Quantum dynamics of two-dimensional vortex pairs with arbitrary total vorticity

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Quantum dynamics of a vortex pair is investigated by considering the pair Hamiltonian within various, unequivalent algebraic frameworks. First the vortex pair spectrum is constructed in the standard contest of the $e(2)$-like dynamical symmetry and its degeneracy is thoroughly examined. Then Berry's phase phenomenon is studied through an su(1,1) realization of the pair Hamiltonian when its parameters are assumed to be time dependent, whereas the Feynman-Onsager quantization conditions are recovered by means of symmetry arguments within a third approach based on a magneticlike description of the vortex pair. Finally, it is shown how recasting the dynamical algebra in terms of two-particle realizations of both su(2) and su(1,1) provides the correct approach for the quantization of the model Hamiltonian accounting for the pair scattering from a disklike obstacle. [S0163-1829(99)09109-2]

I. INTRODUCTION

An attempt to investigate the quantum dynamics (QD) of superfluid vortices was performed by Fetter in Ref. 1. He considered a three-dimensional (3D) vortex model where the vorticity field is nonzero on an array of parallel strings and assumed that the vortex interactions were dominated by transversal string oscillations. The crucial feature is that the pair of coordinates involved by the local description of each point of a vortex line are canonically conjugate allowed to quantize the system via the canonical procedure.

The complexity of the problems inherent in the quantization of the vortex field theory together with their extreme formal character[2] have certainly contributed to discourage, for a long time, the investigation of the quantum aspects of vortex dynamics in the context of condensed-matter physics.

The only exception to this statement[3] is represented by the special case of 2D pointlike massless vortices, which are derived from the standard field theory of 2D ideal fluids as the extreme case where the vorticity field is restricted to a set of isolated points.

Before discussing the motivations that prompt further investigations of VQD it is interesting to review the work devoted to 2D vortex dynamics during the past two decades. The attention raised by its quantum version is principally due to the vexed question whether pointlike vortices might exhibit, when dynamically quantized, statistics with a fractional character. A thorough account of this problem can be found in Refs. 4–7 whereas in Ref. 8 an effective theory of 2D quantum vortices on a Josephson junctions array is worked out from the quantum-phase model Hamiltonian. Further contributions aimed at investigating 2D dynamically quantized vortices concern the influence of the boundary effects on VQD,[9] the ordering problem arising from the quantization process,[10] the possibility that the vortex pair inherits anyonlike statistics when its dynamics is issued from that of two charges in a transverse magnetic field,[11] and the emergence of pointlike vortices with quantized dynamical degrees of freedom from a second quantized $|\Psi|^4$-field theory.[12]

Surprisingly, apart from recent developments, such a condensed survey represents more or less the whole work devoted to the 2D VQD in the past 15 years, despite the huge quantity of theoretical work concentrated, during the same period, on the 2D vortex topic in relation to type-II superconductor physics,[13] the Kosterlitz-Thouless transition theory,[14,15] and the investigations through Feynman's variational approach[16] on vortex emergence from the native superfluid background.

A renewed interest for 2D vortex dynamics has been prompted by the recent experimental developments in the context of both superfluidity and superconductivity.[17] The great improvements concerning the measurement techniques and the observations of microscopic processes should render quantum aspects of vortex dynamics viable to experimental detection.

There are at least four experimental situations that are interesting in respect to VQD. For example, devices where thin films of superfluid ^{4}He are adsorbed on porous materials such as vycor glasses. The porous structure endows ^{4}He films with the multiply connected geometry of a Riemann surface thus confining the vortex gas of superfluid films on a network of 2D cylinderlike connected domains (the surface handles) with very small sizes.[18] The fact that the special geometry makes vortices interacting at the mesoscopic scales of the vycor structure is expected to emphasize the effect of quantizing vortex interactions. This, in fact, as entailed by the standard form of the vortex Hamiltonian, depends on the distance between vortex positions which is the quantity that must be quantized. The depicted scenery is further complicated by the effect of the curvature that, as in the case when boundaries confine the fluid, introduces nonlinear terms in the energy that represent virtual vortex contributions.[9]

The second situation where vortex structures are important is the superconductor physics. Point vortices (vortex lines) arising in 2D (3D) superconductors strongly influence conductance measurements of the current. The presence in the medium of pinning centers makes them undergo quantum tunneling phenomena that strongly affect vortex movements through the medium.[19] The ensuing liquidlike behavior of the vortex system affects the supercurrent decay by inducing voltage variations,[20] but it depends, in turn, on interactions among vortices as well as on interactions of vortices both with the medium impurities and with the medium boundaries. Similar effects are observed in superfluid currents in a ring, while vortex creation phenomena, possibly due to the quantum tunneling effect, occur when superflows cross a microscopic orifice.[21]

Finally, vortex dynamics is largely studied in the Josephson-junction arrays,[22] where experiments reveal a rich scenery of phenomena such as high-energetic vortices exhibiting ballistic motion, vortex-driven voltage turbulence, and vortex lattice melting.

A characteristic feature of the literature on VQD is the fact that the attention has been mainly concentrated on the pairs of identical vortices, namely vortex-vortex (VV) pairs as well as antivortex-antivortex (AA) pairs. Their deep quantum nature ensuing from the fact that both VV and AA interactions involve quantized intervortex distances[4] has originated the idea of characterizing a pair of identical vortices (and, more in general, a gas of identical vortices) by fractional statistics. Concerning this point VA pairs were considered uninteresting since no exotic statistics is expected from distinguishable objects such as vortices and antivortices. In addition to this, VA pairs arising in physical systems exhibit a vortex charge which is equal and opposite to the antivortex charge. This fact implies that the VA distance is not quantized[9] thus making their dynamics apparently trivial.

VA pairs are instead basic for understanding the statistical properties of excited superfluids. These are achieved via mean-field techniques that seem to entail an intrinsically classic scenery. The standard renormalization procedure for obtaining scaling laws is illustrative of this when the habitual assumption is made to consider each VA pair of the vortex gas as immersed in a dielectric background of VA pairs with smaller size.[15] This implies that both VV and AA interactions (the very quantum ones) are embodied, and thus averaged, in the dielectric constant. Since the VA pair Hamiltonian possesses a sort of natural, semiclassical structure in that it does not contains any noncommuting quantities, then quantum effects appear to be a higher-order refinement.

As to the emergence of vortex pairs we wish to recall the main traits of such a phenomenon within superfluid condensates and superconductors. The neutrality characterizing both of them at low temperature favors the occurrence of VA pairs when temperature is raised. These, in fact, violate the neutrality just locally and are energetically favored. On the contrary individual VV and AA pairs represent disfavored excited states since they entail local accumulation of vorticity and, for this reason, a higher energy cost. It follows that AA and VV interactions concern more frequently interactions of identical vortices of different VA pairs rather than pairs of identical vortices. Interactions of identical vortices become dominating in superconductors when an external magnetic field is switched on. This is able to break the vortex charge neutrality and allows for vortex arrays where the vortex density is greater than the antivortex one. The decreasing of the antivortex fraction is compensated by the magnetic field that can be viewed as a macroscopic background charge. The same effect is achieved in superfluids when the condensate undergoes a uniform rotation.[13]

The apparent nonquantum character of VA pairs above mentioned, disappears as soon as one goes beyond the simplified scenery of the mean-field picture. Indeed the VA dynamics is as complex as the VV dynamics when any elementary effect (e.g., the boundary effects) coupling vortices with the environment is taken into account.[13,23] This is further confirmed, at the classical level, when the substrate effects on vortex dynamics are inserted either via an effective potential,[24] or by the explicit introduction of rigid obstacles in the fluid.[25] A possible, unexpected effect is that of providing the customary continuum spectrum of the VA pair, with a discrete character.[9]

The purpose of this paper is to provide a complete treatment of the vortex pair QD both for the VV case and for the VA case (the AA case and the VV case are easily shown to be equivalent) and of attracting the attention on the rich structure characterizing their quantization from the group-theoretic viewpoint in relation to possible applications.

VQD is studied by means of the spectrum generating algebra method[26] which consists essentially in identifying first a complete set of dynamical degrees of freedom forming a Lie algebra (the dynamical algebra) and in constructing then the Hilbert space of the system by exploiting the unitary irreducible representations of the related Lie group. In this sense the vortex system is quite interesting due to the rich variety of ways in which the vortex coordinates can be structured so as to form a dynamical algebra. Remarkably, the possibility of working within different algebraic schemes does not reduce to the mere freedom of choosing among different formal approaches trivially equivalent to each other. Each algebraic scheme, in fact, sheds light on some particular feature of the system.

Classical dynamics of N pointlike vortices is reviewed in Sec. II together with the symmetries characterizing their motion described by the habitual $E(2)$-like symmetry group. The procedure adopted for quantizing the N vortex gas and hence the vortex pair is the standard canonical scheme,[4,10] concerning the pairs of canonically conjugate coordinates assigned to the points where the vorticity field is nonzero. Moreover, the vortex pair dynamics is pictured through a significant geometric form suitable for interpreting the quantum spectra.

In Sec. III the QD of two interacting vortices is examined in the case when the topological charges, namely the vorticities k_1 and k_2 of pointlike vortices are arbitrary. Its integrable character is clearly manifested by the fact that the pair Hamiltonian is a function of the Casimir operator of the $E(2)$-like dynamical algebra. After working out explicity both the eigenstates and the eigenvalues of the energy spectrum, the degeneration due to the high symmetry of the vortex pair dynamics is analyzed. We show how a complete set of eigenfunctions can be worked out thanks to the possibility of observing one among the numerous constants of motion of the system. As in the case of the Landau spectrum for the electrons acted by a transverse magnetic field, a further quantum number describing the system symmetry must be introduced in addition to that describing the pair energy.

