Systematic study of the Wiedemann-Franz law in the quantum-Hall-effect regime

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A systematic study of the Wiedemann-Franz law in the quantum-Hall-effect regime is presented. For this purpose the diffusion thermal conductivity tensor is calculated for a two-dimensional electron gas at low temperatures in a quantizing magnetic field. The range of validity of the Wiedemann-Franz law is investigated performing both analytical and numerical calculations. The analysis shows that for the diagonal component of the thermal conductivity, the Wiedemann-Franz law is violated with decreasing Landau level broadening. Responsible is the coefficient N_{xx} and specifically the effect of the energy derivatives of the diagonal electrical conductivity and consequently the shape of the density of states. For the nondiagonal components we obtain smaller deviations. We give physical interpretations for the resulted behavior. [S0163-1829(99)05408-9]

I. INTRODUCTION

One of the most unclear problems since the discovery of the quantum Hall effect[1] is heat transport. Although a number of studies have been devoted to the evaluation and the measurement of thermopower[2–10] and thermal conductivity (TC),[11–15] the situation is still obscure. Fundamental laws such as the Wiedemann-Franz (WF) law have been questioned as far as their validity is concerned and a convincing systematic analysis for the reasoning of possible deviations[11,15] seems to be lacking in the literature. For the TC there are very few experimental data available. Syme *et al.*[16] have measured the electron contribution in the TC and they found that the WF law holds for the particular 2DEG. Unfortunately their data were taken in the absence of a magnetic field.

Oji[11] evaluated the TC tensor of a two-dimensional electron gas (2DEG) in a quantizing magnetic field using the Kubo formula and the self-consistent Born approximation (SCBA). He reported that the diagonal component of the TC tensor versus the chemical potential has a characteristic ''two-peak'' shape, while WF law is violated. The Lorentz number $\kappa_{xx}/\sigma_{xx}T$ varies erratically with the position of the chemical potential in the energy spectrum. The nondiagonal components $\kappa_{xy}/\sigma_{xy}T$ oscillate about the standard value $\pi^2(k_B/e)^2/3$. As a reasoning he only noticed that these effects originate from the discrete nature of the density of states (DOS).

Blanter *et al.*[15] evaluated the TC tensor by means of the diagrammatic technique within the SCBA for the same regime with impurity disorder. They reported that both the diagonal and the nondiagonal components of the TC tensor exhibit deviations from the WF law. The reason for the WF law violation was attributed to the fact that thermopower components are not small due to rapid variation of the electronic DOS.

We believe that the theoretical studies reported leave the question of the validity of the WF law in the quantum-Hall-effect (QHE) regime still open and a complete and convincing physical reasoning of the behavior obtained is still lacking.

In the present paper we attempt a systematic approach to this problem. We investigate the range of validity of the WF law in the QHE regime calculating analytically and numerically the diffusion TC tensor. Our theoretical approach is applicable at very low temperatures where diffusion dominates over phonon-drag. Ruf *et al.*[17] gave a threshold of 0.6 K as a lower limit for phonon-drag dominance below which diffusion prevails. Recent measurements reported for 2D systems by Crump *et al.*[18] and Bayot *et al.*[19] showed that the diffusion thermopower dominates at temperatures lower than 0.3 K. This was theoretically justified by Karavolas *et al.*[20] A detailed physical interpretation of the results is presented.

The present paper consists of the following: The basic elements of the transport theory are given in Sec. II. Numerical and analytical calculations for the diagonal component of the TC, together with the diagonal component of the electrical conductivity are given in Sec. III. Calculations of the corresponding nondiagonal quantities are presented in Sec. IV. Finally our conclusions are given in Sec. V.

II. TRANSPORT THEORY

The basic equations that govern the response of a typical semiconductor to external stimuli (for example, an electric field $\mathbf{E}$ or a temperature gradient ∇T)[21] are

$$\mathbf{J}=\sigma\mathbf{E_m}+L(-\nabla T), \tag{1a}$$

$$\mathbf{Q}=M\mathbf{E_m}+N(-\nabla T), \tag{1b}$$

where $\mathbf{E_m}$ is the electromotive force, $\mathbf{J}$ is the electric current density, $\mathbf{Q}$ is the thermal current density, σ is the electrical conductivity and L,M,N are the remaining three transport coefficients.

