|\tilde{H}|\delta\psi\rangle \geq 0$ .

### B. A simple example

To better illustrate the situation let us consider the  $N=2$  case. The model (13) is soluble in an elementary way.<sup>17</sup> We assume  $\Gamma_{11}^{(\pm)} = \Gamma_{22}^{(\pm)} \equiv \Gamma^{(\pm)}$  and  $\Gamma_{12}^{(\pm)} = \Gamma_{21}^{(\pm)} = \Gamma^{(\pm)}\beta$ ; moreover,  $\Gamma^{(-)} \geq \Gamma^{(+)}$ . From positivity it follows that  $|\beta| \leq 1$ . The spectrum is given by

$$\begin{aligned} E_{11} &= 2\Gamma^{(-)}, \quad E_{00} = 2\Gamma^{(+)}, \\ E_{t,s} &= (\Gamma^{(-)} + \Gamma^{(+)}) (1 \pm \beta), \end{aligned} \quad (14)$$

with eigenstates given, respectively, by

$$|11\rangle, |00\rangle, 2^{-1/2}(|01\rangle \pm |10\rangle).$$

If  $\Gamma^{(+)} > 0$ , for  $|\beta| \leq (\Gamma^{(-)} - \Gamma^{(+)}) / (\Gamma^{(-)} + \Gamma^{(+)})^{-1} \equiv \beta_c$  one has  $E_2 = E_{00}$ , for  $\beta > \beta_c$  ( $\beta < \beta_c$ ) one finds  $E_0 = E_s$  ( $E_0 = E_t$ ).  $\Gamma^{(+)} = 0 \Rightarrow E_2 = E_{00} = 0$ . Finally, for  $\beta = \pm 1$  one has again  $E_2 = 0$ , with eigenstates given by  $|\psi_{s,t}\rangle$ . In summary, subdecoherent states exist in a subset of the boundary of the  $\Gamma$  manifold. This result is quite general: for *generic*  $\Gamma$ 's one has  $E_N[\Gamma] > 0$ , the subdecoherence condition  $E_N[\Gamma] = 0$  is fulfilled just in a “zero-measure” set of the Hamiltonian models (13). Of course, this is simply due to the fact that a generic  $\Gamma$  gives rise to a Lindblad algebra  $\mathcal{A}$  is too large for admitting a (nontrivial) singlet sector.

Turning back to the general  $N$  case, to exemplify the collective nature of the decoherence-dissipation dynamics let us consider the states ( $N$  even)

$$|\psi_{sym}\rangle \equiv (S^\dagger)^{N/2} |0\rangle,$$

$$|\mathcal{D}\gamma\rangle \equiv \otimes_{(i,j) \in \mathcal{D}} [(|01\rangle - (-1)^{\gamma(i,j)} |10\rangle)]_{ij}, \quad (15)$$

where  $\mathcal{D}$  is a dimer partition of the qubit array, and  $\gamma: \mathcal{D} \rightarrow \{0,1\}$ . The first state in Eq. (15) is simply the totally symmetric  $S^z = 0$  state [belonging to the  $\text{sl}(2)$  multiplet of the vacuum], whereas the  $|\mathcal{D}\gamma\rangle$ 's are products of singlet or triplet pair states depending on the signature  $\gamma$  of the register dimer partition  $\mathcal{D}$ . This latter family of states (15) will play an important role in the following. Notice that, for  $\gamma = \mathbf{0}$ , one gets global  $\text{sl}(2)$  singlets corresponding to zero total angular momentum  $S^z$ . In terms of Hadamard transformations and controlled-not operators<sup>23</sup> the  $|\mathcal{D}\sigma\rangle$ 's can be *synthesized* as follows from a pure product state:

$$|\mathcal{D}\gamma\rangle = \otimes_{(l,m) \in \mathcal{D}} \text{cnot}_{lm} H_l |\gamma(l,m) + 1, 1\rangle_{lm}. \quad (16)$$

With straightforward calculations one finds that the first-order decoherence rates of states (15) are given, respectively, by  $(\tau_\alpha/\tau_0)^{-1} = f_\alpha(\Gamma)$ , ( $\alpha = \text{sym}, \mathcal{D}\gamma$ ) in which  $\tau_0^{-1} = \Gamma_0 N/2$  is the decoherence rate for uncorrelated qubits, and  $(\tilde{\Gamma})_{ij} \equiv \Gamma_{ij}/\Gamma_0$

$$f_{sym} = 1 + \frac{1}{N-1} \Re \sum_{i < j} \tilde{\Gamma}_{ij},$$

$$f_{D\gamma} = 1 - \frac{2}{N} \Re \sum_{(i,j) \in \mathcal{D}} (-1)^{\sigma(i,j)} \tilde{\Gamma}_{ij}, \quad (17)$$

where  $\Gamma_0 = \Gamma_{ii}$  ( $i = 1, \dots, N$ ) The  $f_\alpha$ 's contain the information about the degree of many-qubit correlation in the decay process: if  $\Gamma \propto \mathbf{I}$  one has  $f_\alpha = 1$  and the qubits decohere independently.

### III. APPLICATION TO SEMICONDUCTOR NANOSTRUCTURES

In this section we shall discuss a potential application of the above subdecoherent quantum-encoding strategies to realistic, i.e., state-of-the-art, semiconductor-based nanostructures. Since in semiconductors the primary source of decoherence is known to be carrier-phonon scattering, we will consider as prototypical systems quasi-zero-dimensional (0D) structures, for which the reduced phase space available allows for a significant suppression of phonon-induced energy relaxation and dephasing.

We will choose as a prototype of quantum register an array of semiconductor quantum dots. In particular, we will consider as a quantum dot (QD) a GaAs/Al<sub>x</sub>Ga<sub>1-x</sub>As structure similar to that studied in Ref. 18. Here, various effects due to carrier-carrier interaction will *not* be considered. This is, of course, a potential limitation of our analysis, especially in relation to state preparation/manipulation (not addressed in this paper). Indeed, the latter requires a controllable source of entanglement, i.e., a qubit-qubit interaction that might be provided by ‘‘switchable’’ Coulomb couplings.<sup>19</sup> On the other hand, our coding states will involve *single-electron* occupations only; for such states the intradot Coulomb repulsion is clearly absent, while the interdot one at the distances relevant for our quantum encoding is found to be negligible.<sup>20</sup> Moreover, since the system under consideration is based on intrinsic III-V materials, carrier-impurity scattering is negligible.

Generally speaking, Hamiltonian modifications will result in leakage from the coding subspace only on a longer time scale with respect to the phonon scattering one, i.e., it does not affect the stability classification based on  $\tau_1$  (see Sec. II). Finally, we would like to stress that there exists a whole class of interactions leaving the code invariant.<sup>12</sup>

