Ultrasonic attenuation in clean *d*-wave superconductors

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We consider the attenuation of longitudinal ultrasonic waves in a clean two-dimensional *d*-wave superconductor. We show that the attenuation coefficient is linear in temperature at low temperatures for *all* in-plane directions of the propagation of the ultrasound, and that the coefficient of the linear term can be used to determine the parameters crucial for the low-temperature transport in these compounds.
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Much of the novel physics associated with the superconducting state of the high-T_c cuprates is due to nontrivial angular dependence of the energy gap $\Delta_{\mathbf{k}}$ at the Fermi surface. While in conventional, *s*-wave, superconductors the energy gap is finite for all quasiparticle excitations, it is believed that in many cuprates it has the angular structure of a $d_{x^2-y^2}$ state, with lines of nodes, which leads to a gapless excitation spectrum along certain directions in momentum space. Consequently the temperature dependence of the thermodynamic and transport coefficients in the high-T_c materials is qualitatively different from that of their *s*-wave counterparts: at low temperatures the behavior of the coefficients measuring an average of a particular quantity over the Fermi surface is governed by the low-energy excitations near the nodes of the gap at the Fermi surface, and the resulting temperature dependence is given by power laws rather than by exponentially decaying functions with an activation barrier.[1] While the power laws are entirely determined by the dimension of the manifold where the gap vanishes, in this case by the existence of lines of nodes, the coefficients can be used to determine parameters of the superconducting materials and test the agreement with specific models.

The ratio of the Fermi velocity at the nodes, v_f, and the velocity v_2 associated with the growth of the superconducting gap at the Fermi surface $\Delta(\mathbf{p}_f)=\mathbf{v}_2(\mathbf{p}_f-\mathbf{p}_f^{(n)})$ near the node is a particularly important parameter: both the universal limit of transport coefficients[2] and the temperature dependence of the penetration depth[3] depend solely on v_2/v_f. The numerical value of this ratio has not been clearly determined yet; Lee and Wen[3] obtain $v_f/v_2=6.8$ from the analysis of the penetration depth data,[4] while the measurements of the thermal conductivity[5] yield $v_f/v_2=13.6$. It is therefore important to have additional ways of experimentally determining this parameter.

Here we analyze the electronic attenuation of the longitudinal ultrasound waves in a *d*-wave superconductor and argue that it can be used to measure the ratio v_f/v_2. We consider the clean case $ql\geqslant 1$, where $\mathbf{q}$ is the wave vector of the sound wave, and l is the electron mean free path; this implies that for a linear frequency of 150 MHz, taking the sound velocity $v_s=4\times 10^5$ cm/sec[6] we need $l\geqslant 4\,\mu$m. In previously studied optimally doped $YBa_2Cu_3O_{6.93}$ samples the mean free path was determined to be $l\simeq 0.6\,\mu$m at $T\simeq 20$ K,[7] which is closer to the hydrodynamic limit $ql\ll 1$, where the longitudinal and transverse ultrasonic attenuation have been analyzed.[8,9] However, the ultra-high-purity crystals recently grown in $BaZrO_3$ crucibles[10] are about an order of magnitude cleaner, and experiments demonstrate that at low temperatures $l\simeq 4-5\,\mu$m,[11] indicating that the limit $ql\geqslant 1$ is likely to be achieved. Experimentally the clean limit is manifested in the attenuation rate which is linear, rather than quadratic, in the frequency of the ultrasound waves, and by an angular dependence of the longitudinal attenuation rate, described below, which differs from that predicted in the dirty limit.[8]

Let us first consider the problem qualitatively. In the clean limit α_s is determined by a scattering rate of a phonon by the quasiparticles. Apart from the matrix element of the electron-phonon interaction this scattering rate depends solely on the phase space available for the phonon to decay. As the energy and the wave vector of the sound wave are small on the scale of electronic excitations, the relevant scattering processes are those with negligible energy and momentum transfer, and are given by the condition $\mathbf{v}_g\cdot\mathbf{q}=0$, where $\mathbf{v}_g$ is the group velocity of the quasiparticles. In a normal metal the scattering is restricted to a belt on the Fermi surface where the Fermi velocity $\mathbf{v}_f$ is perpendicular to $\mathbf{q}$. The opening of an *s*-wave gap preserves the direction of the group velocity of the quasiparticles but thermally suppresses the occupancy of the states near the Fermi surface resulting in the exponential suppression of the scattering.[12] For an anisotropic *s*-wave superconductor the attenuation at temperatures lower than the minimal gap still decays exponentially, albeit with an exponent which depends on the direction.[13]