For experimental convenience the above equations are transformed to

$$\mathbf{E_m}=\rho\mathbf{J}+S(-\nabla T), \tag{2a}$$

$$\mathbf{Q}=\pi\mathbf{J}+\kappa(-\nabla T), \tag{2b}$$

where ρ is the resistivity, S is the thermopower, π is the Peltier coefficient, and κ is the TC. Here,

$$\rho=\sigma^{-1}, \tag{3a}$$

$$S=-\sigma^{-1}L, \tag{3b}$$

$$\pi=M\sigma^{-1}, \tag{3c}$$

$$\kappa=N-M\sigma^{-1}L=N+MS. \tag{3d}$$

When a magnetic field is applied, these coefficients become second rank tensors depending on the applied magnetic field. These transport tensors as far as diffusion is concerned have been derived from the Kubo formula by Smrčka and Středa [22] and they are given by

$$\sigma_{ij}=\int_{-\infty}^{\infty}\left(-\frac{\partial f(E)}{\partial E}\right)\sigma_{ij}(E)dE, \tag{4a}$$

$$M_{ij}=-\frac{1}{e}\int_{-\infty}^{\infty}\left(-\frac{\partial f(E)}{\partial E}\right)(E-E_F)\sigma_{ij}(E)dE=L_{ij}T, \tag{4b}$$

$$N_{ij}=\frac{1}{e^2T}\int_{-\infty}^{\infty}\left(-\frac{\partial f(E)}{\partial E}\right)(E-E_F)^2\sigma_{ij}(E)dE. \tag{4c}$$

$f(E)$ is the Fermi-Dirac distribution function, $(-e)$ is the electron charge, and $\sigma_{ij}(E)$ is the zero temperature electrical conductivity for $E=E_F$.

The full theory has been presented elsewhere.[20,23] The matrix elements of the TC tensor are given by

$$\kappa_{xx}=N_{xx}+M_{xx}S_{xx}-M_{yx}S_{yx}, \tag{5a}$$

$$\kappa_{xy}=N_{xy}+M_{xy}S_{xx}+M_{xx}S_{xy}. \tag{5b}$$

The thermopower tensor is given by[20,24]

$$S=\begin{pmatrix}-\rho_{xx}L_{xx}-\rho_{xy}L_{yx} & -\rho_{xy}L_{xx}+\rho_{xx}L_{yx}\\ \rho_{xy}L_{xx}-\rho_{xx}L_{yx} & -\rho_{xx}L_{xx}-\rho_{xy}L_{yx}\end{pmatrix}. \tag{6}$$

WF law is expressed as $\kappa_{ij}=\sigma_{ij}(E)L_0T$. Here, L_0 is the Lorenz number given by

$$L_0=\frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 \tag{7}$$

and k_B is the Boltzmann constant.

To reproduce analytically the WF law and to investigate possible deviations, we have to analyze the behavior of the two terms appearing on the right-hand side of Eq. (3d), evaluating the tensor components appearing in the matrix elements of Eqs. (5a) and (5b).

The N_{ij} tensor is given by

$$\begin{aligned}N_{ij}&=\sigma_{ij}L_0T+\sum_{n=2}^{4}a_n(k_BT)^{2n}\frac{d^{2n-1}}{dE^{2n-1}}H_{ij}(E)\bigg|_{E=E_F}\\&=\frac{1}{e^2T}\int_{-\infty}^{\infty}H_{ij}(E)f(E)dE\\&=\frac{1}{6}(\pi k_BT)^2\frac{dH_{ij}(E)}{dE}+\frac{7}{360}(\pi k_BT)^4\frac{d^3H_{ij}(E)}{dE^3}\\&\quad+\frac{930}{15\,120}(\pi k_BT)^6\frac{d^5H_{ij}(E)}{dE^5}\\&\quad+\frac{7112}{604\,800}(\pi k_BT)^8\frac{d^7H_{ij}(E)}{dE^7},\end{aligned} \tag{8}$$

where a_n are dimensionless numbers[21] and at very low temperatures $\sigma_{ij}(E)=\sigma_{ij}$. In the above analytical expression we have kept only the first three correction terms in the intergral expansion. The first term on the right-hand side of Eq. (8) represents the WF law. Here,