#### A. Free-carrier states in the quantum-dot array

The confinement potential  $V^{0D}$  giving rise to the quasi-0D carrier states in such a QD structure is properly described in terms of a quantum-well (QW) profile  $V^{\parallel}$  along the growth direction of the structure plus a 2D parabolic potential  $V^{\perp}$  in the normal plane. More specifically, a carrier within the  $i$ th QD structure is described by the following single-particle Hamiltonian:

$$h_i = -\frac{\hbar^2 \nabla_{\mathbf{r}}^2}{2m^*} + V^{0D}(\mathbf{r})$$

$$= \left( -\frac{\hbar^2 \nabla_{\mathbf{r}_{\perp}}^2}{2m^*} + V^{\perp}(\mathbf{r}_{\perp}) \right) + \left( -\frac{\hbar^2 \nabla_{\mathbf{r}_{\parallel}}^2}{2m^*} + V^{\parallel}_i(\mathbf{r}_{\parallel}) \right)$$

$$= H^{\perp} + H^{\parallel}, \quad (18)$$

where

$$V^{\perp}(\mathbf{r}_{\perp}) = \frac{1}{2} m^* \omega^2 |\mathbf{r}_{\perp}|^2 \quad (19)$$

is the 2D harmonic-oscillator potential in the  $(x,y)$  plane perpendicular to the  $(z)$  array axis (which coincides with the growth axis of the QD structure), while  $V^{\parallel}_i(\mathbf{r}_{\parallel})$  is a 1D square-well potential centered at  $\mathbf{r}_{\parallel}^i = ia\hat{\mathbf{z}}$  with width  $d$  and infinite walls,<sup>21</sup>  $a$  being the array periodicity, i.e., the interdot distance. This choice for the single-particle Hamiltonian, even though not generally valid, well describes the 0D carrier confinement of the low-energy states in the QD structure, which are the only relevant states for the quantum encoding considered. We would like to point out that the very same QD model turned out to be able to explain, in a quantitative way, the addition spectra reported in Ref. 18.<sup>22</sup>

The Hamiltonian (18) is elementary soluble, its spectrum being the sum of the parallel and perpendicular contributions:

$$\epsilon_{nv} = E_n^{\perp} + E_v^{\parallel} = (n_x + n_y + 1) \hbar \omega + \frac{\pi^2 \hbar^2 v^2}{2m^* d^2}. \quad (20)$$

The corresponding 3D eigenstates will be factorized according to

$$\phi_{i,nv}(\mathbf{r}) = \phi_{n_x, n_y}^{\perp}(\mathbf{r}_{\perp}) \phi_v^{\parallel}(\mathbf{r}_{\parallel} - ia). \quad (21)$$

The total free-carrier Hamiltonian describing our QD array can then be expressed in the (second-quantized) form

$$H_{\mathcal{R}} = \sum_{i,\alpha} \epsilon_{\alpha} c_{i\alpha}^{\dagger} c_{i\alpha}, \quad (22)$$

where the fermionic operators  $c_{i\alpha}^{\dagger}(c_{i\alpha})$  create (destroy) an electron in the  $i$ th QD in state  $\alpha \equiv n_x n_y v$ .

#### B. Carrier-phonon coupling

The Hamiltonian describing the free phonons of a semiconductor crystal is given by<sup>24</sup>

$$H_{\mathcal{E}} = \sum_{\lambda \mathbf{q}} \hbar \omega_{\lambda \mathbf{q}} b_{\lambda \mathbf{q}}^{\dagger} b_{\lambda \mathbf{q}}, \quad (23)$$

where  $\lambda$  and  $\mathbf{q}$  denote, respectively, the phonon mode (e.g., acoustic, optical, etc) and the phonon wave vector.

The coupling of phonons with the electrons in the QD array is described by the following carrier-phonon interaction Hamiltonian:

$$H_{\mathcal{I}} = \sum_{i\alpha, i'\alpha'; \lambda \mathbf{q}} [g_{i\alpha, i'\alpha'; \lambda \mathbf{q}} c_{i\alpha}^{\dagger} b_{\lambda \mathbf{q}} c_{i'\alpha'} + \text{H.c.}], \quad (24)$$

where

$$g_{i\alpha, i'\alpha'; \lambda \mathbf{q}} = \tilde{g}_{\lambda \mathbf{q}} \int \phi_{i\alpha}^*(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} \phi_{i'\alpha'}(\mathbf{r}) d\mathbf{r} \quad (25)$$

are the matrix elements of the phonon potential between the quasi-0D states  $i\alpha$  and  $i'\alpha'$ . The explicit form of the coupling constant  $\tilde{g}_{\lambda \mathbf{q}}$  depends on the particular phonon mode.

### C. The qubit register

In the proposed information-encoding scheme the single *qubit* is given by the two lowest energy levels of the QD structure. Since the width  $d$  of the GaAs QW region is typically of the order of a few nanometers, the energy splitting due to the quantization along the growth direction is much larger than the confinement energy  $\hbar\omega$  induced by the 2D parabolic potential  $V^\perp$  (typically of a few meV). Thus, the two lowest-energy states—state  $|0\rangle$  and  $|1\rangle$ —realizing our qubit are given by products of the QW ground state times the ground or first excited state of the 2D parabolic potential.<sup>25</sup>

More specifically, they are given by

$$\begin{aligned} \langle \mathbf{r} | 0 \rangle_i &= \phi_0^\perp(x) \phi_0^\perp(y) \phi_{i,0}^\parallel(z), \\ \langle \mathbf{r} | 1 \rangle_i &= \phi_0^\perp(x) \phi_1^\perp(y) \phi_{i,0}^\parallel(z), \end{aligned} \quad (26)$$

where

$$\begin{aligned} \phi_0^\perp(x) &= C_0 e^{-a_0 x^2}, \quad C_0 = (2a_0/\pi)^{1/4}, \quad a_0 = \frac{m^* \omega}{2\hbar}, \\ \phi_1^\perp(x) &= C_1 x e^{-a_0 x^2}, \quad C_1 = 2a_0^{3/4} (2/\pi)^{1/4} \end{aligned} \quad (27)$$

are, respectively, the ground and first excited states of the harmonic oscillator in the (perpendicular)  $xy$  plane, and

$$\phi_{i,0}^\parallel(z) = C_z \cos\left[\frac{\pi}{d}(z - ia)\right], \quad C_z = \sqrt{2/d} \quad (28)$$

is the ground state of the  $i$ th quantum-well potential parallel to the array axis [ $\phi_{i,0}^\parallel(z) = 0$  for  $|z - ia| \geq d/2$ ].

Notice that the only dependence on the QD label  $i$  of the qubit states is in the  $z$  component of the wave function.

Since we are restricting ourselves to the low-energy sector  $\alpha = 0, 1$  in the absence of interdot ( $i \neq i'$ ) transitions, the only relevant fermionic bilinears in Eq. (24) are given by  $X_i = c_{i1}^\dagger c_{i0}$  and their conjugates. Consistently with the commutation relations  $[X_i, X_j^\dagger] = \delta_{ij}(n_i^1 - n_i^0) \equiv \sigma_{ij}, \sigma_i^z$ , these bilinears can be described by the spin-1/2 operators  $\sigma_i^\pm$ . Let  $|\mathbf{0}\rangle = \prod_{i=1}^N c_{i0}^\dagger |\text{vac}\rangle$  be the reference state built over the electron vacuum by occupying all the  $|0\rangle_i$ . Our *reduced* Hilbert space containing the computational degrees of freedom is then given by

$$\mathcal{H}_R = \text{span} \left\{ \prod_{i=1}^N X_i^{\alpha_i} |\mathbf{0}\rangle \mid \alpha_i = 0, 1 \right\} \equiv \otimes_{i=1}^N \mathbf{C}^2. \quad (29)$$

Any process inducing transitions out of this subspace will result in a computational error. Let  $\Delta$  be the energy gap between  $|1\rangle$  and the higher excited states (in the present case

$\Delta = \hbar\omega$ ) and  $T$  the environment (i.e., lattice) temperature; this sort of *leakage* error occurs with low probability as long as  $\Delta \gg k_B T$ .

By denoting with  $E \equiv \epsilon_{i,1} - \epsilon_{i,0} = \hbar\omega$  the energy spacing between our two qubit levels, the free-carrier Hamiltonian (22) for our qubit register, i.e., restricted to the low-energy sector  $\alpha = 0, 1$ , can then be written as

$$H_R = E \sum_{i=1}^N \sigma_i^z, \quad (30)$$

where  $\sigma_i^z$  denotes the usual diagonal Pauli matrix acting on the  $i$ th qubit.