Recently this approach has been transferred to a *d*-wave superconductor.[14,15] Assuming that the quasiparticles contributing to the attenuation are still located at the points of the Fermi surface where $\mathbf{v}_f\cdot\mathbf{q}=0$, the authors of Refs. 14 and 15 noticed that the scattering processes sample the local gap in the direction perpendicular to $\mathbf{q}$, and predicted the fourfold oscillations of α_s as a function of the angle of propagation in the plane θ. In particular, since the gap vanishes along the nodal direction, the attenuation in that direction was predicted to be temperature independent and equal to

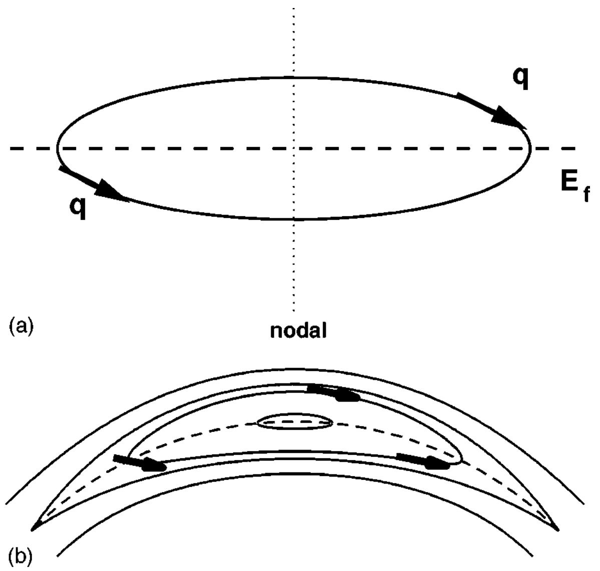

FIG. 1. (a) A typical contour of constant energy near a node of the order parameter and the scattering processes contributing to the attenuation for an arbitrary $\mathbf{q}$. (b) Contours of constant energy $E = a\Delta_0$ for $a=0.2$, 0.8, 1.0, 1.4 with $\Delta_0/\epsilon_f=0.1$, and the scattering processes for $\mathbf{q}$ nearly normal to the node. Arrows *are* parallel.

that at T_c, while the attenuation in other directions was determined to decay exponentially with the activation energy given by the local gap $\Delta(\theta \pm \pi/2)$.

However, for an anisotropic order parameter, not only the magnitude, but also the direction of the group velocity changes with the opening of the energy gap. This change has important consequences for superconductors with gap nodes, especially for compounds with a relatively large value of the superconducting gap amplitude Δ_0 as a fraction of the Fermi energy ϵ_f, and at temperatures below Δ_0^2/ϵ_f. This was first noted in the context of the heavy fermion superconductors by Coppersmith and Klemm,[16] who predicted a maximum in the ultrasonic attenuation for certain directions of the propagation of the ultrasound, and a power-law behavior for low temperatures. Heavy fermion materials are not in the clean limit, and $T_c \sim 1$ K made the desired regime impossible to achieve experimentally. In the cuprates, on the other hand, a high transition temperature and a large ratio of $\Delta_0/T_c \sim 2.5$–4 bring the gap amplitude to about 10–30 % of the Fermi energy, so that the electronic contribution to the attenuation rate is measured at $T \ll \Delta_0^2/\epsilon_f$, where the theoretical analysis for a d-wave superconductor is still lacking.