In Sec. IV the dynamical algebra is assumed to be su(1,1). This leads to reveal unexpected symmetry properties relating dynamics of vortex pairs with different topological charges, and provides the framework for evaluating Berry's phase when k_j and the density ρ are possibly time dependent. Recasting instead the pair Hamiltonian in the magneticlike form of Sec. V allows one to derive the Feynman-Onsager condition,[27,28] on the jth topological charges $k_j \equiv h n_j/m$, where $j=1,2$, and m is the helium atom mass, from pure symmetry considerations.

Finally, Sec. VI is devoted to introducing a two-boson dynamical algebra which, in addition to give a further insight on the specific features both of the VA case and of the VV case, provides the correct way to approach the pair dynamics when a disklike obstacle is present in the fluid. Our algebraic construction reformulates the resulting disk-pair model in terms of a generalized angular momentum dynamics, leads to a clear geometric picture of the pair-obstacle interaction, provides the Hamiltonian in a form exempt from the habitual ordering problem, and establishes the basis for investigating the quantum processes. The interest for such a case is founded as well on the fact that the pair scattering phenomena can be interpreted as pinning effects on vortices due to the impurities of superfluid substrate.

A final comment is in order as to considering generic topological charges k_j, despite the standard choice used in the literature $|n_j|=1$. The motivations are at least two. First of all, employing arbitrary charges k_1 and k_2 involves no formal complication, whereas it allows one to gain an interesting general insight when relating the algebraic structures pertaining to the VV case to those of VA case. Second, the emergence of vortices with $|n_j|>1$ can take place in sufficiently excited superfluid media as well as in close proximity of the medium boundary.[23] Such a situation is certainly interesting since a vortex pair with $|n_1|,|n_2|>1$ provides the basic framework in which to investigate the exchange of *quanta* of vorticity between the pair members via quantum tunneling processes.

II. CANONICAL QUANTIZATION OF THE 2D VORTEX GAS DYNAMICS

Classical dynamics of N pointlike vortices in a frictionless fluid is described by the Hamiltonian[29]

$$H(\mathbf{R_1},\ldots,\mathbf{R_n}) = -\frac{\rho}{4\pi}\sum_{i\neq j} k_i k_j \ln\left(\frac{|\mathbf{R_i}-\mathbf{R_j}|}{a}\right), \tag{1}$$

where the parameter k_j represents the vorticity carried by the jth vortex, ρ is the fluid planar density, and the vector $\mathbf{R_j} = (x_j, y_j)$ describes the jth vortex position in the 2D ambient space in terms of planar coordinates x_j, y_j. The length a represents the vortex core size that is the minimum distance allowed between a vortex ($k_j>0$) and an antivortex ($k_j<0$) before coalescence processes take place. The Hamiltonian equations relative to Eq. (1) are standardly derived via the Poisson brackets[30]

$$\{F,G\} = \sum_j \frac{1}{\rho k_j}\left(\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y_j} - \frac{\partial G}{\partial x_j}\frac{\partial F}{\partial y_j}\right), \tag{2}$$

involving, in turn, the ρk_j-dependent canonical brackets

$$\{x_i, y_j\} = \frac{\delta_{ij}}{\rho k_j}. \tag{3}$$

Then vortex coordinates can be regarded as a complete set of canonically conjugate variables whose *momenta* are defined as $p_j = \rho k_j y_j$. Also, one can easily check that the functions

$$J_z = -\frac{\rho}{2}\sum_m k_m (x_m^2 + y_m^2),$$

$$J_x = \rho \sum_m k_m x_m, \tag{4}$$

$$J_y = \rho \sum_m k_m y_m,$$

$$J_* = \frac{\rho}{2}\sum_{i\neq j} k_i k_j [(x_i - x_j)^2 + (y_i - y_j)^2], \tag{5}$$

fulfilling the equation

$$-\rho J_* = 2 C J_z + J_x^2 + J_y^2, \tag{6}$$

where $C = \rho \Sigma_j k_j$ is related to the total vorticity, are constants of motion and satisfy the classical commutators ($a=x,y,z$)

$$\{J_a, J_*\} = 0, \quad \{H, J_a\} = \{H, J_*\} = 0, \tag{7}$$

$$\{J_x, J_y\} = C, \quad \{J_z, J_x\} = J_y, \quad \{J_y, J_z\} = J_x. \tag{8}$$

It is worth noticing that J_x, J_y, J_z exhibit an $e(2)$-like algebraic structure—hereafter we shall denote it by $e_*(2)$—which is fully reached when $C=0$, namely when the vortex total charge equals the antivortex total charge.

The canonical quantum description of N pointlike vortices is obtained by replacing classical commutators (3) with

$$[x_i, p_j] = \delta_{ij} i\hbar, \tag{9}$$

(we have set $p_j = \rho k_j y_j$ in order to get commutators in the canonical form) that furnishes the quantum version of the algebra (7)

$$[J_x, J_y] = i\hbar C, \quad [J_z, J_x] = i\hbar J_y, \quad [J_z, J_y] = i\hbar J_x. \tag{10}$$

In view of the fact that the constants of motion can be employed for integrating the dynamical equation provided they are in involution, commutators (9) involve that the many-body wave function for the 2D vortex gas is characterized at most by three macroscopic quantum numbers two of which are, of course, H and J_*, while the third one can be arbitrarily chosen among J_x, J_y, and J_z.

A meaningful geometric picture of the system dynamics is achieved by means of such constants of motion when the vortex pair is considered. To this end it is useful to describe the vortex pair through the new set of coordinates

$$x \doteq x_1 - x_2, \quad y \doteq y_1 - y_2, \tag{11}$$

$$X \doteq J_x / C, \quad Y \doteq J_y / C, \tag{12}$$

where $J_x = \rho(k_1 x_1 + k_2 x_2)$ and $J_y = \rho(k_1 y_1 + k_2 y_2)$ follow from Eqs. (4), that reduce the Casimir function (5) to the form

$$J_* = \rho k_1 k_2 (x^2 + y^2). \tag{13}$$

Then, after expressing the coordinates x_j, y_j as

$$x_1=\frac{1}{C}(J_x+\rho k_2 x),\quad y_1=\frac{1}{C}(J_y+\rho k_2 y),$$
$$x_2=\frac{1}{C}(J_x-\rho k_1 x),\quad y_2=\frac{1}{C}(J_y-\rho k_1 y),\tag{14}$$

by means of Eqs. (11) and (12), it is quite easy to recast J_* in the two equivalent forms

$$\frac{J_*}{k_1k_2\rho}=\frac{C^2}{\rho^2k_2^2}\left[\left(x_1-\frac{J_x}{C}\right)^2+\left(y_1-\frac{J_y}{C}\right)^2\right],\tag{15}$$

$$\frac{J_*}{k_1k_2\rho}=\frac{C^2}{\rho^2k_1^2}\left[\left(x_2-\frac{J_x}{C}\right)^2+\left(y_2-\frac{J_y}{C}\right)^2\right],\tag{16}$$

which, without solving the equations of motion, identify completely the classical orbits where the two vortices move along. Such orbits—they are easily recognized to be circumferences—present as a common center the vorticity center $\mathbf{R}_*=(J_x/C,J_y/C)$ and have radii

$$\mathcal{R}_1=\frac{|k_2|}{|k_1+k_2|}\sqrt{\frac{J_*}{k_1k_2\rho}},\tag{17}$$

$$\mathcal{R}_2=\frac{|k_1|}{|k_1+k_2|}\sqrt{\frac{J_*}{k_1k_2\rho}},$$

for the vortex with charge k_1 and k_2, respectively. $\mathbf{R}_*$, $\mathcal{R}_1$, and $\mathcal{R}_2$ are manifestly time-independent quantities in that they just depend on the constants of motion J_x,J_y,J_*.

For $k_2,k_1>0$ vortices rotate in such a way that a common straight line always join them to $\mathbf{R}_*$. The latter, in particular, coincides with the rotation center which is situated between the two vortices. When $k_2\to k_1$ the circumferences merge in one whose center is yet $\mathbf{R}_*$. On the other hand, for $k_2\to 0$ the radius $\mathcal{R}_1$ vanishes so that the vortex with finite vorticity k_1 fall in $\mathbf{R}_*$ thus losing any dynamical role. In this case the weak vortex—that with $k_2\simeq 0$—ends up by running along on a limiting circle with radius $\mathcal{R}_2\to$const. When $k_2\to 0$ from negative values the dynamical situation is almost the same except for the fact that now the vortices stay along a rotating half-line whose extreme is attached to $\mathbf{R}_*$. The emergence of the full VA regime is announced when, for $k_2\to -k_1$, the center $\mathbf{R}_*$ moves away from the vortices and gets a larger and larger distance from them. In the limiting case of the pair with $k_2\equiv -k_1$ the two vortices run along parallel straight lines (i.e., circles with infinitely large radii) and keep a constant relative distance. One easily checks that $\mathcal{R}_1,\mathcal{R}_2\to\infty$.

III. DEGENERACY OF THE PAIR ENERGY SPECTRUM

Quantizing the two-vortex system seems no more complex than quantizing a simple harmonic oscillator (HO)[4,10] even when the topological charges k_1, k_2 of the interacting vortices are arbitrary. In this case, in fact, Hamiltonian (1) reduces to a single logarithmic term whose argument is $|\mathbf{R_1}-\mathbf{R_2}|^2$, while Eq. (5), whereby J_* takes the form $J_*=|\mathbf{R_1}-\mathbf{R_2}|^2/\rho k_1k_2$, entails H written as

$$H(\mathbf{R_1},\mathbf{R_2})=-\frac{\rho}{4\pi}k_1k_2\ln\left(\frac{J_*}{k_1k_2\rho a^2}\right).\tag{18}$$

This feature is specific to the two-body problem and implies that the set of energy eigenvectors exactly coincides with the J_* spectrum. A remarkable freedom is then permitted in selecting the remaining quantum number which labels the degeneracy of the energy states. In fact, any operator of the form $I=aJ_x+bJ_y+cJ_z$ fulfills the equation $[H,I]=0$ establishing the constant of motion status of I. Anyway, a deeper inspection reveals that any invariant $I(a,b,c)$ is obtained either from J_z or from J_y via the transformation $I=gI_0g^{-1}$, $I_0=J_y,J_z$, where g is a unitary transformation obtained by combining appropriately the action of $D_x(\alpha)=e^{i\alpha J_x}$, $D_y(\beta)=e^{i\beta J_y}$, and $D_z(\phi)=e^{i\phi J_z}$. The possibility to reconstruct the algebra of the $E_*(2)$ group (the symmetry group of H) from the elements J_z and J_y, representative of the algebra disjoint sectors, via the adjoint action map, entails two possible pictures of the degeneracy. In this section we examine the vortex pair spectrum relative to two such ways to structure the energy-level degeneracy.