Let us now consider again the carrier-phonon interaction Hamiltonian (24). Within the carrier model considered, wave functions corresponding to different QD's do not overlap; thus, one has  $g_{i\alpha, i'\alpha'; \lambda \mathbf{q}} = 0$  for  $i \neq i'$ , i.e., phonons induce intradot (intraqubit) transitions only. The coupling constants associated to the relevant elementary processes in our qubit register are  $g_{i, \lambda \mathbf{q}} \equiv g_{i, i, 0; \lambda \mathbf{q}}, \bar{g}_{i, \lambda \mathbf{q}} \equiv g_{i, i, 1; \lambda \mathbf{q}}$ . More specifically, starting from the explicit form of the single-particle wave functions  $\phi$  in Eq. (26) one finds  $g_{i, \lambda \mathbf{q}} = \tilde{g}_{\lambda \mathbf{q}} g_x(q_x) g_y(q_y) g_z(q_z, z_i) [\mathbf{q} = (q_x, q_y, q_z)]$ , where

$$g_x(q_x) = \langle \phi_0^\perp | e^{iq_x x} | \phi_0^\perp \rangle = \exp\left(-\frac{q_x^2}{8a_0}\right),$$

$$g_y(q_y) = \langle \phi_1^\perp | e^{iq_y y} | \phi_0^\perp \rangle = i \frac{1}{2a_0^{1/2}} q_y \exp\left(-\frac{q_y^2}{8a_0}\right),$$

$$g_z(q_z) = \langle \phi_{i,0}^\parallel | e^{iq_z z} | \phi_{i,0}^\parallel \rangle = \frac{8\pi^2}{d^3 q_z} \frac{\sin(q_z d/2)}{q_0^2 - q_z^2} e^{iq_z z_i}, \quad (31)$$

and  $q_0 = 2\pi/d$ .

Within these assumptions the carrier-phonon interaction Hamiltonian (24) can be cast in to the form (4):

$$H_I = \sum_{ki} (g_{ki} b_k^\dagger \sigma_i^- + \text{H.c.}), \quad (32)$$

where the bosonic label  $k$  now corresponds to the phonon modes of the crystal, i.e.,  $k = \lambda \mathbf{q}$ .

Following the Born-Markov procedure discussed in Sec. II, one finds the following result for the matrices  $\Gamma$  and  $\Delta$  defining our ME (Ref. 27):

$$\begin{aligned} \Gamma_{ij}^{(\pm)} &= 2\pi \sum_k g_{ki} \bar{g}_{kj} [n_k + \theta(\mp)] \delta(\hbar\omega_k - E), \\ \Delta_{ij}^{(\pm)} &= \mathcal{P} \sum_k \frac{g_{ki} \bar{g}_{kj}}{\hbar\omega_k - E} [n_k + \theta(\mp)]. \end{aligned} \quad (33)$$

Here,  $\theta$  is the customary Heaviside function and  $\mathcal{P}$  denotes the principal part. From these relations it follows that  $\Gamma^{(\pm)}$  and  $\Delta^{(\pm)}$  are Hermitian as expected. Furthermore,  $\Gamma^{(\pm)} \geq 0$  and  $\Gamma^{(-)} \geq \Gamma^{(+)}$ . Since for the QD structures considered the energy splitting  $E$  is typically much smaller than the optical-phonon energy (36 meV in GaAs), the only phonon modes  $k = \lambda \mathbf{q}$  involved are the acoustic ones. In this case, by considering carrier-phonon coupling due to deformation poten-

tial, one has  $\tilde{g}(\mathbf{q}) = \sqrt{\hbar \varepsilon q^2 / 2\rho V c}$ , where  $\varepsilon$  is the scalar lattice deformation,  $\rho$  and  $V$  the crystal mass density and volume, while  $c$  is the sound velocity.

Let us now focus on the explicit form of the function  $\Gamma$  in Eq. (33), i.e.,

$$\begin{aligned} \Gamma_{ij}^{\pm} &= 2\pi \sum_{\mathbf{q}} g_i(\mathbf{q}) \bar{g}_j(\mathbf{q}) [n_{\mathbf{q}} + \theta(\pm)] \delta(\omega_{\mathbf{q}} - \omega) \\ &= \frac{V}{(2\pi)^2} \int d^3\mathbf{q} g_i(\mathbf{q}) \bar{g}_j(\mathbf{q}) [n_{\mathbf{q}} + \theta(\pm)] \delta(\hbar c q - \hbar \omega). \end{aligned} \quad (34)$$

Because of the axial symmetry of the problem and the delta function of energy conservation, the three-dimensional integral over  $\mathbf{q}$  in Eq. (34) is better approached in polar coordinates:  $d^3\mathbf{q} = q^2 d\varphi d(\cos \vartheta) dq$ . One obtains an expression proportional to

$$\int_{-1}^1 dt e^{Q^2 t^2 / 4a} \frac{\cos(Q t z_{ij})}{[(q/Q)^2 - t^2]^2} \frac{1-t^2}{t^2} \sin^2 \left[ \frac{\pi t}{q/Q} \right], \quad (35)$$

with  $q = q_0$ ,  $a = a_0$ , and  $Q = E/\hbar c$ . Moreover,  $z_{ij} = a(i-j)$  is the distance between  $i$ th and  $j$ th QD's. The crucial point is to observe that for  $Q/a_0^{1/2} = \lambda_{\perp}/\lambda_{\parallel}$ , ( $\lambda_{\parallel} \sim Q^{-1}$ ) is large enough, and this integral is dominated by contributions around  $t \equiv \cos \vartheta = 1$ ; therefore,

$$\Gamma_{ij}^{(\pm)} = \Gamma_{11}^{(\pm)} \cos[Q z_{ij}]. \quad (36)$$

Recalling that  $\lambda_{\perp} = a_0^{-1/2}$  is the typical length scale of carrier confinement in the  $x$ - $y$  plane, this behavior is easily understood: due to the energy-conservation constraints ( $q_{\perp}^2 + q_z^2 = |\mathbf{q}|^2 = Q^2$ ), for delocalized in-plane wave functions (with respect to the length scale  $\lambda_{\parallel}$ ), the significant fluctuation of  $\mathbf{q}$  in the considered state is small; therefore,  $q_z \approx Q$ . In other words, due to the exponential suppression—in the overlap integral—of the contributions from phononic modes with nonvanishing in-plane components the system behaves as in the presence of a single effective phonon mode along the  $z$  axis resonant with the qubit excitations. As clearly confirmed by our numerical analysis reported in Sec. IV, this is an extremely important feature of the semiconductor model considered: in spite of its 3D nature and of the presence of a continuum of decoherence-inducing phonon modes, in this regime the carrier subsystem experiences an effectively 1D coherent environment, that in a good approximation can be described by the circular model (CM) analyzed in the Appendix.<sup>27</sup>

This model, parametrized by the dimensionless quantity  $\tilde{Q} \equiv Q a$ , represents a nontrivial example of a register-environment coupling that admits a rich structure (as a function of  $\tilde{Q}$ ) of subdecoherent encodings. From this point of view, it realizes a generalization of the replica symmetric model (pure collective decoherence) discussed in Ref. 12, that is recovered for  $\tilde{Q} = 0$ . Here, we limit ourselves to summarize the main result:

Safe quantum encoding are possible for the models such that  $e^{i\tilde{Q}}$  is a fourth root of the unity, the most efficient case

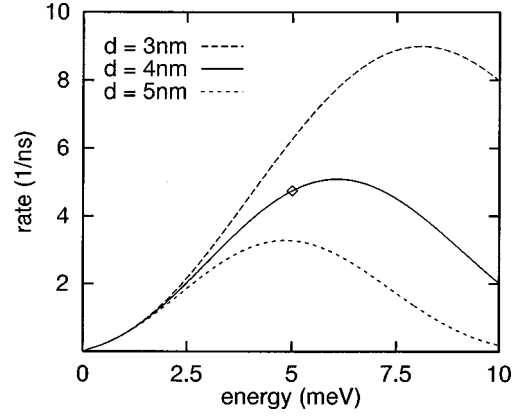


FIG. 1. Carrier-phonon scattering rate for a single QD structure as a function of the energy splitting  $E$  for different values of the QW width  $d$  at low temperature (see text).

being the points  $\tilde{Q} = 0, \text{ mod } \pi$ ; when all the register cells feel the same external coupling the dynamics is maximally collective due to the full permutational symmetry.