$$H_{ij}(E)=\frac{d}{dE}[\sigma_{ij}(E)(E-E_F)^2]. \tag{9}$$

The diagonal component of the electrical conductivity tensor at $T=0$ K is

$$\sigma_{xx}(E)=\frac{e^2}{\pi^2\hbar}\sum_n (n+1/2)(\pi^2l^2\Gamma_nD_n(E))^2, \tag{10}$$

where $D_n(E)$ is the DOS for carriers in the Landau level n, given in a Gaussian form[20,23]

$$D_n(E)=\frac{1}{2\pi l^2}\frac{1}{\sqrt{2\pi}\Gamma_n}e^{-(E-E_n)^2/\Gamma_n^2}. \tag{11}$$

Here, Γ_n is the Landau level broadening, E_n is the energy at the middle of the Landau level given by $E_n=(n+1/2)\hbar\omega_c$, where $\omega_c=eB/m^*$ is the cyclotron frequency, m^* is the electron effective mass, $l=\sqrt{\hbar/eB}$ is the magnetic length, and $1/2\pi l^2$ is the available number of states in each Landau level. Thus, the diagonal component of the electrical conductivity tensor for large magnetic fields, B, is given by[20]

$$\sigma_{xx}=\frac{e^2}{\pi^2\hbar}\sum_n\int dE\left(-\frac{\partial f(E)}{\partial E}\right)(n+\tfrac{1}{2})(\pi^2l^2\Gamma_nD_n(E))^2. \tag{12}$$

The nondiagonal component of the electrical conductivity tensor is given by

$$\sigma_{xy}=-\frac{e}{B}\int dE\left(-\frac{\partial f(E)}{\partial E}\right)\sum_n\int dE\, f(E)D_n(E), \tag{13}$$

where the nondiagonal component of the electrical conductivity tensor at $T=0$ is

