Higher corrections to the mass current in weakly inhomogeneous superfluid ^{3}He-A

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The mass current $\vec{j}$ in the weakly inhomogeneous superfluid A phase of helium-3 is calculated near zero temperature by means of an exact solution of the Dyson-Gorkov equation with linearized order parameter. Two general representations for $\vec{j}$ are obtained in the form of a series and an integral. The standard representation $\vec{j}_0$ for the mass current is known up to the first order in gradients of the order parameter. Moreover, there are indications that higher (quadratic) corrections to $\vec{j}_0$ are possible. We consider three static orientations of the orbital angular momentum $\hat{l}$ with respect to its curl. For the resulting representations, we obtain asymptotic expansions in the gradients of $\hat{l}$ in the London limit (i.e., when the coherence length is smaller than the length in the $\hat{l}$-vector texture). The correcting terms to $\vec{j}_0$ are obtained up to the third order. We numerically estimate the coefficients at the quadratic terms and show that these terms cannot be ignored. Moreover, new cubic contributions that include the logarithm of the London parameter are presented. [S0163-1829(99)12705-X]

I. INTRODUCTION

Presently, the superfluidity of helium-3 is a focus of intensive theoretical and experimental studies.[1] Although considerable attention has been given recently to such problems as quantized vorticity and interfaces,[1–4] the weakly inhomogeneous A-phase of helium-3 (^{3}He-A) still calls for theoretical investigation. This phase arises due to the p-wave spin triplet BCS pairing[5,6] and possesses highly unusual properties.[7] A peculiarity of this phase can be seen, e.g., from the structure of the mass current $\vec{j}$, which, as generally accepted, is of the first order in gradients,[5–7]

$$\vec{j}_0=\rho\,\vec{v}_s+\frac{1}{4m}\,\mathrm{rot}\,(\rho\,\hat{l})+\vec{j}_{\mathrm{an}}\quad (T=0), \tag{1}$$

$$\vec{j}_{\mathrm{an}}=-\frac{1}{2m}\,\mathrm{C}_0\,\hat{l}\,(\hat{l}\cdot\mathrm{rot}\,\hat{l}).$$

Here the first two terms are standard for a nodes-free p-wave superfluid, while the famous anomalous $\vec{j}_{\mathrm{an}}$ testifies that there exist two nodes in the gap on the Fermi surface for the real ^{3}He-A.[2,7] In Eq. (1), ρ is the liquid density, m is the atom mass, $\vec{v}_s$ is the superfluid velocity, $\hat{l}$ is the weakly inhomogeneous orbital angular momentum vector (here ''hat'' is used to denote a unit vector), and $\mathrm{C}_0\approx\rho$. Equation (1) was obtained by many authors in different ways, e.g., by solving the Gorkov[8–10] or matrix kinetic[11] equations, or directly, with the use of the ground state wave function (see Ref. 12).

Although Eq. (1) and the corresponding physical picture have been broadly discussed and accepted,[13–21] indications can be found that higher corrections to Eq. (1) might occur. Thus, Volovik and Mineev considered the free energy of ^{3}He-A and obtained one of these corrections in the form $\chi_{\mathrm{orb}}D\hat{l}_a\vec{\partial}\hat{l}_a$, where $D=\partial_t+\vec{v}_s\cdot\vec{\partial}$.[13] Later, a quasiclassical approach[22] was developed in order to check the applicability of the generalized gauge-transformation approximation used in Ref. 13. The existence of the quadratic terms

$$\rho\,\xi_0|\hat{l}\times\mathrm{rot}\hat{l}|\left(A(\hat{l}\cdot\vec{v}_s)\,\hat{l}+\frac{C}{m}(\mathrm{rot}\,\hat{l}-(\hat{l}\cdot\mathrm{rot}\,\hat{l})\,\hat{l})\right)$$

was established in Ref. 22 (ξ_0 is the zero-temperature coherence length, A and C are constants). With the aim to obtain a new verification of the presence of $\vec{j}_{\mathrm{an}}$ in Eq. (1), Combescot and Dombre developed a microscopic calculation,[10] which allowed them to conclude that, at $T=0$, the quadratic correction $|\hat{l}\times\mathrm{rot}\,\hat{l}|(\mathrm{rot}\,\hat{l})_\perp$ exists in the current perpendicular to $\hat{l}$ and the terms

$$|\hat{l}\times\mathrm{rot}\,\hat{l}|(\vec{v}_s-(1/4m)\mathrm{rot}\,\hat{l})_{||},\quad |\hat{l}\times\mathrm{rot}\,\hat{l}|(\partial_1\hat{l}_2+\partial_2\hat{l}_1)$$

exist in the current parallel to $\hat{l}$. Clearly, the corrections found in Ref. 22 are covered by those from Ref. 10. However, the corresponding numerical coefficients have not been worked out successfully either in Ref. 10 or in Ref. 22. As is noted in Ref. 10, these quadratic corrections would cause difficulties of the superfluid hydrodynamics of ^{3}He-A at $T=0$. As indicated in Ref. 22, unravelling the problem of higher corrections to Eq. (1) turns out to be of importance in understanding subtleties of the physics of ^{3}He-A.