The existence of infinitely many “magic” points is clearly due to the unphysical nature of the CM that allows for undamped interactions between objects separated by arbitrary large distances. In realistic systems (as the ones investigated in this paper) the cosine dependence of the  $\Gamma$  matrix can only be approximated and the periodicity with respect to the cell distance eventually destroyed by some overimposed decay. In a way, the present situation is very similar to having a string of (two-level) atoms in a cavity coupled with a single resonant electromagnetic mode.<sup>28</sup>

#### IV. SIMULATION OF SUBDECOHERENT DYNAMICS IN A QD ARRAY

In this section we will present our numerical analysis of subdecoherent quantum encoding for realistic QD structures.

##### A. Carrier-phonon scattering in a single QD structure

As a starting point, let us discuss the role of carrier-phonon interaction in a single QD structure. Figure 1 shows the total (emission plus absorption) carrier-phonon scattering rate at low temperature ( $T = 10$  K) as a function of the energy spacing  $E$  for three different values of the GaAs QW width ( $d = 3, 4,$  and  $5$  nm).<sup>26</sup> Since the energy range considered is smaller than the optical-phonon energy (36 meV in GaAs), energy-conservation scattering with LO phonons is not allowed. Therefore, the only phonon mode  $\lambda$  that contributes to the rate of Fig. 1 is that of acoustic phonons. Again, due to energy conservation, the only phonon wave vectors involved must satisfy  $|\mathbf{q}| = E/\hbar c_s \equiv q, c_s$  being the GaAs sound velocity. It follows that by increasing the energy spacing  $E$  the wave vector  $q$  is increased, which reduces the carrier-phonon coupling entering in the electron-phonon interaction and then the scattering rate. This well-established behavior, known as phonon bottleneck,<sup>29</sup> is typical of a quasi-0D structure. As shown in Fig. 1, for  $E = 5$  meV—a standard value for many state-of-the-art QD structures—the carrier-phonon scattering rate is already suppressed by almost three orders of magnitude compared to the corresponding bulk values.<sup>8,9</sup>

In addition to the bottleneck scenario discussed so far, for a given value of the energy spacing  $E$  we see that for small values of  $d$  we have an increase of the carrier-phonon rate. In spite of the reduction of the 3D volume available to the carrier states, the overall coupling is increased, basically due to the progressive relaxation of momentum conservation along the growth ( $z$ ) direction.

### B. Short-time analysis

We will now show that by means of a proper information encoding, i.e., a proper choice of the initial multisystem quantum state, and a proper design of our QD array, we can strongly suppress phonon-induced decoherence processes, thus further improving the above single-dot scenario. To this end, let us consider a four-QD array, which is the simplest noiseless qubit register (see the Appendix). From the short-time expansion discussed in Sec. II A, we have numerically evaluated the decoherence rate for such QD array choosing as energy splitting  $E=5$  meV and the QW width  $d=4$  nm (see Fig. 1). As the initial state we have chosen the singlet  $|\psi_{\mathcal{D}_1,0}\rangle$  [see Eq. (15)] defined by the dimer partition  $\mathcal{D}_1=\{(1,2),(3,4)\}$ . We stress that when the CM approximation (see the Appendix) is not exactly fulfilled, different singlets have different decoherence rates. Indeed, the larger the distance  $z_{ij}$  between the pair elements in the dimer covering, the greater the deviation from the strictly periodic behavior. Thus, from Eq. (17) it follows, for instance, that the singlet corresponding to the dimer partition  $\mathcal{D}_2=\{(1,3)(2,4)\}$  has a greater decoherence rate than  $|\psi_{\mathcal{D}_1,0}\rangle$ . The decoherence rate obtained from our numerical calculation is shown as a solid line in Fig. 2(a) as a function of the interdot distance  $a$ . The uncorrelated-dot decoherence rate is also reported as a dashed line for comparison. As suggested by the analysis of the circular model presented in the Appendix, in spite of the 3D nature of the sum over  $\mathbf{q}$  entering the calculation of the function  $\Gamma_{ii'}^{(\pm)}$  [see Eq. (35)], the decoherence rate exhibits a periodic behavior over a range comparable to the typical QD length scale. In the circular-model approximation (and for  $T=0$ ) one obtains  $\tau_1^{-1}[|\psi_{\mathcal{D}_1}\rangle] \approx 2\Gamma_{00}^{(-)}[1 - \cos(Qa)]$ , from which it follows that for  $a_n = 2n\pi/Q$ , ( $n \in \mathbf{n}$ ) the considered state is stable. This effect—which would be natural for a 1D phonon system—stems from the exponential suppression in the overlap integral of the contributions of phononic modes with the nonvanishing in-plane component previously discussed. This 1D behavior is extremely important since it allows, by suitable choice of the interdot distance  $a$ , to realize a symmetric regime in which all the dots experience the same phonon field and, therefore, decohere collectively. Figure 2(a) shows that for the particular QD structure considered, case C should correspond to a decoherence-free evolution of a singlet state, which is not the case for A and B (see symbols in the figure).

In order to better understand how this sort of effective 1D behavior depends on the material parameters considered, we have repeated the subdecoherence analysis of Fig. 2(a) by artificially increasing the GaAs effective mass. More specifically, Figs. 2(b) and 2(c) present the same decoherence analysis, respectively, for values of 5 and  $10m^*$ . As we can see, by increasing the effective-mass value the 1D character

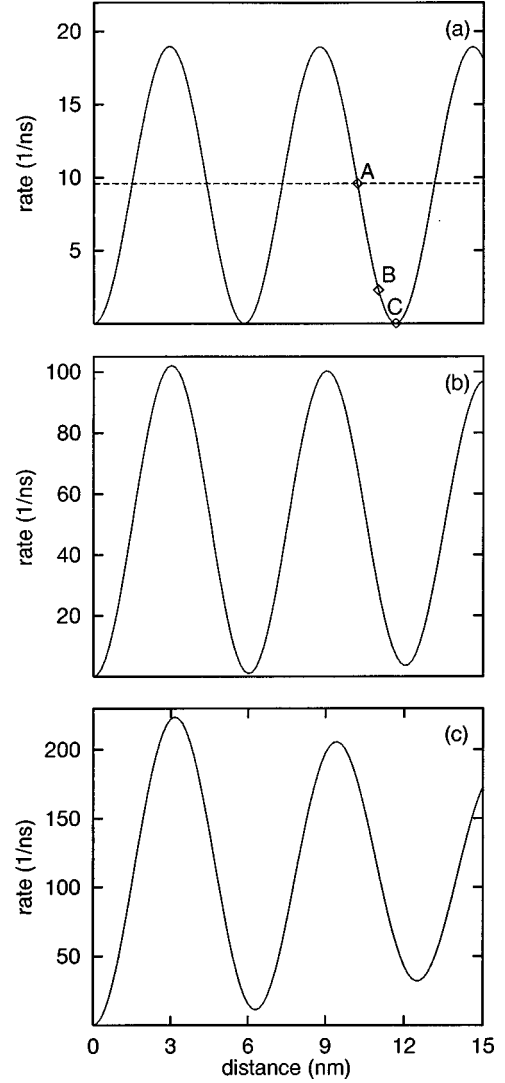


FIG. 2. (a) Phonon-induced decoherence rate for a four-QD array (solid line) as a function of the interdot distance  $a$  compared with the corresponding uncorrelated dot rate (dashed line); (b) same as in (a) but with an artificial effective mass of  $5m^*$ ; (c) same as in (a) but with an artificial effective mass of  $10m^*$  (see text).