$$\sigma_{xy}(E)=\sum_n\int dE\, D_n(E)f(E). \tag{14}$$

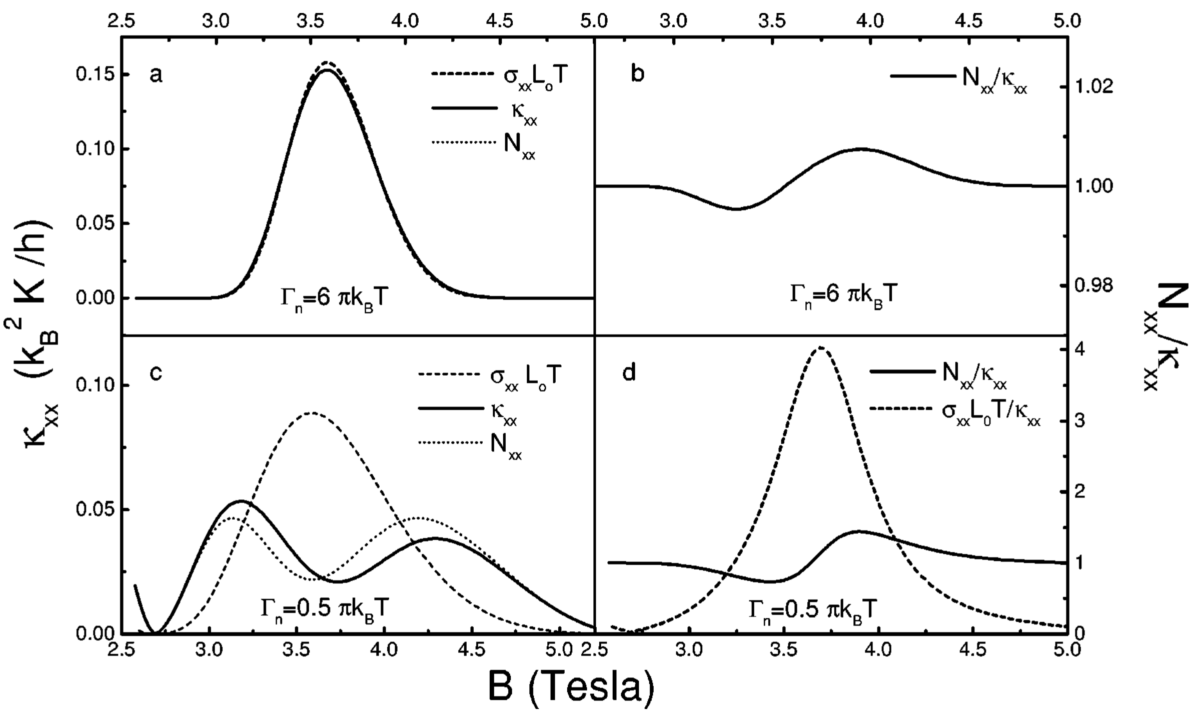

FIG. 1. κ_{xx}, $\sigma_{xx}L_0T$, and N_{xx} versus the magnetic field at $T=0.15$ K, $\Gamma_n=6\pi k_BT$ (a), and $\Gamma_n=0.5\pi k_BT$ (c). The ratio N_{xx}/κ_{xx} for $\Gamma_n=6\pi k_BT$ (b) and the ratios N_{xx}/κ_{xx} and $\sigma_{xx}L_0T/\kappa_{xx}$ are also presented for $\Gamma_n=0.5\pi k_BT$ (d).

III. DIAGONAL COMPONENT OF THE THERMAL CONDUCTIVITY

We calculated numerically σ_{xx}, κ_{xx}, and N_{xx} using Eqs. (12), (5a), and (4c). We have taken $T=0.15$ K. Our results are shown in Fig. 1. In Fig. 1(a) we plot κ_{xx} together with $\sigma_{xx}L_0T$ and N_{xx} versus the magnetic field for large Landau level broadening ($\Gamma_n=6\pi k_BT$). The two curves κ_{xx} and N_{xx} coincide while the two curves representing κ_{xx} and $\sigma_{xx}L_0T$ are very close. In Fig. 1(b) the ratio N_{xx}/κ_{xx} versus the magnetic field for the same Landau-level broadening is plotted. The deviation of this ratio from 1 is almost negligible. In Fig. 1(c) we plot κ_{xx}, $\sigma_{xx}L_0T$, and N_{xx} versus the magnetic field for a smaller Landau level broadening ($\Gamma_n=0.5\pi k_BT$). κ_{xx} and N_{xx} behave similarly showing a ''two- peak'' behavior. This is in accordance with the behavior obtained by Oji[12] and Blanter *et al.*[15] In contrast with Fig. 1(a), $\sigma_{xx}L_0T$ does not follow the κ_{xx} behavior. In Fig. 1(d), the ratio N_{xx}/κ_{xx} versus the magnetic field for the same Landau level broadening deviates from 1.

The deviation from WF law becomes clear looking at the ratio $\sigma_{xx}L_0T/\kappa_{xx}$. The fact that the ratio N_{xx}/κ_{xx} is close to unity shows that the WF law is violated not because of the thermopower [see Eq. (3d)] suggested by Blanter *et al.*,[15] given that $N_{xx}/\sigma_{xx}L_0T$ is not always 1, as Blanter *et al.* accepted, but due to the effect of N_{xx} and specifically of the energy derivatives of the diagonal electrical conductivity [Eqs. (8) and (9)].

When a Landau level is half-filled, the $E-E_n$ term vanishes. Then N_{xx} is given by

$$N_{xx}=L_0T\sum_n (n+\tfrac{1}{2})\frac{e^2}{8\pi\hbar^2}\left[1-\frac{14}{10}\left(\frac{\pi k_BT}{\Gamma_n}\right)^2+\frac{11\,160}{5030}\left(\frac{\pi k_BT}{\Gamma_n}\right)^4-\frac{106\,680}{75\,600}\left(\frac{\pi k_BT}{\Gamma_n}\right)^6\right]. \quad (15)$$

Equation (15) shows that as the broadening becomes smaller, the higher order in $\pi k_BT/\Gamma_n$ correction terms become important. This equation allows us to find a threshold for the validity range of the WF law. For a *Gaussian DOS*, a

$$\Gamma_n>5\pi k_BT \quad (16)$$

gives the first correction term to be only the 5.6% of the main contribution (the WF law), a value well below the usual experimental error.[16]