To begin with we assume that the topological charges fulfill the inequalities $k_1>0$, $0\leqslant|k_2|\leqslant k_1$ and notice how the ranges allowed are capable of describing any possible pair. Also, let us introduce the compact notation $\mathbf{D}=\mathbf{R_1}-\mathbf{R_2}$, and express it by means of the set of canonical conjugate variables [see Eqs. (11), (12)]

$$X=J_x/C,\quad x=x_1-x_2,$$
$$P=J_y,\quad p=\frac{k_1k_2\rho}{k_1+k_2}y,\tag{19}$$

whose momenta p and P satisfy the commutators $[X,p]=[x,P]=0$ and $[x,p]=[X,P]=i\hbar$. By using such variables, the logarithm argument is easily turned into the HO form

$$\mathbf{D}^2=y^2+x^2=\frac{1}{w^2}(p^2+w^2x^2),\tag{20}$$

where the frequency w reads $w=k_1k_2\rho/(k_1+k_2)$. This implicates that the wave functions

$$\Psi_n(x;|w|)=\frac{1}{\sqrt{2^n\pi n!\,l}}\,e^{-x^2/2l^2}H_n(x/l),\tag{21}$$

where $l^2=\hbar/|w|$, that satisfies the secular equation $(p^2+w^2x^2)\Psi_n=\hbar|w|(2n+1)\Psi_n$, represent the eigenfunctions of $\mathbf{D}^2$ with eigenvalues given by

$$S_n(w)=(\hbar/|w|)(2n+1).\tag{22}$$

A complete set of eigenfunctions $\Psi_{nK}(x,X)$ is finally obtained when the further observable $P=J_y$ is considered together with $\mathbf{D}^2$. The second quantum number K in

$$\Psi_{nK}(x,X)\doteq\Psi_n(x)\Phi_K(X),\tag{23}$$

where the plane wave $\Phi_K(X)\doteq e^{iKX}/\sqrt{2\pi}$ fulfills the equation $P\Phi_K(X)=\hbar K\Phi_K(X)$, establishes the position $Y=P/C$ of the center of vorticity along the y axis in the ambient space.

As an alternative, a complete set of eigenstates can be constructed by resorting to the quantum number related to the conserved quantity J_z. This is easily achieved by exploiting Eq. (6) in that, after turning it to the form $J_z=-(\rho J_* + J_x^2+J_y^2)/2C$, it is evident that the HO-like wave function

$$\Psi_m(X;|C|)\equiv\frac{1}{\sqrt{2^m\pi m!L}}\,e^{-X^2/2L^2}H_m(X/L), \qquad (24)$$

where $L=\sqrt{\hbar/|C|}$, diagonalizes $J_x^2+J_y^2=P^2+C^2X^2$. Hence the eigenvalues associated with J_z have the form

$$\Lambda_m(n;w)=-\hbar[\mathrm{sg}(w)n+m+1], \qquad (25)$$

where $\mathrm{sg}(w)\equiv w/|w|$ and we have exploited the fact that $\rho J_*/C=\hbar(2n+1)w/|w|$ and $C=\rho(k_1+k_2)>0$, due to our initial assumptions.

The descriptions of the energy spectrum degeneracy just examined, involve a significant geometric-quantum picture. Using the quantum numbers (n,K) implies that the two vortices are confined along the two circumferences (15), (16)—the angular coordinates cannot be specified due to the uncertainty principle—whose radii

$$\mathcal{R}_j=\frac{|k_1k_2|}{|k_j||C|}\,\sqrt{S_n(w)}, \qquad (26)$$

where $j=1,2$, are now labeled by the integer n due to Eq. (22). The center of such circumferences, which coincides with the vorticity center, has $Y=\hbar K/C$ while X is, as expected, undetermined.

On the other hand, when the pair (n,m) is employed the *locus* allowed for the vorticity center changes from noncompact to compact. The latter, in fact, is now confined on the circle of radius

$$X^2+Y^2=[\hbar(2m+1)/|C|]^{1/2},$$

labeled by m, instead of a straight line labeled by K. Once again the uncertainty principle prevents one from getting any further information both on the position of the vorticity center and on the vortex position along the circles of radii $\mathcal{R}_1$ and $\mathcal{R}_2$.

In the extreme case when $C=k_2+k_1\to 0$ the canonical scheme based on Eqs. (19) breaks down due to the divergence of the factor $1/(k_1+k_2)$. In particular, the singular behavior of the pure VA case is distinguished by the fact that J_x and J_y end up by coinciding with x and y, respectively, which now commute since $\rho k_1[x,y]\equiv[J_x,J_y]=i\hbar C\equiv 0$. Such circumstances impose the introduction of a more appropriate set of canonically conjugate variables. We thus define

$$\eta=-\rho kx, \quad \xi=\rho ky,$$
$$\mathcal{Y}=\frac{1}{2}(y_1+y_2), \quad \mathcal{X}=\frac{1}{2}(x_1+x_2), \qquad (27)$$

where $k_2=-k_1=-k$, which turns out to be completely disjoint from those employed in the case when $k_2\neq -k_1$, and obeys the standard relations $[\mathcal{X},\xi]=[\mathcal{Y},\eta]=i\hbar$. Here $\mathcal{X}$, $\mathcal{Y}$, ξ, and η do not play prefixed roles so that, depending on the interactions involved by the dynamical problem, they can be regarded either as *momenta* or as position variables. When further interactions are excluded from H, the simplest choice is that where $\mathcal{X}$ and $\mathcal{Y}$ are looked upon as coordinates which implies that the energy eigenfunctions have the form

$$\Phi_{\mathbf{K}}(\mathcal{X},\mathcal{Y})=\frac{1}{2\pi}\,e^{i(\mathcal{X}K_{\mathcal{X}}+\mathcal{Y}K_{\mathcal{Y}})}. \qquad (28)$$

Here $\hbar K_{\mathcal{X}}$ and $\hbar K_{\mathcal{Y}}$ are the eigenvalues of $\xi=-i\hbar\partial_{\mathcal{X}}$ and $\eta=-i\hbar\partial_{\mathcal{Y}}$, respectively. Information on $\mathcal{X}$ and $\mathcal{Y}$ are, of course, completely missing, that is, the pair cannot localize anywhere in the ambient space.

Some applications can be now illustrated. The quantum-mechanical problem just solved provides the formal tools requested for investigating the scattering processes of the VV pair (as well as of the AA pair) and VA pair dynamics in the presence of 2D potential wells simulating the confining action of the defects placed on the superfluid medium substrate.[24] The effective model Hamiltonian

$$\mathcal{H}=\frac{g}{4}[(x_1+x_2)^2+(y_1+y_2)^2]+H,$$

where g represents the strength of the phenomenological confining action, for $k_1=+k_2=k$ and $k_1=-k_2=k$ yields the Hamiltonian

$$\mathcal{H}=-\frac{g}{\rho k}(J_z+J_*/4k)-\frac{\rho}{4\pi}k^2\ln\left[\frac{J_*}{k^2\rho a^2}\right], \qquad (29)$$

and

$$\mathcal{H}=g(\mathcal{X}^2+\mathcal{Y}^2)+\frac{\rho}{4\pi}k^2\ln\left[\frac{\xi^2+\eta^2}{(\rho ka)^2}\right], \qquad (30)$$

respectively. In view of the spectral problems solved above, one finds that Hamiltonian (29) has a spectrum which is readily obtained from Eqs. (22) and (25), whereas Hamiltonian (30) is clearly related to a 2D Coulomb problem where two particles with opposite charges are studied within the center-of-mass reference frame and are endowed with a reduced mass $\mu=\rho^2k^2/2g$. Notice how it is now natural to exchange the roles of $\mathcal{X}$, $\mathcal{Y}$, and ξ, η assigned above. The treatment of such a Coulomb system will be reconsidered in Sec. VI where a more adequate algebraic scheme will be introduced.

More in general, Hamiltonians of the form $\mathcal{H}=f(I)+H$, where $f(I)$ is a generic function of the operator $I(a,b,c)$ defined above, are easily diagonalized by reducing I either to J_z or to J_y, depending on the values taken by a, b, and c. For example, the situations where $\mathcal{H}$ exhibits a term I proportional either to J_z, or to J_x (J_y) can be interpreted as the way to picture the effect of a macroscopic velocity field responsible for a uniform rotation around the plane origin, in the first case, and inducing the vortex dragging along the y (x) axis, in the second case.