in (a) is progressively suppressed. This can be clearly understood as follows: the increase of the effective mass leads to a stronger and stronger localization of the 2D harmonic-oscillator wave functions which, in turn, can easily interact with transverse ( $xy$ ) phonon modes  $\mathbf{q}$ .

As far as the unitary component of the Liouvillian is concerned, one can easily show that [for any  $|\psi\rangle$  eigenstate of  $S^z$ ]  $F(t) = |\langle \psi | e^{-iH_{\mathcal{R}}'t} | \psi \rangle|^2 = 1 - (t/\tau_U)^2 + o(t^3)$ , where

$$\frac{2}{\tau_U^2} = \langle \psi | \delta H_{\mathcal{R}}^2 | \psi \rangle - \langle \psi | \delta H_{\mathcal{R}} | \psi \rangle^2. \quad (37)$$

Figure 3 shows  $\tau_U^{-1}[|\psi_{\mathcal{D}_1}\rangle]$  as a function of the interdot distance  $a$ .

We find an oscillatory behavior similar to that of Fig. 2(a); it stems from the fact that (for the material parameters considered)  $\Delta_{ij}^{\pm} \approx \Gamma_{00}^{\pm} \sin[Q(i-j)a + \varphi]$ , with  $\varphi \ll \pi/2$ . Thus, for values of  $a$  corresponding to a subdecoherent dynamics



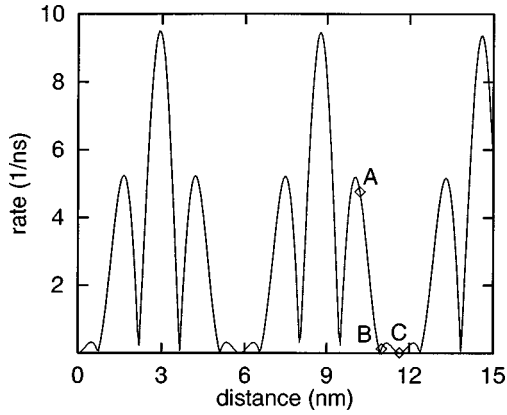


FIG. 3.  $\tau_U^{-1}[\langle\psi_{D_1}\rangle]$  as a function of  $a$  (see text).

[see point C in Fig. 2(a)], the  $\Delta$  contribution, also known as the polaronic shift, is negligible as well.

### C. Time-dependent solution of the master equation

In order to extend the above short-time analysis, we have performed a direct numerical integration of the master equation (see Sec. II A), thus obtaining the reduced density matrix  $\rho$  as a function of time. Also the Lamb-shift terms discussed in Sec. II have been taken into account. Starting from the same GaAs QD structure considered so far, we have simulated the above noiseless encoding for a four-QD array. Figure 4 shows the fidelity as a function of time as obtained from our numerical solution of the master equation. In particular, we have performed three different simulations—for the same initial state  $|\psi_{D_1,0}\rangle$ —corresponding to the different values of  $a$  depicted in Fig. 2(a). Consistently with our short-time analysis, for case C we find a strong suppression of the decoherence rate, which extends the sub-nanosecond time scale of the B case (corresponding to the uncorrelated dot rate) to the microsecond time scale.

Another quantity that properly describes the environment-induced corruption of information is the linear entropy  $\delta[\rho] \equiv \text{tr}(\rho - \rho^2)$ . Its production rate is also directly connected to  $\tau_1$ ; indeed, for an initial pure preparation we have  $\delta(t) = 2t/\tau_1 + o(t^2)$ . The time evolution of the linear entropy, as obtained from our numerical solution of the ME, is

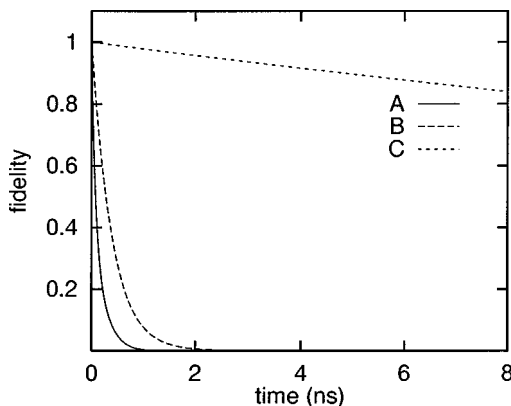


FIG. 4. Fidelity  $F$  as a function of time as obtained from a direct numerical solution of the master equation for the relevant case of a four-QD array (see text).

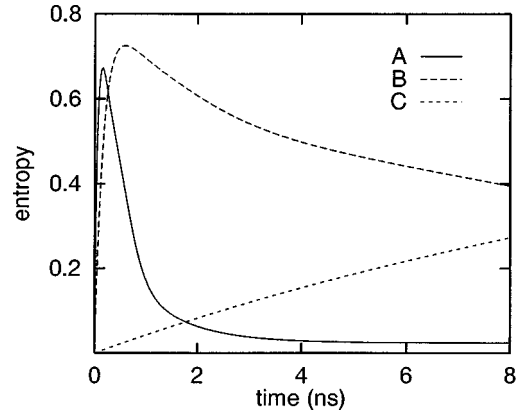


FIG. 5. Linear entropy as a function of time as obtained from a direct numerical solution of the master equation for the relevant case of a four-QD array (see text).

reported in Fig. 5. We can clearly recognize an initial transient (of the order of  $\tau_1$ ) in which the register, getting entangled with the environment, decoheres; this is followed by a subsequent slower relaxation dynamics.

The time-dependent analysis of Figs. 4 and 5 confirms that by means of the proposed encoding strategy one can realize a decoherence-free evolution over a time scale comparable with typical recombination times in semiconductor materials.<sup>8</sup>

## V. SUMMARY AND CONCLUSIONS

We have investigated a possible semiconductor-based implementation of the subdecoherent quantum-encoding strategy, i.e., error avoiding, recently proposed in Ref. 12. The goal is the suppression of phase-breaking processes in a quantum register realized by the lowest energy *charge* excitations of a semiconductor QD array.<sup>13</sup> In this case, the primary noise source is given by electron-phonon scattering, which is considered to be the most efficient decoherence channel in such a system.<sup>8,9</sup>

The main result is that, in spite of the 3D nature of carrier-phonon interaction in our QD structure, by means of a proper quantum encoding as well as of a proper tailoring of the semiconductor structure, one can, in principle, increase the coherence time by several orders of magnitude with respect to the bulk value. This would allow us to realize a coherent quantum-mechanical evolution on a time scale longer compared to that of ultrafast optical spectroscopy. From this point of view this result might constitute an important step toward a solid-state implementation of quantum computers. On the other hand, it certainly represents a first nontrivial example of a solid-state quantum system for which one can apply quantum error-avoiding strategies.