We first proceed with the qualitative analysis. Below T_c the excitation spectrum has the form $E(\mathbf{k}) = \sqrt{\zeta_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$, where $\zeta_{\mathbf{k}}$ is the quasiparticle energy in the normal state with respect to ϵ_f, so that the direction of the group velocity, given by a tangential line to a $E(\mathbf{k}) = \text{const}$ contour, does not coincide with the direction of the Fermi velocity at the same point. For $T \ll T_c$ only the contours with $E \ll \Delta_0$, near the gap nodes, are important for scattering. Then, as shown in Fig. 1(a), the contour of $E = \text{const}$ is approximately an ellipse $\zeta(\phi) = \pm \sqrt{E^2 - \Delta'^2(\phi - \phi_n)^2}$, where ϕ parametrizes the Fermi surface, and Δ' is the angular derivative of the gap at the node, and for *any* wave vector $\mathbf{q}$ there are two quasiparticle scattering processes which contribute to the attenuation. Since α_s is proportional to the phase-space volume available for scattering an immediate conclusion is that the attenuation is linear in T at low temperatures in *all* the directions except normal to the nodes.

A second observation concerns the role of the relatively large ratio Δ_0/ϵ_f. This can be accounted for by introducing the curvature of the Fermi surface, as shown in Fig. 1(b). Then the group velocity of the quasiparticles with energies $E \ll \Delta_0$ or $E \gg \Delta_0$ does not change significantly. For energies $E \sim \Delta_0$, however, the competition between the curvature of the contour $E = \text{const}$ and that of the underlying Fermi surface opens up additional phase space for scattering of the ultrasound propagating within a narrow window of angles nearly perpendicular to the node; as seen in Fig. 1(b) three rather than two scattering processes become possible. For a d-wave gap the curvatures become comparable for $E \sim \Delta'^2/2\epsilon_f \simeq 2\Delta_0^2/\epsilon_f$, which implies that the additional scattering is most effective for $T \simeq 2\Delta_0^2/\epsilon_f$, a relatively high temperature, resulting in an attenuation coefficient larger than that at T_c. For $T \lesssim T_c$ the additional scattering is effective, while at lower temperatures the occupancy of these states is thermally suppressed and they do not contribute to the attenuation. We therefore expect a maximum in α_s, below the superconducting transition followed by a transition to linear, in T, behavior at lower temperature T^*. The position of the maximum, as well as crossover temperature T^* scale with Δ_0^2/ϵ_f, while the angular window within which it exists scales as Δ_0^2/ϵ_f^2.

This analysis implies that the main conclusions of Ref. 16 are qualitatively applicable to d-wave superconductors. We now investigate the issue in more detail focusing on the low-temperature regime. Within a standard BCS-like theory the ultrasonic attenuation is given by[16,17]

$$\alpha_s(T,\mathbf{q}) = A\frac{\Omega}{4T}\int d\mathbf{k}\frac{1}{\cosh^2(E_{\mathbf{k}}/2T)} \times\left(1+\frac{\zeta_{\mathbf{k}}\zeta_{\mathbf{k+q}} - \Delta_{\mathbf{k}}\Delta_{\mathbf{k+q}}}{E_{\mathbf{k}}E_{\mathbf{k+q}}}\right)\delta(E_{\mathbf{k}} - E_{\mathbf{k+q}}), \quad (1)$$

where the constant A depends on the scattering matrix element and the frequency of the ultrasound Ω, and the integration is over the Brillouin zone. Expanding in $\mathbf{q}$ and taking into account that the angular average of the gap derivative vanishes, we obtain

$$\alpha_s(T,\mathbf{q}) = A\frac{\Omega}{2T}\int d\mathbf{k}\frac{1}{\cosh^2(E_{\mathbf{k}}/2T)}\frac{\zeta_{\mathbf{k}}^2}{E_{\mathbf{k}}^2}\delta(\mathbf{v}_g\cdot\mathbf{q}), \quad (2)$$

where the group velocity $\mathbf{v}_g$ is given by

$$\mathbf{v}_g(\mathbf{k}) = \nabla_{\mathbf{k}}E_{\mathbf{k}} = \frac{1}{E_{\mathbf{k}}}\left(\zeta_{\mathbf{k}}\mathbf{v}_f + \Delta_{\mathbf{k}}\frac{\partial \Delta_{\mathbf{k}}}{\partial \mathbf{k}}\right). \quad (3)$$

We now consider a model two-dimensional d-wave superconductor with the wave vector $\mathbf{q}$ in the plane at an angle θ from the x axis. For $\mathbf{q}$ not *exactly* in the nodal or antinodal direction $\mathbf{v}_f\cdot\mathbf{q}=0$ does not satisfy the condition imposed by the delta function; we exclude these two specific cases from the analysis below, although we allow θ to be arbitrarily