In Fig. 2 we present N_{xx} versus the magnetic field evaluated both analytically, using Eq. (15), and numerically, from Eq. (4c), for different Landau level broadenings in the range $[6\pi k_BT,\pi k_BT]$. Figure 2(a) (for $\Gamma_n=6\pi k_BT$) shows a very good agreement between the analytical and numerical results achieved taking into account only the main contribution (first term) of the analytical expansion given by Eq. (15). The three correction terms make negligible contributions. In Fig. 2(b) (for $\Gamma_N=3\pi k_BT$) the contribution in the expansion from the first-order correction, $(\pi k_BT/\Gamma_n)^2$, is important and as the value of the Landau level broadening becomes smaller [$\Gamma_N=2\pi k_BT$ in Fig. 2(c) or $\Gamma_N=\pi k_BT$ in Fig. 2(d)] more correction terms have to be taken into account to improve the agreement between the analytical and the numerical results. From Fig. 2(d) it becomes obvious that for small Γ_n even the inclusion of three correction terms is inadequate to approximate the numerical result and consequently many more correction terms are needed.

For a semielliptical DOS,[25]

$$D_n(E)=\frac{1}{2\pi l^2}\sum_n \{1-[(E-E_n)/\Gamma_n]^2\}^{1/2}, \quad (17)$$

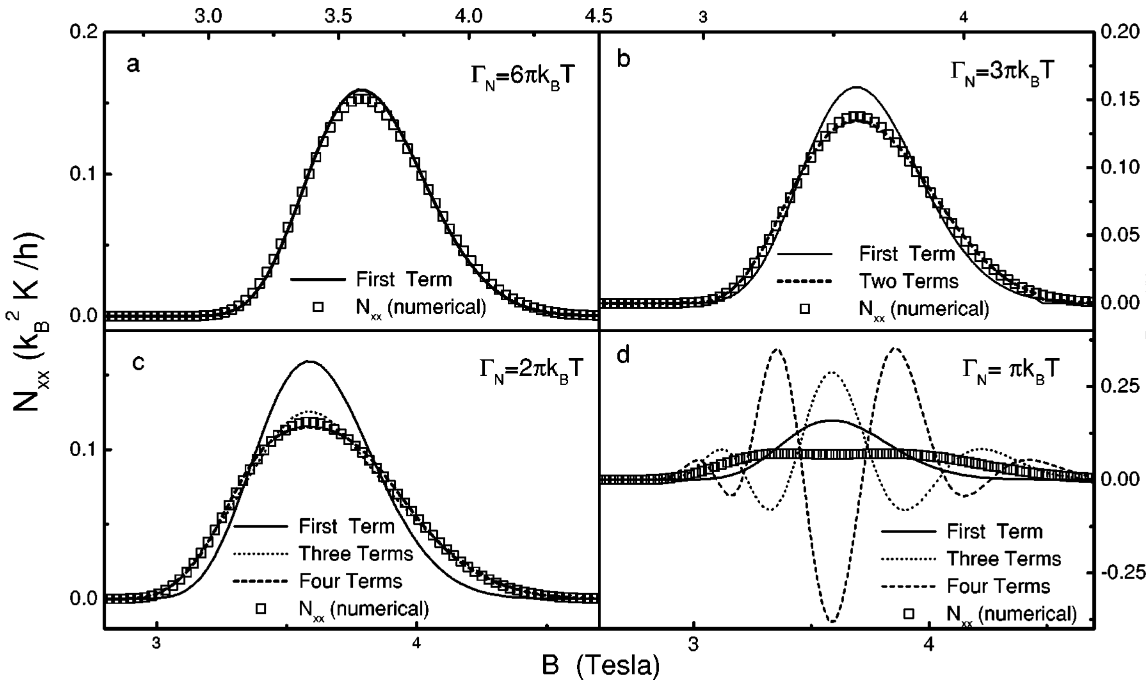

FIG. 2. N_{xx} calculated both numerically and analytically at $T=0.15$ K (with an increasing number of terms with decreasing Landau level broadening) versus the magnetic field for $\Gamma_n=6\pi k_BT$ (a), $\Gamma_n=3\pi k_BT$ (b), $\Gamma_n=2\pi k_BT$ (c), and $\Gamma_n=\pi k_BT$ (d).

$$\frac{d^2\sigma_{xx}(E)}{dE^2}=-\frac{1}{\Gamma_n^2} \quad \text{and} \quad \frac{d^4\sigma_{xx}(E)}{dE^4}=-\frac{3}{\Gamma_n^4}, \tag{18}$$

resulting to correction terms of substantially lower values compared with the corresponding terms for a Gaussian DOS.