It is curious that two years after,[10] Combescot and Dombre declared in Ref. 23 that all the integrals found in Ref. 10 as the coefficients at the non-analyticities are, mainly, zero. In spite of the fact that Ref. 10 was devoted to the case $T=0$, and the Gorkov equation was solved exactly (after linearization of the order parameter), an intermediate high-temperature approximation was not avoid to obtain a manageable formula for the ξ-integrated Green's function. The difference between the approximate and exact formulas was regarded as a reason of the presence of the second order corrections to Eq. (1). However, the strategy of Ref. 10 did not permit one to benefit from the exact solution, and, thus, to state definitely that the coefficients are finite, nonzero, and not negligible. To a certain extent, this was the reason of the appearance of the refutation in Ref. 23. As to the higher contributions at $T=0$ (e.g., the logarithmic terms deduced in the present paper), the regular expansion procedure was not formulated clearly in Ref. 10.

To obtain an unambiguous procedure of asymptotic expansion for $\vec{j}$ is an important technical problem. The solution of this problem would lead to a more deep understanding of ^{3}He-A, and a cure for mathematical difficulties arising in Ref. 10 was suggested in Refs. 24 and 25. Namely, a different method of solving the Dyson-Gorkov equation was used. This method immediately produced new representations for the fermionic Green's function and, thus, for $\vec{j}$. It is well known in mathematical physics that the Green's function of a Sturm-Liouville operator can be written either as an integral or as a series in the eigenfunctions. The approach under consideration uses the second possibility. After the paper in Ref. 26, it became clear that this way looks suitable because the new representations for $\vec{j}$ admit $T\to 0$ accurately and can rigorously be studied by the Laplace method (steepest descent) provided the characteristic length in the texture is much longer than the coherence length ξ_0 (the London limit). As the result, it becomes possible to deduce the corrections to Eq. (1) in the form of an asymptotic series in the gradients of $\hat{l}$. It is crucial that the coefficients in question acquire a manageable form.

The present paper provides support for the results of Refs. 10 and 13; it also contains, as special cases, the results of these papers (as well as of Ref. 22) about the terms quadratic in gradients, and enables us to calculate the corresponding numerical coefficients. Moreover, new third order logarithmic contributions are found at the same footing as the quadratic ones. To catch higher terms, one should go beyond the linearization of the order parameter. With the exception of Ref. 25, only in the paper of Ref. 23 an attempt was made to estimate the numerical coefficients at the quadratic corrections to Eq. (1). Since the present investigation has the same starting point as that in Ref. 10 while the technical procedure is different (it also differs from the iterative approach of Ref. 22), it becomes possible to choose between the Refs. 10 and 23 in favor of the first one.

The present paper completes the series of papers.[24–26] Some technical details omitted here can be found in Ref. 26. The paper is organized as follows. Section II contains the outline of the problem and, actually, it is the same as in Ref. 10. Basic approximations and notation are almost unaffected here, and the reader is referred to Ref. 10 for details. Section III is concerned with the solution of the ordinary nonhomogeneous differential equation related to the Dyson-Gorkov equation and with the calculation of the mass current in the form of a series. Section IV deals with the integral representations for that series as well as with various limits for them. The supplementary term to $\vec{j}_0$ (1) in the form of an integral is found in Sec. V at $T=0$, and three special mutual orientations of rot $\hat{l}$ and $\hat{l}$ are considered to expand it up to the third order. Apart from the quadratic terms predicted in Ref. 10, new cubic contributions are found. These terms contain the logarithm of the London parameter. The numerical coefficients at the second order terms are estimated. The content of Sec. V is the main achievement of the present paper. Since it is difficult to study the problem of higher corrections in the general form, three typical choices of the static order parameter enable us to clarify the situation by means of specific calculations. In Sec. VI, we compare the results of the present paper with those of Ref. 23 to understand the reason of the negative statement in Ref. 23. The discussion in Sec. VII concludes the paper. Hopefully, the present investigation will be useful for any systematic microscopic approach to correct observables in ^{3}He-A.