A final comment is in order as to the two limiting cases $k_1\to\infty$ with finite k_2 (the vortex with k_1 recover the classical status since $[x_1,y_1]$ is vanishing), and $k_2\to 0$ with finite k_1 (the vortex with k_2 gets an ultraquantum status since $[x_2,y_2]\to\infty$). At the classical level such limits make the k_1 vortex tend to stillness, while the k_2 vortex goes on to rotate with a frequency proportional to $w \to \rho k_2$. In particular a greater and greater period occurs when $k_2 \to 0$. While the realism of a situation where $k_2 \simeq 0$ is hard to maintain, since there is no experimental evidence of fractional quanta of vorticity, the case when k_1 is large can be easily interpreted as the situation where a small cluster of vortices interacts with solitary vortices. Moreover, while the first case implies diverging spectrum gaps $S_{n+1} - S_n$ due to Eq. (22), the second one leads to $S_n = \hbar(2n+1)/\rho k_2$. In particular $k_1 \to \infty$ is related to the dynamics of a solitary vortex moving around a disklike obstruction with radius R contained inside the 2D ambient space. Hamiltonian H_D of Appendix A embodying the effects of the disk is illustrative of the manner in which this is realized when the vortex complex coordinate z_1 is such that $|z_1| \simeq R$.

IV. THE su(1,1) APPROACH

A different approach to the quantization of the vortex pair is provided by the xp realization of the algebra su(1,1) which allows one to regard $\mathbf{D}^2 = |\mathbf{R_1} - \mathbf{R_2}|^2$ as the compact generator of su(1,1). After recalling that the algebra generators are given by[31]

$$J_1 = \frac{w^2 x^2 - p^2}{4\hbar |w|}, \quad J_2 = -\frac{px + xp}{4\hbar}, \quad J_3 = \frac{w^2 x^2 + p^2}{4\hbar |w|},$$

and satisfy the commutation relations

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2, \tag{31}$$

the equation $\mathbf{D}^2 \equiv 4\hbar J_3/|w|$ ensuing from Eq. (20) shows that the spectrum of J_3 is that involved by the vortex dynamics. It shows as well how $\mathbf{D}^2$ no longer plays the role of the Casimir operator now being nontrivially acted both by J_1 and by J_2. In spite of this the present algebraic framework inherits both X and $P = CY$ (and hence any function depending on them) as dynamical constants of motion from the $e_*(2)$ algebra of Sec. III. As for Hamiltonians (29), (30), again one can take advantage of this fact for constructing vortex models with Hamiltonian of the form $\mathcal{H} = F(X, Y) + H$ accounting for the background medium influence. For example, describing the drag action on the pair vorticity center due to the flow stream lines in the presence of a saddle point simply requires $F(X,Y) = \gamma(X^2 - Y^2)$, γ being some suitable dimensional parameter.

The su(1,1) scheme is useful to discuss the statistics of the pair system. In Ref. 6 the same algebra was constructed step by step starting from the Weyl-Heisenberg algebra $\{\mathbf{I}, x, p\}$, in order to relate the VV pair statistics to the unitary irreducible representations (UIR) of the xp realization. A direct way to obtain them is that of calculating the eigenvalues of the su(1,1)-Casimir operator

$$C = {J_3}^2 - {J_1}^2 - {J_2}^2 = l(l+1)\mathbf{I},$$

where $\mathbf{I}$ is the identity operator. One easily finds that the allowed values for l are $l = -1/4$ and $l = -3/4$ which select two UIR's in the set of the SU(1,1) supplementary series. The solutions of the secular equation $J_3 f_\nu(x;l) = \nu f_\nu(x;l)$ (see Ref. 32) read

$$f_\nu(x;l) = (-)^s D_{ls} \left(\frac{2}{l}\right)^{1/2} \left(\frac{x}{l}\right)^{\alpha + 1/2} e^{-x^2/2l^2} L_s^\alpha(x^2/l^2),$$

where l is the same dimensional parameter employed in Eq. (21), $D_{ls} = [s!/\Gamma(s-2l)]^{1/2}$ is the normalization factor and L_s^α are the Laguerre polynomials, whereas α and the nonnegative integer s are related to l and ν by $\alpha = -(2l+1)$ and $\nu = s - l$, respectively. By exploiting the general formula[33]

$$L_s^\alpha(z^2) = \frac{(-)^s H_{n(\alpha)}(z)}{2^{n(\alpha)} s! z^{\alpha + 1/2}},$$

where $n(\alpha) = 2s + \alpha + 1/2$, relating Laguerre polynomials to Hermite polynomials $H_{n(\alpha)}$ when $\alpha = \pm 1/2$, one finds that $f_\nu(x;l) \equiv \Psi_n(x;|w|)$ namely the functions (21). The representations corresponding to $l = -3/4$ and $l = -1/4$ are thus associated with symmetric and antisymmetric eigenfunctions, respectively, in that $\Psi_n(-x;|w|) = (-)^n \Psi_n(x;|w|)$ and

$$f_{s+1/4}(x; -1/4) = \Psi_{2s}(x;|w|),$$

$$f_{s+3/4}(x; -3/4) = \Psi_{2s+1}(x;|w|).$$

This establishes when the pair has either a fermionic character or a bosonic character with respect to the transformation $(\mathbf{R}_1, \mathbf{R}_2) \to (\mathbf{R}_2, \mathbf{R}_1)$ changing x, y (namely p) in $-x$, $-y$. No conclusion, however, is permitted until the second quantum number requested for the complete description of the pair, is considered. To this end consider the state $\Psi_{nm} = \Psi_n(x;|w|)\Psi_m(X;|C|)$ obtained from formulas (24), (21). When the vortex exchange is equivalently enacted via the substitution $(k_1, k_2) \to (k_2, k_1)$, this implicates, in particular, that $X \to X' \equiv X + x(k_2 - k_1)/C$. Hence, while the charge exchange does not affect $\Psi_n(x;|w|)$ for any value of k_1, k_2, thus exhibiting an unexpected type of symmetry involving nonidentical charges, the usual situation is re-established by the presence of $\Psi_m(X;|C|)$ which is trivially symmetric only when $X' = X$, i.e., $k_1 = k_2$.

A further aspect that makes interesting to adopt the su(1,1) scheme is connected to the effect of the unitary action of $D_\phi = \exp(i\phi J_2)$ on the canonical variables x, p. In passing we point out how this is the distinctive trait of the xp description which, as opposite to the $e_*(2)$ scheme, does not involve for $\mathbf{D}$ the role of a constant, structureless object. The D_ϕ action is given by[32]

$$D_\phi x D_\phi^\dagger = e^{-\phi/2} x, \quad D_\phi p D_\phi^\dagger = e^{\phi/2} p$$

and implies that $D_\phi^\dagger \Psi_n(x;r) = \Psi_n(x; re^{-\phi})$ for any wave function (21). Equipped with such formulas, one easily shows D_ϕ to succeed in connecting the dynamics characterized by (k_1, k_2) and $w = k_1 k_2 \rho/(k_1 + k_2)$, with any other having different vorticities (K_1, K_2) and $W \equiv K_1 K_2 \rho/(K_1 + K_2)$. Notice that whenever two cases are related then they both must stay either in the VV sector, or in the VA sector.

To exploit the D_ϕ-action effects, we first define the modified Schrödinger problem $i\hbar \partial_\tau \Phi_\tau = H(W)\Phi_\tau$, with time τ, where the pair Hamiltonian $H(W) = (-\rho K_1 K_2/4\pi) \ln[(x^2 + p^2/W^2)/a^2]$ [see Eqs. (13), (18), and (20)] depends on the vorticities K_j. Then, exploiting the fact that

$$W^2x^2+p^2=e^{-\phi}D_\phi(w^2x^2+p^2)D_\phi^\dagger, \tag{32}$$

where $W=we^{-\phi}$, and setting $\Phi_\tau\equiv e^{i\theta(\tau)}D_\phi\Psi_t$, with $t=t(\tau)$, we recast the Schrödinger problem in the form

$$i\hbar\left[i\frac{d\theta}{d\tau}+D_\phi^\dagger\partial_\tau D_\phi+\partial_\tau)\right]\Psi_t$$
$$=\left[-\rho\phi K_1K_2/4\pi+\frac{K_1K_2}{k_1k_2}H(w)\right]\Psi_t, \tag{33}$$

where θ is obtained by imposing $\hbar d\theta/d\tau\equiv\rho\phi K_1K_2/4\pi$. This, in turn, reproduces the initial problem with the charges k_j

$$i\hbar\partial_t\Psi_t=H(w)\Psi_t, \tag{34}$$

whose solutions can be derived by means of wave functions (21), when both the condition $d\phi/dt\equiv0$ and the time rescaling $\tau=(k_1k_2/K_1K_2)t$ are assumed. Therefore, replacing w with W, which represents a generic change $k_j\to K_j$, is compensated by substituting Φ with its transformed version Ψ.

The analysis developed shows how evaluating the Berry phase[35] when a vortex with constant charge interacts with a vortex cluster situated at a distance (much larger) from the cluster size. The vortex with slowly varying vorticity can be regarded as the pointlike approximation of the cluster exhibiting vortex creation-annihilation processes.

As to this case, suppose that the problem relative to $H(W)$ has W with K_2 depending on the time τ. Then, in the spirit of the adiabatic approximation approach,[35] we go back to Eq. (33) and solve it, with the ket notation, by setting the following two equations:

$$\langle\Psi_t|[i\hbar\partial_t-H(w)]|\Psi_t\rangle\equiv0,$$

$$\hbar\frac{d\theta}{d\tau}\equiv-\hbar\langle\Psi_t|J_2|\Psi_t\rangle\frac{d\phi}{d\tau}+\frac{\rho}{4\pi}K_1K_2(\tau)\phi(\tau),$$

where the fact that ϕ is a function of the time-dependent charge K_2 provides the nonvanishing term $D_\phi^\dagger\partial_\tau D_\phi=iJ_2(d\phi/d\tau)$. The first equation is obtained by absorbing the dependence on the time τ in the time t via the equation $dt/d\tau=[K_1K_2(\tau)/k_1k_2]$ where both K_1 and k_j are independent of τ. It involves the usual time-independent pair dynamics whose exact solutions are given by Eq. (21). On the other hand, the second equation, expressing the standard approximation of the adiabatic scheme, explicitly provides $\theta(\tau)$ via integration which, as expected, plays the role of geometric contribution to Berry's phase. It is found that

$$|\Phi_{t(\tau)}\rangle\simeq e^{i[\theta(\tau)-\tau E_n/\hbar]}e^{i\phi(\tau)J_2}|\Psi_n\rangle,$$

where $E_n=-(\rho/4\pi)k_1k_2\ln[S_n(w)/a^2]$ [see Eq. (22)], ϕ is determined by $\phi=-\ln(w/W)$, and $|\Psi_n\rangle$ is given by Eq. (21). Effects due to a possible time dependence of both the density ρ and the core size a can be treated along the same lines (see Ref. 5).