As already discussed in Sec. III, carrier-phonon scattering is not the only source of decoherence in semiconductors. In conventional bulk materials also, carrier-carrier interaction is found to play a crucial role. However, state-of-the-art QD structures—often referred to as semiconductor macroatoms<sup>10</sup>—can be regarded as few-electron systems basically decoupled from the electronic degrees of freedom of the environment. For the semiconductor QD array considered, the main source of Coulomb-induced “noise” may

arise from the interdot coupling. However, since such Coulomb coupling vanishes for large values of the QD separation and since the proposed encoding scheme can be realized for values of  $a$  much larger than the typical Coulomb-correlation length (see Fig. 2), a proper design of our quantum register may rule out such additional decoherence channels.<sup>30</sup>

The actual implementation of the suggested encoding relies, of course, on precise quantum-state synthesis and manipulations. This further step, not addressed in this paper, represents the most challenging open issue concerning the ultimate usefulness of the proposed coding strategy.

### ACKNOWLEDGMENTS

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### APPENDIX: CIRCULAR-INTERACTION MODEL

This appendix is devoted to the formal analysis of a model with periodic (environment-induced) interactions between register cells. We set  $\Gamma_{ij}^{(\pm)} = \Gamma^{(\pm)} \cos[Q(i-j)]$ ; the resulting model will be referred to as the CM. The dimensionless parameter  $Q$  is taken to be given by the product of a characteristic wave vector (corresponding to an effective one-phonon field) times the intercell distance. The effective Hamiltonian (13) takes the form  $\tilde{H} = \sum_{\alpha=\pm} H^{(\alpha)}(Q)$ , with

$$H_Q^{(\alpha)} = \frac{1}{2} \Gamma^{(\alpha)} (S_Q^{-\alpha} S_{-Q}^{\alpha} + S_{-Q}^{-\alpha} S_Q^{\alpha}), \quad (\text{A1})$$

where  $S_Q^{\alpha} = \sum_{j=1}^N e^{iQj} \sigma_j^{\alpha}$  ( $\alpha = \pm, z$ ), are the present Lindblad operators. They fulfill the following commutation relations:

$$\begin{aligned} [S_Q^{\pm}, S_{Q'}^{\mp}] &= \pm 2S_{Q+Q'}^z, \\ [S^z, S_Q^{\pm}] &= \pm S_Q^{\pm}. \end{aligned} \quad (\text{A2})$$

For  $Q=0 \pmod{2\pi}$  one recovers the global  $\mathfrak{sl}(2)$  algebra spanned by the  $S^{\alpha}$ 's, to which the  $S_Q^{\alpha}$ 's are connected by the following unitary transformations  $U_Q \equiv \exp(iQ \sum_{j=1}^N j \sigma_j^z)$ . Indeed, we have  $S_Q^{\alpha} = U_{\alpha Q} S^{\alpha} U_{\alpha Q}^{\dagger}$  ( $\alpha = \pm$ ) (notice that  $U_Q^{\dagger} = U_{-Q}$ ). In terms of these unitary transformations and of the  $Q=0$  Hamiltonian  $H_0 = \Gamma^{(-)} S^+ S^- + \Gamma^{(-)} S^- S^+$  the CM model (A1) reads

$$H_Q = 2^{-1} \sum_{\eta=\pm} U_{\eta Q} H_0 U_{\eta Q}^{\dagger}. \quad (\text{A3})$$

From Eq. (A2) it follows that, for *any generic*  $Q$ , the two terms in the above equation do not commute: the model is nontrivial, i.e., nonintegrable.

Next, the proposition shows that the analytic structure of the CM strongly depends on the input parameter  $Q$ , for particular  $Q$  values it is quite simple and its subdecoherent coding efficiency is optimal.

*Proposition 1.* One has the following integrable points.

(i)  $Q=0 \pmod{2\pi} \Rightarrow H^{(\alpha)}(2\pi) = \Gamma^{(\alpha)} S^{-\alpha} S^{\alpha}$ , replica symmetry.

(ii)  $Q = \pi \pmod{2\pi} \Rightarrow H^{(\alpha)}(\pi) = \Gamma^{(\alpha)} S^{-\alpha}(\pi) S^{\alpha}(\pi) = U_{\pi} H^{(\alpha)}(2\pi) U_{\pi}^{\dagger}$ .

(iii) if  $Q = \pi/2, 3/2\pi \pmod{2\pi}$  one has  $\Gamma_{i,i+2n}^{(\alpha)} = \Gamma^{(\alpha)} (-1)^n$  and  $\Gamma_{i,i+2n+1}^{(\alpha)} = 0$ . The odd- and even-site sublattices decouple, and for each sublattice case (ii) is recovered.

Notice that for cases (i) and (ii)  $2Q=0 \pmod{2\pi}$ ; then the  $\eta=+$  and  $\eta=-$  terms in Eq. (A3) are identical; the model is then unitarily equivalent to the  $Q=0$  case. The latter is clearly diagonalized in the  $S^2, S^z$  eigenbasis and its spectrum is given by  $E = \sum_{\alpha=\pm} E^{(\alpha)}(J, M, r)$ , where

$$E^{(\alpha)}(J, M, r) = \Gamma^{(\alpha)} [J(J+1) - M(M+\alpha)], \quad (\text{A4})$$

$J = J_{\min}, \dots, N/2; M = -J, \dots, J; r = 1, \dots, n(J, N)$ , in which  $J_{\min} = 0 (J_{\min} = 1)$  for  $N$  even (odd), and  $n(J, N)$  denotes the multiplicity of the  $\mathfrak{sl}(2)$  representation labeled by  $J$  (Ref. 12)

$$n(J, N) = \frac{N!(2J+1)}{(N/2+J+1)!(N/2-J)!}. \quad (\text{A5})$$

If  $N$  is even and  $0 < \Gamma^{(+)} \leq \Gamma^{(-)}$ , the lowest eigenvalue is  $E_0 = 0$  with degeneracy  $n(0, N)$ , the ground-state manifold being the *singlet* sector of the global  $\mathfrak{sl}(2)$ . At zero temperature one has  $\Gamma^{(+)} = 0$ ; therefore all the lowest-weight  $\mathfrak{sl}(2)$  vectors  $|J, -J\rangle$  are ground states of  $\tilde{H}$ .

Let us consider the  $N$ th roots of the unit (with  $N$  even)

$$\mathcal{Z}_N = \left\{ e^{iQ_j/Q} \sigma_j^{\pm}, j=0, \dots, N-1 \right\}. \quad (\text{A6})$$

This (multiplicative) group is, of course, isomorphic to the (additive) group  $\mathbf{Z}/N\mathbf{Z} = \{0, \dots, N-1\}$ ; thus, we shall use the same notation for both. Here,  $\mathcal{Z}_N$  is considered a subgroup of  $S_N$ . The latter as a natural action on  $\mathcal{H}_{\mathcal{R}}$  given by the linear extension of  $p: \otimes_{j=1}^N |\sigma_j\rangle \mapsto \otimes_{j=1}^N |\sigma_{p(j)}\rangle$ , ( $p \in S_N$ ).