Thus, due to the sensitivity of these derivatives on the shape of the DOS, the study of the deviations from the WF law could indicate which type of DOS is the most appropriate.

We are interested in carriers of energy difference k_BT from both sides of the Fermi energy. If the Landau level broadening is quite large, the Gaussian DOS can be approximated as linear in the region $[E_F-k_BT,E_F+k_BT]$ and thus the higher derivatives of the electrical conductivity are not important. For small Landau level broadening, the above approximation fails and the energy derivatives start to play an important role.

IV. NONDIAGONAL COMPONENT OF THE THERMAL CONDUCTIVITY

We have also calculated numerically σ_{xy}, κ_{xy}, and N_{xy} using Eqs. (13), (5b), and (4c) at $T=0.15$ K. Our results are shown in Fig. 3. In Fig. 3(a) we plot κ_{xy}, $\sigma_{xy}L_0T$, and N_{xy} for large Landau level broadening ($\Gamma_n=6\pi k_BT$). The three curves coincide. In Fig. 3(b) for the same broadening we plot the ratio N_{xy}/κ_{xy}. The deviations of this ratio from 1 are almost negligible. In Fig. 3(c) we plot κ_{xy}, $\sigma_{xy}L_0T$, and N_{xy} and in Fig. 3(d) we plot N_{xy}/κ_{xy} and $\sigma_{xy}L_0T/\kappa_{xy}$ for small Landau level broadening ($\Gamma_n=0.5\pi k_BT$). The TC follows roughly the behavior of N_{xy}.

Thus, it is clear that the coefficient N_{xy} is responsible for the resulted smaller deviations of the nondiagonal components from the WF law compared with the diagonal. The contribution of the thermopower is noticeable only near the center of the Landau level.

Using the Sommerfeld expansion and taking into account only the first two terms, we obtain

$$N_{xy}=\sigma_{xy}L_0T+\frac{14}{10B(\pi k_B)(2\pi l^2)\sqrt{2\pi}} \times\left(\frac{\pi k_BT}{\Gamma_n}\right)^3 L_0e(E-E_n)e^{-(E-E_n)^2/\Gamma_n^2}. \tag{19}$$

Away from the middle of the Landau level, the exponential dependence on $(E-E_n)$ diminishes N_{xy} strongly. Near the middle of the Landau level is the linear term $E-E_n$, which makes the contribution of this correction term unimportant. Thus, it is the interplay of these two terms that results in the validity of the WF law even for small values of the Landau level broadening. In contrast with Fig. 3(a), in Fig. 3(c) $\sigma_{xy}L_0T$ does not follow the κ_{xy} behavior. The larger deviations appear half-way from the middle of the Landau level. This behavior is consistent with the preceding analysis. In Fig. 3(d) the ratios N_{xy}/κ_{xy} and $\sigma_{xy}L_0T/\kappa_{xy}$ versus the magnetic field for the same Landau-level broadening are close to 1 for the whole range of the magnetic field. It is shown that $\sigma_{xy}L_0T/\kappa_{xy}$ deviates stronger from 1 than the ratio N_{xy}/κ_{xy}.

The above analysis shows that the WF law is valid for a larger extent of the Landau level broadening in the case of the nondiagonal TC in contrast with the diagonal components. The form and the magnitude of the N_{xy} variation compared with κ_{xy} and the corresponding behavior of $\sigma_{xy}L_0T$ indicates that N_{xy} is responsible for the appearance of the plateaulike feature in κ_{xy}, [Fig. 3(c)] with a small contribution due to the thermopower. In the nondiagonal case we are interested in lower order energy derivatives of the DOS than in the diagonal. Thus, the result is less sensitive in the shape of the DOS than before. We have to de-