V. MAGNETIC FORM OF TWO-VORTEX DYNAMICS

Hamiltonian (18) can be easily turned into a magneticlike form[34] by introducing the momenta $P_x=\rho|k_1k_2|^{1/2}x$ and $P_y=\rho|k_1k_2|^{1/2}y$. Such a picture allows one to recover the Feynman-Onsager quantization condition on the charges k_j in an alternative way. The momenta just introduced, whose range of validity covers both the case $k_1>0$, $k_2>0$ and the case $k_1>0$, $k_2<0$, lead to rewrite $\mathbf{D}^2$ as

$$\mathbf{D}^2=\frac{1}{|k_1k_2|\rho^2}(P_x^2+P_y^2), \tag{35}$$

where P_x, P_y obey the commutator $[P_x,P_y]=+(-)i\hbar C$ when $k_2>0$ $(k_2<0)$, and allows one to identify the total vorticity C as the parameter playing the role of the magnetic field. Likewise, since $[P_x,J_y]=[P_y,J_x]=0$, and $[J_x,J_y]=iC$, it is quite natural to regard J_x and J_y as the generators of magnetic translations pertaining to the present contest. They, in fact, generates the Euclidean transformations of the vortex coordinates

$$D_y(\lambda_x)x_iD_y^\dagger(\lambda_x)=x_i+\lambda_x,$$
$$D_x^\dagger(\lambda_y)y_iD_x(\lambda_y)=y_i+\lambda_y, \tag{36}$$

responsible for the displacements of the vortex pair, where $D_y(\lambda_x)=\exp(i\lambda_xJ_y/\hbar)$, and $D_x(\lambda_y)=\exp(i\lambda_yJ_x/\hbar)$.

After that one can proceed along two independent lines. First, one can look upon x_1 and y_2 as position variables thus defining $P_1\equiv\rho k_1y_1$, $P_2\equiv-\rho k_2x_2$ as their canonically conjugate momenta. On the other hand, the opposite choice, where x_2 and y_1 are position variables and $P_2=\rho k_2y_2$, $P_1=-\rho k_1x_1$ the respective momenta, is equally natural. The same twofold choice characterizes the case of a planar charge acted by a transverse magnetic field. In fact, the momentum space picture is always allowed as an alternative way to describe the system in the coordinate space. The interchangeable role of the vortex variables makes vanishing such a distinction for the vortex pair system where the ambient space contains both the momentum space and the configuration space.

Assuming now to operate within the first of the above schemes, we implement the diagonalization of $\mathbf{D}^2$ in the Landau gauge.[34] To this end Eq. (35) must be recast in the more adequate version depending on x_1, y_2, P_1 and P_2

$$\mathbf{D}^2=\frac{1}{\rho^2k_1^2}\mathcal{P}_1^2+\frac{1}{\rho^2k_2^2}\mathcal{P}_2^2, \tag{37}$$

where $\mathcal{P}_1=P_1-\rho k_1y_2$, and $\mathcal{P}_2=P_2+\rho k_2x_1$ have been singled out so as to fulfill the conditions $[\mathcal{P}_j,J_y]=[\mathcal{P}_j,J_x]=0$, and $[\mathcal{P}_1,\mathcal{P}_2]=-iC$. Then, by acting on Eq. (37) through the gauge transformation $\exp(i\rho k_1x_1y_2/\hbar)$ which turns it into the standard harmonic-oscillator form, the Landau gauge eigenvectors are found to be

$$\Phi_{n,q}(x_1,y_2)=e^{i/\hbar(\rho k_1x_1-\hbar q)y_2}\Psi_n(x_1-\hbar q/C;\Omega),$$

where $\Psi_n(x;\Omega)$ is obtained from wave function (21) when $|w|$ is substituted with $\Omega=|Ck_1/k_2|$, whereas the associated eigenvalues reproduce the spectrum $S_n(w)$ defined by Eq. (22). It is easily shown as well that J_x is diagonalized by $\Phi_{n,q}(x_1,y_2)$ and exhibits $\hbar q$ as eigenvalues.

Now, the invariance of H under the action of both $D_x(\lambda_y)$, and $D_y(\lambda_x)$, λ_y, $\lambda_x\in\mathbf{R}$, can be displayed in the Hilbert space through the formulas

$$D_x(\lambda_y)\Phi_{n,q}(x_1,y_2)=e^{iq\lambda_y}\Phi_{n,q}(x_1,y_2),$$

$$D_y(\lambda_x)\Phi_{n,q}(x_1,y_2)=\Phi_{n,q-\alpha}(x_1,y_2),$$

where $\lambda_x=\alpha/C$, and D_y appears to act as a raising (lowering) operator on the quantum number q if $\alpha<0$ $(\alpha>0)$. Hence D_y is able to explore the range of the energy spectrum degeneracy related to the nth Landau-like level.

The final step of the magnetic procedure consists in stating the flux quantization as a consequence of imposing the expression

$$D_x(\lambda_y)D_y(\lambda_x)D_x^\dagger(\lambda_y)D_y^\dagger(\lambda_x)=\mathbf{I}\exp\left[\frac{i\lambda_x\lambda_y}{\hbar}C\right] \quad (38)$$

—it represents the displacement of the charge along a rectangular loop on the plane $\{(x_1,y_2)\}$—to reduce to the *identity* operator $\mathbf{I}$. The right-hand side of Eq. (38) is carried out by means of the Baker-Campbell-Hausdorff formula[26] $e^{a+b}=e^{-1/2[a,b]}e^a e^b$ and becomes $\mathbf{I}$ if

$$\frac{\rho\lambda_x\lambda_y}{h}(k_1+k_2)\equiv\frac{M_{\rm tot}}{h}(k_1+k_2)=N_* \quad (39)$$

with $N_*\in\mathbf{Z}$. The quantity $M_{\rm tot}$ is the superfluid mass enclosed in the box of area $\lambda_x\lambda_y$. Upon introducing the helium atomic mass $m_H=M_{\rm tot}/N_A$, one derives from Eq. (39) the constraint

$$k_1+k_2=(N_*/N_A)\frac{h}{m_H} \quad (40)$$

on the pair total vorticity. Such a result can be readily extended to a many-vortex system in that the translation symmetry holds independently from the number of interacting vortices considered, as follows from (the quantum version of) Eqs. (4) and (7). Since circulation operator (38) only depends on the algebraic properties of symmetry generators, the extension is simply performed by replacing k_1+k_2 with $\Sigma_j k_j$ in the previous formula.

A first interpretation of N_* follows from Eq. (40) when the Feynman-Onsager condition on the vorticity quantization is taken into account.[27,28] In fact, assuming $k_j=hn_j/m_H$ with $n_j\in\mathbf{Z}$ successfully solves Eq. (40) and implies that $N_*=N_A\Sigma_j n_j$. On the other hand, Eq. (40) naturally contemplates the Feynamn-Onsager condition as a possible solution which, in conclusion, appears to emerge as a pure consequence of the symmetries characterizing the vortex system.

Further information concerning the meaning of N_* is obtained when considering the system in a rectangular box. A standard requirement consists in enforcing the cylinderlike geometry in the ambient space via the further condition $D_x(\lambda_y)=\mathbf{I}$ on the y-translation symmetry, where λ_y has been identified with one of the two macroscopic dimensions characterizing a 2D superfluid sample of area $\lambda_y\lambda_x$. In the Hilbert space, this amounts to stating the quantization condition $q=2\pi s/\lambda_y$ with $s\in\mathbf{Z}$, involving the periodicity condition $D_x(\lambda_y)\Phi_{n,q}(x_1,y_2)=\Phi_{n,q}(x_1,y_2)$.

We have thus recovered for the vortex pair dynamics the description in terms of Landau levels and of their degeneracy: s enumerates the straight lines (parallel to the y axis) characterized by the fact that $X=J_x/C=$const representing the 1D domains of the ambient space where the vorticity center is allowed to stay. Moreover, assuming that the eigenvalues $\hbar q/C$ of $X\equiv J_x/C$ take values inside $[0,\lambda_x]$ entails

$$0<q\leqslant C\lambda_x\lambda_y/h=N_*,$$

so that N_* turns out to be the parameter measuring the degeneracy as in the magnetic case. In the case when two equal vortices occupy an area $\lambda_x\lambda_y$ having a macroscopic size, then the degeneracy $N_*=2N_A$ is macroscopically large since N_A is the number of atoms contained inside that area.

As expected, in view of the analysis of the VA case developed in Sec. III, when $C=k_1+k_2\to 0$ no magnetic scenery can be realized this reflecting the fact that $[J_x,J_y]=0$ and the impossibility to relate finite areas of the plane with the free-particle character of the VA dynamics quantum states.

VI. VORTEX PAIR INTERACTION WITH DISK LIKE OBSTRUCTION

Another way to quantize the vortex pair dynamics is that based on expressing $\mathbf{D}^2=|\mathbf{R_1}-\mathbf{R_2}|^2$ by the two-particle operator realizations either of the algebra su(2), or of algebra su(1,1). This requires that vortices are considered as individual objects and involves the use of commutators (9) for x_j and $p_j=\rho k_j y_j$. The purpose of this section is to show how such an approach is particularly suitable to deal with the case when the vortex dynamics takes place in the presence of circular obstacles with reflecting walls.