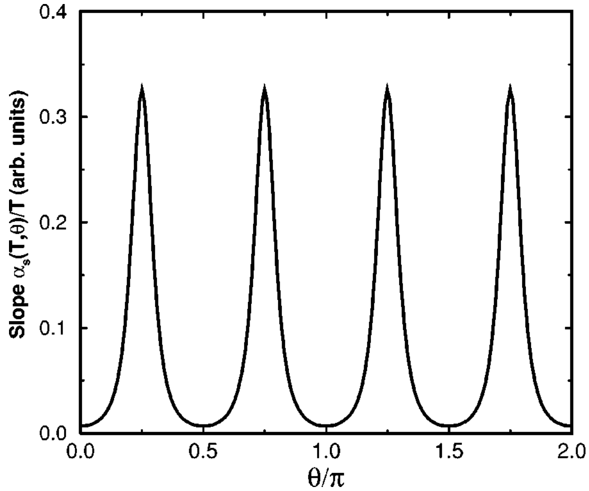

FIG. 2. Angular dependence of the low-temperature slope of the attenuation for $v_2/v_f = 0.2$.

close to these directions, so that the restriction has no bearing on the generality of the results. In the low-temperature regime $T \ll T_c$

$$\frac{\alpha_s(T,\theta)}{\alpha(T_c)} \approx \ln 2 \frac{v_2}{v_f}\frac{T}{\epsilon_f} \times \left(\frac{\sin^2(\theta-\pi/4)}{[\cos^2(\theta-\pi/4)+(v_2/v_f)^2\sin^2(\theta-\pi/4)]^{3/2}} + \frac{\cos^2(\theta-\pi/4)}{[\sin^2(\theta-\pi/4)+(v_2/v_f)^2\cos^2(\theta-\pi/4)]^{3/2}} \right). \tag{4}$$

Even though there is a normalization constant $\alpha(T_c)$ here, the ratio of the attenuation coefficients at different angles is a function of v_2/v_f only and therefore provides a direct measurement of this important parameter. Notice also that the slope vanishes as $v_2/v_f \to 0$. Equation (4) represents the main result of this paper, and in Fig. 2 we show the slope as a function of the direction in the plane.

To obtain an explicit temperature dependence and compare our results with those of Ref. 16 in more detail we now consider a superconductor with a cylindrical Fermi surface. Then the attenuation is given by

$$\alpha_s(T,\theta) = \frac{A'}{2T}\frac{\Omega}{v_f q}\int_0^{2\pi}\frac{d\phi}{|\cos(\theta-\phi)|}\frac{\zeta_0^2(\phi)}{E(\phi)}\cosh^{-2}\left(\frac{E(\phi)}{2T}\right), \tag{5}$$

where $\zeta_0(\phi) = \Delta_0^2 \sin 4\phi \tan(\theta-\phi)/2\epsilon_f$, and $E(\phi) = \sqrt{\zeta_0^2(\phi)+\Delta^2(\phi)}$. Notice that the contribution to the integral of the regions $\phi \approx \theta \pm \pi/2$, where $\mathbf{v}_f \cdot \mathbf{q} = 0$ and $\cos(\theta-\phi)$ tends to zero, is suppressed exponentially by the thermal function as $\tan(\theta-\phi)$, and consequently $E(\phi)$, becomes large.

In Fig. 3 we present the results for the ultrasonic attenuation normalized by its value at the transition temperature for near-nodal (as the nodes are at $m\pi/4$, the direction normal to a node coincides with the nodal direction) and antinodal orientation of the wave vector $\mathbf{q}$ for different values of the ratio

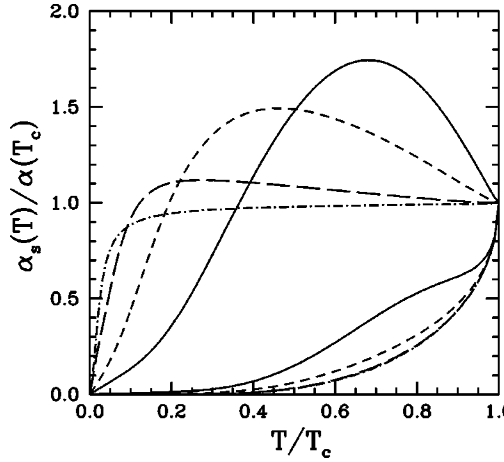

FIG. 3. Temperature dependence of the attenuation coefficient in the near-nodal ($\theta = 0.78$, upper curves) and antinodal ($\theta = 1.57$, lower curves) directions for ratios of Δ_0/ϵ_f of 0.25 (solid line), 0.1 (dashed line), 0.025 (long-dashed line), and 0.0025 (dot-dashed line).