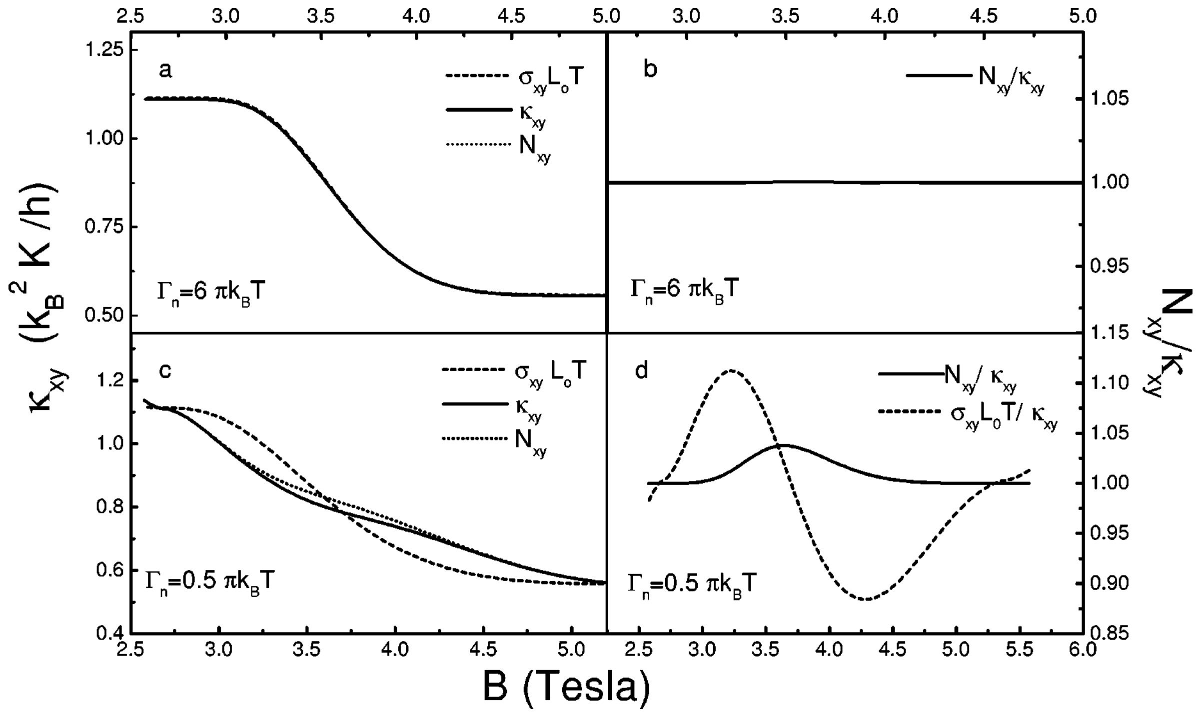

FIG. 3. κ_{xy}, $\sigma_{xy}L_0T$, and N_{xy} versus the magnetic field at $T=0.15$ K, $\Gamma_n=6\pi k_BT$ (a), and $\Gamma_n=0.5\pi k_BT$ (c). The ratio N_{xy}/κ_{xy} for $\Gamma_n=6\pi k_BT$ (b) and the ratios N_{xy}/κ_{xy} and $\sigma_{xy}L_0T/\kappa_{xy}$ are also presented for $\Gamma_n=0.5\pi k_BT$ (d).

crease even more the Landau level broadening to obtain noticeable deviations from the WF law.

The different behavior of the κ_{xx} and κ_{xy} is not surprising. The physical reason lies in the number of the electrons that take part in the energy transport. In contrast with κ_{xx}, where we are interested only in electrons with energy near the Fermi energy, in κ_{xy} we are interested in the total number of electrons. Thus the change of the broadening has less of an effect on the mean electron energy and thus on the energy transported than for the diagonal case.

V. CONCLUSIONS

We have evaluated the diagonal and nondiagonal components of the transport coefficient tensors, σ_{ij}, κ_{ij}, and N_{ij}, for a 2DEG in a quantizing magnetic field to study the validity of the WF law in the QHE regime. The systematic analysis, analytical and numerical, shows that for the diagonal component of the TC the WF law is violated with decreasing Landau level broadening. This is also confirmed by the fact that as the value of the Landau level broadening becomes smaller, more correction terms in the analytical expressions have to be taken into account to improve the agreement between the analytical and the numerical results. For the nondiagonal components we obtain considerably smaller deviations of the WF law. Responsible for the resulting behavior is the coefficient N_{ij} and specifically the effect of the energy derivatives of the electrical conductivity and consequently the shape of the DOS. The experimental investigation of these results will be of great interest.

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