To begin with we consider the VV dynamics, where $k_1,k_2>0$, and show its version in terms of two-particle generators of su(2). These, when expressed via canonical variables x_j, p_j, have the form

$$V_3=\frac{1}{4\hbar}\left(r_1p_1^2-r_2p_2^2+\frac{x_1^2}{r_1}-\frac{x_2^2}{r_2}\right), \quad (41)$$

$$V_1=\frac{1}{2\hbar}\left(\sqrt{r_1r_2}p_1p_2+\frac{x_1x_2}{\sqrt{r_1r_2}}\right), \quad (42)$$

$$V_2=\frac{1}{2\hbar}\left(\sqrt{\frac{r_2}{r_1}}x_1p_2-\sqrt{\frac{r_1}{r_2}}x_2p_1\right), \quad (43)$$

where $r_j\equiv 1/\rho k_j$, and fulfill the standard commutation relations of su(2)

$$[V_1,V_2]=iV_3,\quad [V_2,V_3]=iV_1,\quad [V_3,V_1]=iV_2, \quad (44)$$

whereas their Casimir operator is given by $V_0\doteq V_3^2+V_2^2+V_1^2\equiv V_4^2-1/4$, with

$$V_4=\frac{1}{4\hbar}\left(r_1p_1^2+r_2p_2^2+\frac{x_1^2}{r_1}+\frac{x_2^2}{r_2}\right). \quad (45)$$

Then, the fact that $\mathbf{D}^2$ can be expressed as

$$\mathbf{D}^2=4\hbar\left[\frac{r_1+r_2}{2}V_4-\frac{r_2-r_1}{2}V_3-\sqrt{r_1r_2}V_1\right], \quad (46)$$

makes it possible to reformulate the VV dynamics within the su(2) scheme where the dynamical variables are now represented by the angular momentum components V_j just defined.

The choice of the algebra su(1,1) instead characterizes the VA case $[(k_1>0,\ k_2<0)]$ and accounts for the change of sign of k_2. Its generators A_1, A_2, and A_3 fulfill the standard commutators of su(1,1)

$$[A_1,A_2]=-iA_3, \quad [A_2,A_3]=iA_1, \quad [A_3,A_1]=iA_2, \tag{47}$$

and the equation for the Casimir operator $A_0 \doteq A_3^2-A_2^2-A_1^2 \equiv A_4^2-1/4$. In particular, the explicit form of A_4 and A_3 is achieved by the substitution $r_2\to -r_2$ in V_3 and V_4, respectively, while A_1 and A_2 are derived by replacing p_2 with $-p_2$ in V_1 and V_2, respectively. The operators thus obtained—notice that such substitutions can be recast in terms of a process of analytic continuation connecting su(2) to su(1,1)—allows one to express $\mathbf{D}^2$ as

$$\mathbf{D}^2=4\hbar\left[\frac{r_1-r_2}{2}A_4+\frac{r_1+r_2}{2}A_3-\sqrt{r_1r_2}A_1\right], \tag{48}$$

where $r_j\equiv 1/\rho|k_j|$. The linear character of both Eqs. (46) and (48) allows one to readily obtain the diagonal form of $\mathbf{D}^2$ by means of unitary transformations. Evidence of this is expressed by means of the formulas

$$\mathbf{D}^2=4\hbar R_\pm\left[\frac{r_1+r_2}{2}V_4\pm\frac{r_1+r_2}{2}V_3\right]R_\pm^+, \tag{49}$$

where $R_\pm \doteq e^{i(\beta\pm\pi/2)V_2}$, with $tg\beta=(r_2-r_1)/\sqrt{4r_1r_2}$, and

$$\mathbf{D}^2=4\hbar R_\eta\left[\frac{r_1-r_2}{2}A_4+\frac{r_2-r_1}{2}A_3\right]R_\eta^+, \tag{50}$$

where $R_\eta \doteq e^{i\eta A_2}$ and $th\,\eta=\sqrt{4r_1r_2}/(r_1+r_2)$, whose basic feature is that of depending only on V_4, V_3, and A_4, A_3. As a consequence of the fact that $[V_4,V_j]=0$ and $[A_4,A_j]=0$ since (the eigenvalues of) V_4 and A_4 are c numbers labeling the representations of the respective groups, then the spectral problem is reduced to the standard one of diagonalizing A_3 and V_3.

The form of the vortex dynamics suggested by formulas (46) and (48) is that of two interacting oscillators. In particular, the terms V_3, V_4 and A_3, A_4 describe two independent harmonic oscillators on their elliptic trajectories in the phase space; in view of the double nature of vortex coordinates pointed out in Sec. V, such trajectories also represent circular solitary motions of vortices around the ambient plane origin. Due to the presence of V_1 and A_1 which introduce vortex interactions, a more structured dynamics takes place which involves the bounded states classically described in Sec. III.

In spite of the similarity of their diagonalization process a crucial difference, however, distinguishes the VA case from the VV case. The latter, in fact, presents two diagonal forms of $\mathbf{D}^2$, for any value of k_1 and k_2, related to the sign $(\pm)$ of Eq. (49), whereas the VA case always exhibits a unique diagonal form of $\mathbf{D}^2$. Since

$$\tfrac{1}{2}(r_1+r_2)(V_4\pm V_3)=\begin{cases}\dfrac{r_1+r_2}{4\hbar}(r_1p_1^2+x_1^2/r_1),\\[2mm] \dfrac{r_1+r_2}{4\hbar}(r_2p_2^2+x_2^2/r_2),\end{cases} \tag{51}$$

where the first and the second expressions correspond to the case with $(+)$ and $(-)$, respectively, then the eigenfunctions exhibit the two independent versions

$$\Psi_{nq}^+(x_1,x_2)=R_+\Psi_n(x_1;r_1)\Psi_q(x_2;r_2), \tag{52}$$

$$\Psi_{nq}^-(x_1,x_2)=R_-\Psi_n(x_2;r_2)\Psi_q(x_1;r_1), \tag{53}$$

where the wave functions $\Psi_p(x_j,r_j)$, with $p=n,q$, diagonalize the (harmonic oscillator) secular equation $(r_jp_j^2+x_j^2/r_j)\Psi_p=\hbar(2p+1)\Psi_p$. The effect of the action of $R_\pm$ in Eqs. (52), (53) and hence the explicit form of $\Psi_{nq}^\pm(x_1,x_2)$ are calculated explicitly in Appendix B. Their relation with the classical single vortex picture based on Eqs. (15), (16) can be established by noticing that the circle coordinates identify with the coordinates exhibited by the final explicit form of Eqs. (52), (53).

While $\Psi_{nq}^\pm(x_1,x_2)$ appear to have a form differing from that of the eigenstates found within different procedures, the eigenvalues of $\mathbf{D}^2$ and V_4 given by

$$S_n=\hbar(r_1+r_2)(2n+1), \quad \Lambda_q(n)=\frac{1}{2}(n+q+1), \tag{54}$$

respectively, where $n\in\mathbf{N}$ and $q\in\mathbf{N}$, are consistent with Eqs. (22) and (observe that $V_4=-J_z/2\hbar$) (25), respectively.

One should observe how the form of both Eqs. (52) and (53) ensues from the choice of employing the eigenvalues of V_3+V_4 (V_4-V_3) for describing the degeneracy of the nth level, when the diagonal form of $\mathbf{D}^2$ is V_4-V_3 (V_4+V_3). Actually a full arbitrariness affects the previous choice since any function $F(x_2,p_2)$ $[F(x_1,p_1)]$ commutes with V_3+V_4 (V_4-V_3). This is consistent with the scenarios encountered in Secs. III and IV.

Coming now to the VA case, its distinctive feature is that of providing, when $r_2\to r_1$, a limiting situation where the unitary transformation of Eq. (50) is not able to take A_3-A_1 into A_3, such operators pertaining to disjoint sectors of su(1,1). This consistently matches the fact that, while A_3 is endowed with a discrete spectrum, A_3-A_1 exhibits instead the continuous spectrum characterizing noncompact generators.[32] The dramatic change of the spectrum occurring when $k_2\to -k_1$ is the consequence of the transition from the regime of confinement [described at the classical level in Sec. II by circumferences (15), (16)] to a situation where the VA pair freely drifts through the ambient plane.

The standard, compact sector of the VA spectrum can be readily worked out from the diagonal core of Eq. (50) $A_3-A_4=(r_2p_2^2+x_2^2/r_2)/2\hbar$. It is found that the eigenvalues and the eigenvectors of $\mathbf{D}^2$ are given by

$$S_n=\hbar(r_2-r_1)(2n+1),$$

$$\Psi_{nq}(x_1,x_2)=R_\eta\Psi_n(x_2;r_2)\Psi_q(x_1;r_1),$$

respectively, while eigenvalues (25) once more are matched by the A_4 spectrum involved by states $\Psi_{nq}(x_1,x_2)$.