The operators  $S_m^{\alpha} \equiv S^{\alpha}(Q_m)$  satisfy to the commutation relations  $[S_m^{\alpha}, S_n^{\beta}] = K_{\gamma}^{\alpha\beta} S_{n+m}^{\gamma}$ ,  $K_{\gamma}^{\alpha\beta}$  are the  $\mathfrak{sl}(2)$  structure constants. They span the ( $\mathcal{Z}_N$  graded) Lie algebra

$$\mathcal{A}_N \equiv \text{span}\{S_m^{\alpha}/\alpha=z, \pm, m \in \mathcal{Z}_N\} \cong \oplus_i^N \mathfrak{sl}(2)_i. \quad (\text{A7})$$

Let  $\mathcal{A}_N^Q$  be the Lindblad operators algebra for a generic  $Q$ . The following proposition gives a characterization of it when  $Q$  varies.

*Proposition 2.*

(i) For a generic  $Q$  (i.e.,  $e^{iQ} \notin \mathcal{Z}_N$ ) one has  $\mathcal{A}_N^Q \cong \mathcal{A}_N$ , whereas for  $e^{iQ} \in \mathcal{Z}_N$  one finds

$$\mathcal{A}_N^Q \cong \text{span}\{S_{2pn}^z, S_{n(2p+1)}^{\pm}/p \in \mathcal{Z}_N\}.$$

(ii)  $\mathcal{A}_N^0 \cong \mathcal{A}_N^{\pi} \cong \mathfrak{sl}(2)$ .

(iii)  $\mathcal{A}_N^{\pi/2} = \mathfrak{sl}(2)_e \oplus \mathfrak{sl}(2)_o$ , where

$$\begin{aligned} \mathfrak{sl}(2)_e &\equiv \text{span}\left\{ \sum_{j=1}^{N/2} (-1)^j \sigma_{2j}^{\alpha} \right\}_{\alpha}, \\ \mathfrak{sl}(2)_o &\equiv \text{span}\left\{ \sum_{j=0}^{N/2-1} (-1)^j \sigma_{2j+1}^{\alpha} \right\}_{\alpha}. \end{aligned} \quad (\text{A8})$$

(iv)  $e^{iQ_j} \in \mathcal{Z}_N^* \equiv \mathcal{Z}_N - \mathcal{Z}_4 \Rightarrow \dim \mathcal{A}_N^Q = 3N/2$ .

*Proof.* One can check that  $\mathcal{A}_N^Q = \text{span}\{S_{2pQ}^z, S_{Q(2p+1)}^\pm/p \in \mathcal{Z}\}$ . If  $Q$  is rationally independent from  $2\pi$  the numbers  $e^{i2pQ}, e^{iQ(2p+1)}$  densely fill the unit circle, from which  $\mathcal{A}_N \subset \mathcal{A}_N^Q$ .<sup>31</sup> Points (ii) and (iii) follow from Proposition 1, and (iv) can be checked by a simple calculation.

Notice that  $e^{i\pi/p} \in \mathcal{Z}_N$  iff  $N=0 \pmod{2p}$  ( $p=1,2$ ). Remembering that  $|\psi\rangle \in \text{Ker } H_Q \Leftrightarrow |\psi\rangle$  is annihilated by *all* the generators of  $\mathcal{A}_N^Q$ —and then that the smaller is the algebra the greater is the code—Proposition 2 seems to indicate that the “magic”  $Q$ ’s possibly relevant for subdecoherent encoding are just the ones such that  $e^{iQ} \in \mathcal{Z}_N$ .

Let us now consider the  $Q$  dependence of the symmetry structure of our model.

*Lemma.* Let  $\mathcal{G}_Q$  the (maximal) symmetry group of  $H_Q$ , one has: (i)  $\mathcal{G}_0 \cong \mathcal{G}_\pi = \mathcal{S}_N$ , (ii)  $\mathcal{G}_{\pi/2} \cong \mathcal{G}_{3\pi/2} = \mathcal{S}_{N/2} \times \mathcal{S}_{N/2}$ , and (iii) for  $e^{iQ} \in \mathcal{S}^1 - \mathcal{Z}_4$  one has  $\mathcal{G}_Q = \mathcal{Z}_N$ .

Pictorially, one can say that in the CM the register has a regular polygon topology that for the special points  $Q=0, \pi$  ( $Q=\pi/2, 3/2\pi$ ) collapses to a point (dimer) gaining in this way a larger permutational symmetry. This dynamical clustering is associated with a greater subdecoherent coding efficiency.<sup>14</sup> The next proposition summarizes in a formal manner the present situation.

*Proposition 3.* Let  $N$  be even.

- (i)  $e^{iQ} \in \mathcal{Z}_N \Leftrightarrow d_N(Q) > 0$ .
- (ii)  $d_N(Q_0) = d_N(Q_{N/2}) = n(0, N)$ .
- (iii) If  $N=0 \pmod{4} \Rightarrow d_N(Q_{N/4}) = d_{N/2}(Q_0)^2$ .
- (iv)  $e^{iQ_j} \in \mathcal{Z}_N^* \Rightarrow d_N(Q_j) = 1$ .
- (v) When  $e^{iQ_j} \in \mathcal{Z}_N^*$  the null space is spanned by the vectors

$$|\psi_j\rangle = \otimes_{i=1}^{N/2} (|01\rangle - (-1)^j |10\rangle)_{i, i+N/2}. \quad (\text{A9})$$

- (vi) Let  $N$  be odd then  $d_N(Q) = 0 \forall Q$ .

*Proof.* The cases  $e^{iQ} = \pm 1$  are isomorphic and have been previously discussed. Notice that, if  $H \geq 0$  one has  $\langle \psi | H | \psi \rangle = 0 \Leftrightarrow H | \psi \rangle = 0$ . Moreover,  $|\psi_j\rangle \equiv |\bar{D}, \gamma_j\rangle$ , where  $\bar{D}$  is the unique dimer partition of the array with  $|l-k| = N/2$  and  $\gamma_j(l, k) = j \pmod{2} \forall (l, k) \in \bar{D}$ . From the second of Eqs. (17) one finds that

$$\langle \psi_j | H_{Q_j} | \psi_j \rangle \sim 1 - \frac{2}{N} \sum_{j=1}^{N/2} (-1)^j \cos(\pi j) = 0,$$

from which the sufficiency parts of (i) and (v) follow. If  $e^{iQ} \notin \mathcal{Z}_N$  from Propositions 2 and 3, one has that if  $|\psi\rangle \in \text{Ker } H_Q$  then it is in the singlet sector of  $\mathcal{A}_N^Q$  (Proposition 2). Since  $\mathbf{C}^{2^n}$  is an irreducible (nontrivial) representation space of  $\mathcal{A}_N$ , such a sector is empty. Points (ii) and (iii) follow directly from Propositions 2 and 3. Since the  $S_{\pm Q}^\alpha$ ’s transform according 1D  $\mathcal{G}_Q$  irreps, from representation theory it follows that  $d_N(Q)$  (i.e., multiplicity of the 1D  $\mathcal{A}_N^Q$  irrep) is equal to the dimension of an irrep of the symmetry group  $\mathcal{G}_Q$ . But for  $e^{iQ} \in \mathcal{Z}_N^*$  one has  $\mathcal{G}_Q \cong \mathcal{Z}_N$ , (abelian) therefore, its irreps are 1D, from which point (iv) follows.<sup>32</sup> Finally, point (vi) simply stems from the fact that the necessary condition  $S^z |\psi\rangle = 0$  cannot hold for odd  $N$ .