II. OUTLINE OF THE PROBLEM

Since our objective is to calculate the mass current $\vec{j}$ by means of the normal Green's function, we start with the Dyson-Gorkov standard matrix equation

$$\partial_\tau g(\vec{k},\vec{k}') - \int d^3k'' H(\vec{k},\vec{k}'')\, g(\vec{k}'',\vec{k}') = (2\pi)^3 \delta^{(3)}(k-k')\, \delta(\tau-\tau'). \qquad (2)$$

Here τ is the ''imaginary'' time of thermal approach, $g(\vec{k},\vec{k}')$ is the 2×2 matrix of normal and anomalous two-point Green's functions, and $H(\vec{k},\vec{k}'')$ has the conventional BCS form,

$$H(\vec{k},\vec{k}'') = \begin{pmatrix} \xi_{k''}\delta^{(3)}(k-k'') & (2\pi)^{-3}\Delta(\vec{k},\vec{k}'') \\ (2\pi)^{-3}\Delta^*(\vec{k}'',\vec{k}) & -\xi_{k''}\delta^{(3)}(k-k'') \end{pmatrix},$$

where $\xi_k \equiv (k^2-k_F^2)/2m$, k_F is the Fermi momentum, and $\Delta(\vec{k},\vec{k}'')$ is the order parameter of ^{3}He-A. We shall calculate the current by the formula

$$\vec{j} = \beta^{-1}\sum_\omega (2\pi)^{-3}\int d^3k\, \vec{k}\, g_{11}. \qquad (3)$$

It is convenient to pass in Eqs. (2) and (3) to the mixed coordinate-momentum representation[12,27,28] as follows:

$$H(\vec{k},\vec{r}) = (2\pi)^{-3}\int d^3q\, H(\vec{k}+\vec{q}/2, \vec{k}-\vec{q}/2)\, e^{i\,\vec{q}\cdot\vec{r}},$$

$$g_{\vec{k}}(\vec{r}) = (2\pi)^{-3}\int d^3q\, g(\vec{k}+\vec{q},\vec{k})\, e^{i\,\vec{q}\cdot\vec{r}},$$

where $\vec{r} = \frac{1}{2}(\vec{r}_1+\vec{r}_2)$ is the center of mass coordinate and the momentum $\vec{k}$ is conjugate to $\vec{r}_1-\vec{r}_2$. Using the Fourier expansion in τ, we obtain from Eq. (2) the equation

$$i\omega\, g_{\vec{k}}(\vec{r}) - \left[H\left(\vec{k} - i\vec{\partial}_r - \frac{i}{2}\vec{\partial}_y, \vec{y} \right) g_{\vec{k}}(\vec{r}) \right]\Bigg|_{\vec{y}=\vec{r}} = \mathbf{1},$$

where $\mathbf{1}$ is the unit matrix and ω is the Matsubara fermionic frequency. Assuming that $|\vec{k}|\approx k_F$ and, thus, $\xi_{(k-i\partial)} \approx \xi_k - i\, c_F\, \hat{k}\cdot\vec{\partial}$, we obtain to the lowest order in gradients:

$$i\omega g-\begin{pmatrix} \xi-ic_F\hat{k}\cdot\vec{\partial} & \Delta(\vec{k},\vec{r}) \\ \Delta^*(\vec{k},\vec{r}) & -\xi+ic_F\hat{k}\cdot\vec{\partial} \end{pmatrix} g=\mathbf{1}, \qquad (4)$$

where $g\equiv g_{\vec{k}}(\vec{r})$, $\xi\approx c_F(k-k_F)$, and c_F is the Fermi velocity. The order parameter $\Delta(\vec{k},\vec{r})$ has the form $\delta[\hat{k}\cdot\hat{\Delta}_1(\vec{r})+i\hat{k}\cdot\hat{\Delta}_2(\vec{r})]$, where δ is the gap amplitude, $\hat{k}$ is unit reciprocal vector and the orbital momentum vector is given by $\hat{\Delta}_1\times\hat{\Delta}_2=\hat{l}$.

The resulting approximate equation (4) can nicely be treated as one-dimensional since the spatial differentiations are performed along the directions indicated by $\hat{k}$. In Ref. 28, the gradient expansion method is presented to study the dynamics of spatially inhomogeneous systems provided the inhomogeneities are slow in comparison with the relevant length scales. As a result, the three-dimensional problem splits into one-dimensional subsystems. The proofs required to justify Eq. (4) can be found in Ref. 28.