For $r_1 = r_2$, the eigenstates of $A_3 - A_1$ might be easily expressed by means of the plane-wave eigenstate of Sec. III which, however, fails in diagonalizing the constant of motion A_4. To fulfill such a requirement one must consider functions of variables (27) depending, in particular, on $\mathcal{X}$ and $\mathcal{Y}$. Then $\mathbf{D}^2$ can be viewed as a 2D Laplacian operator $\mathbf{D}^2 = (1/k\rho)^2(\xi^2 + \eta^2) = -(\hbar/k\rho)^2(\partial^2_{\mathcal{X}} + \partial^2_{\mathcal{Y}})$ whose eigenvectors, in the su(1,1) setup, are given by the Lindblad-Nagel states[37,31]

$$\Psi_{\epsilon s}(\mathcal{R},\theta) = e^{is\theta} J_s(\mathcal{R}\sqrt{8\epsilon/r\hbar}), \tag{55}$$

where $\mathcal{R}^2 = \mathcal{X}^2 + \mathcal{Y}^2$, $\mathrm{tg}\theta = \mathcal{Y}/\mathcal{X}$, and the parameter $\epsilon \geqslant 0$ is the continuous eigenvalue of the secular equation

$$(A_3 - A_1)\Psi_{\epsilon s}(\mathcal{R},\theta) = \epsilon \Psi_{\epsilon s}(\mathcal{R},\theta), \tag{56}$$

where $A_3 - A_1 \equiv (r/4\hbar)(\xi^2 + \eta^2)$ (see Appendix C). Concerning the index s, formulas of Appendix C, besides showing the form of operators A_j in terms of variables (27), allow one to recognize $e^{is\theta}$ as the factor of $\Psi_{\epsilon s}(\mathcal{R},\theta)$ responsible for associating the eigenvalue $\lambda_s \doteq -s/2$ with A_4. Also, according to the Casimir formula $A_0 = A_4^2 - 1/4 \equiv J(J+1)$, the index J labeling the SU(1,1) turns out to have the form $J \equiv -(|s|+1)/2$ involving negative integer or half-integer values[37] when s takes integer values.

The geometric meaning of the eigenvalue λ_s deserves some comment. When external interactions do not affect the VA dynamics the pair proceeds along a rectilinear trajectory which is orthogonal to the vector $\mathbf{D}$ joining its two vortices. Then the quantity $4r\hbar A_4 = \mathbf{B}\cdot\mathbf{D} = DB\cos\beta$, where the vector $\mathbf{B} \doteq \frac{1}{2}(\mathbf{R}_1 + \mathbf{R}_2)$ represents the position of the pair on the plane, allows one to evaluate the deviation $|\mathbf{B}\wedge\mathbf{D}|/D = [B^2 - (4\hbar r A_4/D)^2]^{1/2}$ of the pair trajectory from the plane origin. Whenever the two vortices have the same distance from the plane origin then $A_4 = 0$ since $\beta = \pm\pi/2$, and the pair trajectory crosses the plane origin.

The algebraic approach based on the two-particle representation just discussed is involved by the model Hamiltonian H_* describing the vortex pair dynamics in the presence of a circular boundary with radius R. The general form of H_* where vortices have arbitrary charges k_1, k_2 is given in Appendix A, whereas the two extreme cases $k_1 = -k_2 = k$ and $k_1 = k_2 = k$ are described by

$$H_*(z_1,z_2) = \frac{\rho}{2\pi} k^2 \ln \mathcal{A}, \tag{57}$$

where the vortex position vectors $\mathbf{R}_j$ have been replaced with the complex variables $z_j \doteq x_j + iy_j$, and $\mathcal{A}$ reads

$$\mathcal{A} = R^{2u-4}(|z_1|^2 - R^2)(|z_2|^2 - R^2)\frac{|z_1 - z_2|^{2u}}{|z_1\overline{z_2} - R^2|^{2u}}$$

with $u=1$ and $u=-1$ in the case VA and VV, respectively.

Indeed the dynamics relative to H_* can be profitably represented within the angular momentum picture introduced above, when in the VA (VV) case the logarithm argument is expressed as a function $\mathcal{A}(\{A_j\})[\mathcal{A}(\{V_j\})]$ of a 3D vector $\mathbf{A}$ $(\mathbf{V})$ (see Appendix A). The first consequence is that of simplifying the description of the system whose dynamical equations are now obtained via the classic (up to the standard factor $i\hbar$) commutators (44) and (47). Also, the integrable character of the dynamics with the disk-pair interaction turns out to be accounted by a constant of motion (A_4 or V_4, depending on the case studied) which is geometrically meaningful. A second, *a priori* less evident, effect comes out within the quantization process, where a desirable result is that of producing operators unaffected by ordering problems. Indeed this is the case when, based on the angular momentum description and in view of the logarithm property $\ln \mathcal{A} \equiv -\ln \mathcal{A}^{-1}$, the logarithm argument of Eq. (57) is reduced to

$$\mathcal{A}^{-1} = \frac{\nu/2}{A_3 - A_1} + \frac{\nu^2}{(A_3-\nu)^2 - A_4^2}, \tag{58}$$

and

$$\mathcal{A}^{-1} = \frac{\nu^2}{v^2 - V_3^2} - \frac{\nu^2}{v^2 - 2\nu(V_1 - V_4) - V_3^2}, \tag{59}$$

where $\nu \doteq R^2/2\hbar r$ and $v \doteq V_4 - \nu$. On this account formulas (58), (59) provide the most convenient way to construct the operator version of H_*. A similar situation already occurred in Ref. 9 where the Hamiltonian of a pair interacting with a rectilinear boundary turned out to possess at least one version able to avoid ordering problems after the quantization process. This fact strongly suggests that some nontrivial, hidden character pertains to the system presently considered as well as to the one discussed in Ref. 9. As to this point the main indication is certainly that relative to the surviving of a constant of motion, despite the analytic complexity induced by the presence of the boundary.

The extension of the work involved by the diagonalization process of $\mathcal{A}^{-1}$ requires a separate treatment that we shall develop elsewhere. Nevertheless, based on the spectral analysis of the free vortex pair performed above, it is possible to evaluate perturbatively the spectrum of the pair dynamics when the vortex separation is much smaller than the distance from the disk, namely the condition $\mathbf{D}^2 \ll |z_j|^2$ holds for true. At the classical level, this implies that $V_3^2, V_4 - V_1 \ll V_4^2$, entailing quantically the condition $S_n \ll \Lambda_q(n)$ on eigenvalues (54). Then one easily finds that

$$\langle \mathcal{A}^{-1} \rangle \simeq \frac{\nu^3(2n+1)}{[\Lambda_q(n)+\nu]^4}\left[1 + \frac{\langle V_3^2\rangle - \nu(n+1/2)}{[\Lambda_q(n)+\nu]^2}\right], \tag{60}$$

where the expectation value notation is referred to the state $\Psi^{\pm}_{nq}$ in the ket notation $|n,q\rangle$.

On the other hand, in the VA case, one has $A_3 - A_1, A_4^2 \ll A_3^2$ so that the second term of Eq. (58) can be treated as a perturbation. When this is rewritten as $\mathcal{C}^{-1} = [(A_3 - \nu - A_4)^{-1} - (A_3 - \nu + A_4)^{-1}]/(2A_4)$ with $\mathcal{C} \equiv (A_3 - \nu)^2 - A_4^2$, thanks to the fact that $[A_4, A_j] = 0$, and the further condition $x^2 + y^2 \ll R^2 \ll \mathcal{X}^2 + \mathcal{Y}^2$ is assumed, then it is possible to work out (see the formulas of Appendix C) the zero-order approximation

$$\frac{1}{\mathcal{C}} \simeq \frac{\hbar r}{A_4}\left[\frac{1}{\mathcal{R}^2 - u_+^2} - \frac{1}{\mathcal{R}^2 - u_-^2}\right],$$

where $\mathcal{R}^2=\mathcal{X}^2+\mathcal{Y}^2$, and $u_{\pm}^2=R^2\pm 2\hbar r A_4$. By using states (55) its expectation value can be expressed as

$$\mathcal{I}(\epsilon,s)\equiv\left\langle\frac{1}{\mathcal{C}}\right\rangle\simeq-\sum_{\delta=\pm}\int_0^\infty\frac{2r\hbar J_s^2(\mathcal{R}\mu_\epsilon)\mathcal{R}d\mathcal{R}}{\delta(u_\delta^2-\mathcal{R}^2)},$$

where $\mu_\epsilon^2\equiv 8\epsilon/(r\hbar)$, and the term A_4 in u_δ has been replaced by its eigenvalue $\lambda_s=-s/2$, namely the second quantum number of states (55). It results that

$$\langle\mathcal{A}^{-1}\rangle\simeq\frac{\nu^2}{2\epsilon}+2r\hbar\sum_{\delta=\pm}\frac{\pi}{2}sJ_s(\mu_\epsilon u_\delta)N_s(\mu_\epsilon u_\delta),\quad(61)$$

where J_s and N_s are Bessel functions of the first and second type, respectively.[38]

VII. CONCLUSIONS

In the present paper we have considered various algebraic schemes as independent frameworks where is possible to treat the quantum dynamics of the vortex pair both for the VV case and for the VA case. From the physical viewpoint, their equivalence has been checked by showing how, for each approach, the diagonalization process leads to the same energy spectrum. The aspect concerning the spectrum degeneracy (scarcely mentioned in the literature and, to our knowledge, never studied thoroughly) has been particularly deepened.

The introduction of a second quantum number enumerating the degenerate states has been discussed in Sec. III, where the vortex distance $\mathbf{D}=|\mathbf{R_1}-\mathbf{R_2}|^2$ [i.e., the logarithm argument of Hamiltonian (18)] is identified with the Casimir operator of the symmetry algebra $\mathbf{e}_*(2)$. Any element $I(a,b,c)$ of $\mathbf{e}_*(2)$ can be used for describing the degeneracy since $[I,H]=0$, although the (unitarily) independent choices are just two, namely $I=J_y$ and $I=J_z$. The Hilbert space basis relative to both J_y and J_z have been provided explicitly and the geometric structure in the ambient space of the corresponding spectra has been discussed in detail. These appear to mimic the stripe structure of the Landau gauge and the Corbino disk structure of the symmetric gauge, respectively, for a charge acted by a transversal magnetic field.

The parallel with the magnetically acted charge has been completely developed in Sec. V. The Feynman-Onsager quantization of the vortex charges is in fact reconstructed first by making explicit the magnetic form of $\mathbf{D}$ in terms of generalized momenta $\mathcal{P}_1$ and $\mathcal{P}_2$ (the magnetic field is represented by the total vorticity C), then by using the standard symmetry arguments leading to the magnetic flux quantization.