To understand in a more constructive fashion why the  $|\psi_j\rangle$  are the (only) subdecoherent states for  $e^{iQ_j} \in \mathcal{Z}_N^*$ , let us consider the following state  $|\psi\rangle \in \mathcal{C}$  [the (global) sl(2) singlet sector] such that (i)  $U_{2Q} |\psi\rangle = |\psi\rangle$ . Then  $U_Q |\psi\rangle = U_Q^+ |\psi\rangle = U_- |\psi\rangle \equiv |\tilde{\psi}\rangle$ . This means that if  $|\tilde{\psi}\rangle \in \cap_{\alpha=\pm} U_{\alpha Q} \mathcal{C}$  it follows that  $|\tilde{\psi}\rangle$  is annihilated by  $S_{\pm Q}^\alpha = U_{\pm Q} S^\alpha U_{\pm Q}^\dagger$ , ( $\alpha=z, \pm$ ), and therefore, by  $H_Q$ . It is now easy to check that the states  $|\psi_j\rangle$  of Proposition 4 are just  $U_{Q_j} |\bar{D}, \mathbf{0}\rangle$ , the dimer partition  $\bar{D}$  being the only one allowing for condition (i) to be fulfilled.

<sup>1</sup>For reviews, see D. P. DiVincenzo, *Science* **270**, 255 (1995); A. Ekert and R. Josza, *Rev. Mod. Phys.* **68**, 733 (1996).

<sup>2</sup>K. Kraus, *Lect. Notes in Physics*, Vol. 190 (Springer, Berlin, 1983).

<sup>3</sup>W. G. Unruh, *Phys. Rev. A* **51**, 992 (1995); P. W. Shor, W. H. Zurek, I. L. Zurek, I. L. Chuang, and R. Laflamme, *Science* **270**, 1633 (1995).

<sup>4</sup>See, for example, E. Knill and R. Laflamme, *Phys. Rev. A* **55**, 900 (1997), and references therein.

<sup>5</sup>P. W. Shor, *37th Symposium on Foundations of Computing* (IEEE, New York, 1996), pp. 56–65; *Phys. Rev. A* **57**, 127 (1998).

<sup>6</sup>C. Monroe *et al.*, *Phys. Rev. Lett.* **75**, 4714 (1995); Q. A. Turchette *et al.*, *ibid.* **75**, 4710 (1995); J. I. Cirac and P. Zoller, *ibid.* **74**, 4091 (1995); N. A. Gershenfeld and I. L. Chuang, *Science* **275**, 350 (1996).

<sup>7</sup>A. P. Heberle, J. J. Baumberg, and K. Kohler, *Phys. Rev. Lett.* **75**, 2598 (1995).

<sup>8</sup>J. Shah, *Ultrafast Spectroscopy of Semiconductors and Semiconductor Nanostructures* (Springer, Berlin, 1996).

<sup>9</sup>T. Kuhn, in *Theory of Transport Properties of Semiconductor Nanostructures*, edited by E. Schöll (Chapman & Hall, London, 1998), p. 173.

<sup>10</sup>For reviews, see e.g., M. A. Kastner, *Rev. Mod. Phys.* **64**, 849 (1992); *Phys. Today* **46**, (1), 24 (1993).

<sup>11</sup>D. Loss and D. P. Di Vincenzo, *Phys. Rev. A* **59**, 120 (1998).

<sup>12</sup>P. Zanardi and M. Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997); *Mod. Phys. Lett. B* **25**, 1085 (1997).

<sup>13</sup>P. Zanardi and F. Rossi, *Phys. Rev. Lett.* **81**, 4752 (1998).

<sup>14</sup>P. Zanardi, *Phys. Rev. A* **57**, 3276 (1998).

<sup>15</sup>G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).

<sup>16</sup>The environment can, nevertheless, affect the register through the action of  $\delta H_{\mathcal{R}}$ .

<sup>17</sup>The CM Hamiltonian for  $N=2$  can be written in terms of global  $S^{\alpha}$ ’s.

<sup>18</sup>S. Tarucha *et al.*, *Phys. Rev. Lett.* **77**, 3613 (1996).

<sup>19</sup>A. Barenco, D. Deutsch, A. Ekert, and R. Josza, *Phys. Rev. Lett.* **74**, 4083 (1995).

<sup>20</sup>M. Rontani, F. Rossi, F. Manghi, and E. Molinari, in *Proceedings of the ICPS-24, Jerusalem, Israel, 1998* (World Scientific, Singapore, in press).

<sup>21</sup>This assumption is made for computational simplicity. Realistic (i.e., finite) barriers would give rise to interdot hopping. Nevertheless, due to the exponential decay of interdot overlap integrals, the effect is negligible also for rather small interdot distances.

- <sup>22</sup>M. Rontani, F. Rossi, F. Manghi, and E. Molinari, *Appl. Phys. Lett.* **72**, 957 (1998).
- <sup>23</sup>Here  $H_l$  denotes the Hadamard transformation [i.e.,  $\langle x|H|y\rangle = (-1)^{xy}$ ,  $(x,y=0,1)$ ] on the  $l$ th qubit. Moreover  $\text{cnot}_{lm}$  is the controlled-not transformation acting on the  $l$ th and  $m$ th qubits. If  $\oplus$  denotes addition *modulo 2* one has  $\text{cnot}|x,y\rangle = |x,x\oplus y\rangle$ ,  $(x,y=0,1)$ .
- <sup>24</sup>Here, only coupling to GaAs bulk phonons has been considered. This, of course, is a simplifying approximation that neglects any interface effect on the phonon dispersion, such as confinement of optical modes in the wells and in the barriers, and the presence of interface modes [H. Rucker, E. Molinari, and P. Lugli, *Phys. Rev. B* **45**, 6747 (1992)]. However, while these modifications have important consequences for phonon spectroscopies (like Raman scattering), they are far less decisive for energy-relaxation and dephasing phenomena. Indeed, by now it is well known that the total scattering rates are sufficiently well reproduced if the phonon spectrum is assumed to be bulklike [E. Molinari, in *Confined Electrons and Photons: New Physics and Applications*, edited by E. Burstein and C. Weisbuch (Plenum, New York, 1994)].
- <sup>25</sup>With “first excited state” here we mean any linear combination of the two energetically degenerate states.
- <sup>26</sup>Notice that these matrices are related by a kind of Kramers-Kronig dispersion relation. This stems from the fact that they are basically a real and imaginary part of a single electron propagator obtained by tracing out the phonon field (Ref. 13).
- <sup>27</sup>It is worthwhile to stress that this result appears to be largely independent on the detailed shape of the qubit wave functions [see Eq. (35)]. Indeed, the only requirement that appears to be crucial is the  $x$ - $y$  delocalization, with respect to the effective phonon wavelength.
- <sup>28</sup>P. Horak and K. Gheri, *Phys. Rev. A* **53**, R1970 (1996); K. Gheri, P. Horak, and H. Ritsch, *J. Mod. Opt.* **44**, 605 (1997).
- <sup>29</sup>M. Sugawara, K. Mukai, and H. Shoji, *Appl. Phys. Lett.* **71**, 2791 (1997).
- <sup>30</sup>The reader may wonder whether the increase of  $a$  makes the various qubits interacting with independent (i.e., uncorrelated) phonon baths. We observe that a lower bound for the acoustic-phonon lifetime in GaAs is tenths of picoseconds, which corresponds to a coherence length much larger than the typical interdot distance  $a$  relevant for noiseless encoding. This makes the phonon dynamics fully coherent on the space scale of our QD register.
- <sup>31</sup> $\mathcal{A}_N^Q$  should more properly defined as the *closure* of the algebra generated commuting the  $S_Q^\pm$ .
- <sup>32</sup> $p \in \mathcal{Z}_N, p \cdot |\psi_j\rangle = (-1)^{p(j+1)} |\psi_j\rangle$ .