of Δ_0/ϵ_f, which are very similar in general features to those obtained numerically in Ref. 16. In the numerical work we have used the BCS ratio $\Delta_0 = 2.14 T_c$, a larger coefficient would push both the maximum and the onset of linear behavior towards higher temperatures. For the near-nodal direction there is a clearly defined maximum of the attenuation at high temperatures, the position of the maximum scales with Δ_0^2/ϵ_f as predicted above. The attenuation is linear at low temperatures. For the antinodal direction the decay of $\alpha_s(T)$ is qualitatively close to the exponential behavior, although the dependence on the ratio Δ_0/ϵ_f is clearly seen, and the shoulder on the curve for the largest value of the ratio indicates that the additional phase space for scattering has become available even for $\mathbf{q}$ near the antinode. For both angles the results for the smallest ratio of $\Delta_0/\epsilon_f = 0.0025$ are indistinguishable on the scale of this graph from the generalized BCS prediction[13–15] that $\alpha_s(T,\theta)/\alpha_n = f[\Delta(\theta + \pi/2)]$, where f is the Fermi function. We also obtain that

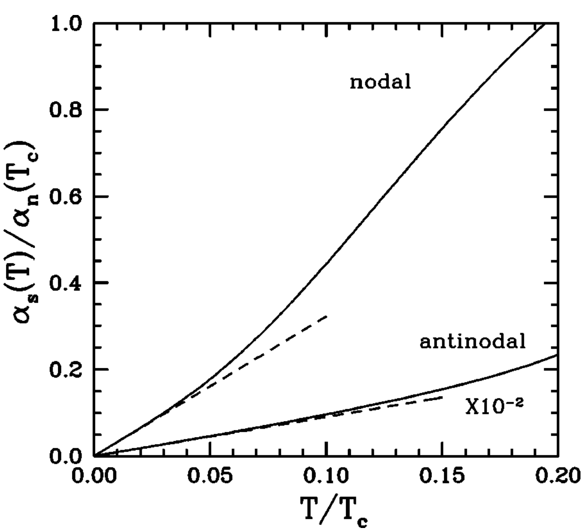

FIG. 4. Low-temperature dependence of the attenuation coefficient in the near-nodal ($\theta = 0.78$) and antinodal ($\theta = 1.57$) directions for $\Delta_0/\epsilon_f = 0.1$. Dashed line: analytic result from Eq. (4).

for a fixed value of the ratio Δ_0/ϵ_f, the window within which the maximum can be observed is narrowly centered around the nodal direction, as expected. Finally, Fig. 4 demonstrates the linear, in temperature, behavior predicted here for $T \ll T_c$, which is in a sharp contrast to the exponential decay of the Fermi function, and agrees remarkably well with the result given in Eq. (4).

The maximum of α_s is at high temperature where the electronic contribution to the attenuation is difficult to measure. While our results are qualitatively the same for a different geometry of the Fermi surface, such as a tight binding, the position of the maximum may be shifted; we also note that for a tight-binding Fermi surface close to half filling the peak does not exist: large flat regions of the Fermi surface away from the nodes contribute to the attenuation in the near-nodal direction above T_c, and in the superconducting state the loss of phase space in the gapped regions cannot be compensated for by the increase in scattering near the nodes. Moderate impurity scattering is expected to ''average'' the attenuation over a small range of angles. As seen from Fig. 2, this can change the slope near the node, but has little effect on the slope away from the node, and this result should be robust with respect to scattering by dilute impurity centers. Therefore measurement of the low-temperature slope provides a direct measurement of the parameter v_2/v_f, which determines the low-temperature behavior of the cuprates. We also note that we expect the general considerations of this work, including the breaking of particle-hole symmetry illustrated in Fig. 1(b), to be important for other transport coefficients, such as Hall effect, although a different and detailed analysis is needed there.

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