The reduction of $\mathbf{D}$ to the harmonic-oscillator Hamiltonian with the canonical variables x and p has inspired instead the use of the su(1,1) scheme of Sec. IV, where $\mathbf{D}$ no longer plays the role of the Casimir operator. Action (32) of the noncompact generator J_2 on $\mathbf{D}$, which identifies with the compact su(1,1) generator J_3, allows one to recognize a continuous symmetry relating Hamiltonian with different vorticity pairs (k_1,k_2). In view of such a symmetry it is possible to calculate both the wave function and the geometric correction of the Berry phase when the Hamiltonian has time-dependent parameters.

In Sec. VI $\mathbf{D}$ has been rewritten by means of two-particle realizations of su(2) and su(1,1) expressed in terms of canonical coordinates x_j, p_j for the VV case and the VA case, respectively. In this contest the energy degeneracy is accounted for by the operators A_4 and V_4 labeling the algebra representation. Such a scheme appears to be really suitable for the purpose of treating the dynamics of vortex pairs in the presence of a disklike obstruction in that it allows one to express the pair Hamiltonian within an angular momentum picture, which is both quite compact and capable of avoiding the ordering problems despite the analytic complexity introduced by the vortex-obstruction interaction. The dynamics of such a case—here the energy spectrum has been evaluated just perturbatively—deserves further investigations since it provides a nonphenomenological approach to study pinning effects due to the impurities of the medium.

More in general, the possibility of using various algebraic approaches to treat QVD fully displays its importance when considering, as the natural development of the results achieved here, the constructions of models coupling vortices with the environment, namely external systems such as the superfluid background, the walls confining the superfluid, defects responsible for vortex scattering, thermal excitations, and so on.

In passing, we notice once more how the quantum number describing the degeneracy could play a relevant role as the dynamical variable to be activated by the interactions with the environment. In this sense it is quite natural to expect that such a number is involved in the energy spectrum of the coupled system thus eliminating the energy spectrum degeneracy.

Several ways to realize the coupling with the environment can be established depending on the algebraic framework where the vortex dynamics is accounted. The resulting coupled dynamics should be sensitively conditioned by the choice performed. In particular, some scheme might appear, due to its intrisic features, more appropriate than another one depending on the physical contest where it is employed [this is the case, e.g., of the $\mathbf{e}_*(2)$ scheme which clearly turns out to be not adequate for describing the interaction with the disklike obstruction].

A similar situation was discussed in Ref. 39 for a charge acted by a transversal magnetic field and interacting with the background phonons. Indeed in that case the choice of a certain particular dynamical algebra for the charge Hamiltonian was able to endow the coupled model with the chaotic character requested by the experimental observations.

A further reason for the interest in considering independent algebraic descriptions of vortex dynamics is related to a possible implementation of the time-dependent variational principle procedure for many-vortex systems within a quantum picture based on a coherent states picture.[26] The combination of such methods has been successfully employed to investigate the quantum dynamics of many-body systems.[40] The main ingredient of such a variational procedure is represented by a macroscopic wave function for the ensemble particles whose construction is profitably realized in terms of generalized coherent states. The usefulness of the analysis developed here becomes evident by recalling that the definition of such states is basically founded on identifying a suitable algebraic framework (the dynamical algebra defined in the Introduction) containing the dynamical degrees of freedom of the ensemble.

Indeed we believe that the analysis performed in the present work can be fruitfully employed for constructing models for the vortex-environment interaction as well as for treating the quantum dynamics of a gas of vortices.

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APPENDIX A

The standard way to account for the presence of boundaries confining the medium where vortices move is based on the virtual charge method.[36,9,25] Such a method makes it possible to work out the vortex pair Hamiltonian incorporating the effects of a circular boundary, which reads

$$H(z_1,z_2)=-\frac{\rho}{2\pi}k_1k_2\ln\left[\frac{R^2|z_1-z_2|^2}{|z_1\overline{z_2}-R^2|^2}\right]$$
$$+\frac{\rho}{2\pi}k_1^2\ln\left(\frac{|z_1|^2-R^2}{R^2}\right)+\frac{\rho}{2\pi}k_2^2\ln\left(\frac{|z_2|^2-R^2}{R^2}\right).$$

When one of the two vortices touches the disk boundary (i.e., $|z_j|\to R$) then the first term is going to zero, whereas $\ln[(|z_j|^2-R^2)/R^2]$ becomes infinitely negative, as is expected whenever a vortex annihilates a vortex with an opposite charge. The latter, in the present case, is the virtual vortex accounting for the boundary effect. Then, after performing a suitable energy rescaling, the remaining logarithm represents the interaction of an isolated vortex with the circular reflecting wall.

In the extreme cases $k_1=k_2$ and $k_2=-k_1$ the Hamiltonian reduces to the form (57). Since $\mathcal{A}$ is constituted by several factors where the canonical variables appear to be mixed in a very complex way, a dramatic ordering problem should affect the quantum version of $\mathcal{A}$. It is almost surprising instead to discover that $\mathcal{A}^{-1}$ is exempt from such a problem. One finds, in fact, the expressions

$$\mathcal{A}^{-1}=\frac{R^2}{|z_1-z_2|^2}+\frac{R^4}{(|z_1|^2-R^2)(|z_2|^2-R^2)},$$

$$\mathcal{A}^{-1}=\frac{R^4}{(|z_1|^2-R^2)(|z_2|^2-R^2)}-\frac{R^4}{|z_1\overline{z_2}-R^2|^2},$$

in the VA case and the VV case, respectively, that, after considering the further formulas

$$|z_1-z_2|^2=\begin{cases}4\hbar r(A_3-A_1),\\ 4\hbar r(V_4-V_1),\end{cases}$$

$$|z_1\overline{z_2}-R^2|^2=\begin{cases}4\hbar^2r^2(A_3^2-A_4^2)+R^4-4\hbar rR^2A_1,\\ 4\hbar^2r^2(V_4^2-V_3^2)+R^4-4\hbar rR^2V_1,\end{cases}$$

$$(|z_1|^2-R^2)(|z_2|^2-R^2)=\begin{cases}(2r\hbar A_3-R^2)^2-4\hbar^2r^2A_4^2,\\ (2r\hbar V_4-R^2)^2-4\hbar^2r^2V_3^2,\end{cases}$$

involving explicitly the algebra generators, clearly exhibit the absence of any ordering problem. The expressions constituting the denominators of $\mathcal{A}^{-1}$ are in fact linear combinations of powers of the generators of the algebra involved in the two-particle quantization scheme. The explicit form of the operator $\mathcal{A}^{-1}$ is easily obtained by resorting to the Laplace integral representation of the function $f(x)=1/x$ (see Ref. 9).

APPENDIX B

The action of $R_\pm$ on the canonical variables x_j, p_j can be derived from the formulas

$$U_\psi\begin{pmatrix}x_1\\x_2\\p_1\\p_2\end{pmatrix}U_\psi^+=\begin{cases}x_1\cos\psi+x_2e^{\mu}\sin\psi,\\ x_2\cos\psi-x_1e^{-\mu}\sin\psi,\\ p_1\cos\psi+p_2e^{-\mu}\sin\psi,\\ p_2\cos\psi-p_1e^{\mu}\sin\psi,\end{cases}$$

where $U_\psi=\exp(-i2\psi V_2)$. To this aim, it is important to consider the two transformations

$$\mathcal{U}_j(\mu)x_j\mathcal{U}_j^\dagger(\mu)=e^{\mu}x_j,\quad \mathcal{U}_j(\mu)p_j\mathcal{U}_j^\dagger(\mu)=e^{-\mu}p_j,$$

where $\mathcal{U}_j(\mu)=\exp[i\mu(x_jp_j+p_jx_j)/(2\hbar)]$, involving the two decompositions

$$U_\psi=\mathcal{U}_2(\mu)e^{-i2\psi L_3}\mathcal{U}_2^\dagger(\mu),\quad U_\psi=\mathcal{U}_1^\dagger(\mu)e^{-i2\psi L_3}\mathcal{U}_1(\mu),$$

with $L_3=(x_1p_2-x_2p_1)/\hbar$. Such decompositions and the fact that $\mathcal{U}_j(\mu)\Psi_n(x_j;r_j)=\Psi_n(x_j;r_je^{-2\mu})$ (see, e.g., Ref. 32), allows one to obtain the explicit form of eigenfunctions (52) and (53). When assuming $e^{\mu}=\sqrt{r_1/r_2}$ one finds

$$\Psi_{nq}^+(x_1,x_2)=\Psi_q(\alpha_2X;r_2)\Psi_n(x/\alpha_2;r_1),$$

$$\Psi_{nq}^-(x_1,x_2)=\Psi_q(\alpha_1X;r_1)\Psi_n(-x/\alpha_1;r_2),$$

respectively, where $\alpha_j=\sqrt{C/\rho k_j}$, and coordinates (11), (12) have been used.

APPENDIX C

Replacing r_2 with $-r_2$ in V_3, V_4 and p_2 with $-p_2$ in V_1, V_2 [see formulas (41)–(43) and (45)] provides the su(1,1) operators A_4, A_3 and A_1, A_2, respectively. The final form

$$A_3=\frac{r}{8\hbar}\left[\xi^2+\eta^2+\frac{4}{r^2}(\mathcal{X}^2+\mathcal{Y}^2)\right],$$

$$A_1=\frac{r}{8\hbar}\left[\frac{4}{r^2}(\mathcal{X}^2+\mathcal{Y}^2)-(\xi^2+\eta^2)\right],$$

$$A_2=-\frac{1}{2\hbar}(\xi\mathcal{X}+\eta\mathcal{Y}),\quad A_4=\frac{1}{2\hbar}(\xi\mathcal{Y}-\eta\mathcal{X}),$$

is achieved when the special coordinates (27) of the case $k_2=k_1$ are employed.

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