Since we are interested in $\vec{j}$ at arbitrary point, say, $\mathcal{O}$, we define the spherical coordinates ρ,θ,ϕ centered at this point and linearize the slowly varying order parameter,

$$\Delta(\vec{k},\vec{r})\approx\Delta(\hat{k},\rho=0)+\alpha\rho\equiv\alpha(\rho_0+\rho)+i\Delta, \qquad (5)$$

where $\Delta\equiv\operatorname{Im}\Delta(\hat{k},\rho=0)$ and $\alpha\rho_0$ denotes $\operatorname{Re}\Delta(\hat{k},\rho=0)$. As the physical result is assumed to be independent of the choice of $\mathcal{O}$, it can be calculated at any $\vec{r}$ with $\vec{r}\to\mathcal{O}$ in final formulas. Therefore, we solve Eq. (4) at $\vec{r}=\rho\hat{k}$, where $\hat{k}\cdot\vec{\partial}$ is simply $\partial/\partial\rho$ and substitute $\rho=0$ in the result.[10] Moreover, we consider our problem for the coherence length $\xi_0=c_F/\delta$ much smaller than the length of the variation of the orbital vector $\hat{l}$,

$$\frac{1}{\chi^2}\equiv\xi_0|\vec{\partial}\otimes\hat{l}|\ll 1 \qquad (6)$$

(the London limit). The parameter α in Eq. (5) depends on the angle variables, on the components of the superfluid velocity $\vec{v}_s$, and on the first derivatives of $\hat{l}$ taken at $\mathcal{O}$ (Appendix A). It can be seen from Eq. (6) that the condition $\alpha\rho\lesssim\delta$ ensuring the linearization (5) implies $\rho/\xi_0\lesssim\chi^2$, and holds the better the larger χ^2.

Changing the variable $x=(\alpha/c_F)^{1/2}(\rho+\rho_0)$ and eliminating ξ from the LHS of Eq. (4), we obtain

$$(i\omega+\mathcal{H})\,G=e^{ix\xi(\alpha c_F)^{-1/2}}\mathbf{1}, \qquad (7)$$

where

$$\mathcal{H}=i\sqrt{\alpha c_F}\sigma_3\frac{d}{dx}-\sqrt{\alpha c_F}\sigma_1 x+\Delta\sigma_2 \qquad (8)$$

is the Hamiltonian expressed in terms of the Pauli matrices and αc_F is a positive real number. In this case, Eq. (3) takes the form

$$\vec{j}=k_F^3(8\pi^3c_F)^{-1}\int d\Omega\,\hat{k}\left(\beta^{-1}\sum_{\omega}\mathcal{J}\right), \qquad (9)$$

where $\mathcal{J}$ is the ξ-integrated Green's function,

$$\mathcal{J}(x)=\int d\xi\,e^{-ix\xi(\alpha c_F)^{-1/2}}G_{11}(x), \qquad (10)$$

and $G(x)$ is determined from Eq. (7). We take $\rho=0$ in the final formulas, i.e., $x=x_0$, where $x_0\equiv\operatorname{Re}\Delta(\hat{k},\rho=0)/\sqrt{\alpha c_F}$.

III. SOLUTION OF THE NONHOMOGENEOUS EQUATION IN THE SERIES FORM

To solve Eq. (7), we represent $G(x)$ as the product of the matrices

$$G=\sqrt{2}\,u\begin{pmatrix} h_1 & h_2 \\ f_1 & f_2\end{pmatrix}, \quad u\equiv\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}, \qquad (11)$$

where $h_{1,2}\equiv h_{1,2}(x)$ and $f_{1,2}\equiv f_{1,2}(x)$ are now to be determined. The conjugation $u^{-1}\sigma_1 u=\sigma_2$ (cycl.perm.) by the unitary matrix u on the Pauli matrices transforms $\mathcal{H}$ (8) to $\mathcal{H}_{\rm em}$,

$$u^{-1}\mathcal{H}u=\mathcal{H}_{\rm em}, \quad \mathcal{H}_{\rm em}=\begin{pmatrix} \Delta & i\sqrt{\alpha c_F}a^- \\ -i\sqrt{\alpha c_F}a^+ & -\Delta\end{pmatrix}, \qquad (12)$$

where $a^\pm=x\mp d/dx$. The operator $\mathcal{H}_{\rm em}$ resembles the Hamiltonian of a spinning electron in a constant homogeneous magnetic field, and its eigenvalues E_0, $\pm E_n$ and eigenfunctions $\hat{\Psi}_0$, $\hat{\Psi}_n^{\pm}$ $(n\geqslant 1)$ can be found in Appendix B.

We use Eq. (12) to pass from Eq. (7) to the equation

$$(i\omega+\mathcal{H}_{\rm em})\begin{pmatrix} h \\ f\end{pmatrix}=\delta(x-x')\,e^{ix\xi(\alpha c_F)^{-1/2}}\begin{pmatrix}\frac{1}{2} \\ \frac{1}{2}\end{pmatrix}, \qquad (13)$$

with the Dirac δ-function in the RHS. Now the unknowns h and f depend on x and x' and the required entry of G is given by

$$G_{11}(x)=\int dx'(h(x,x')+f(x,x')). \qquad (14)$$

It is natural to solve Eq. (13) by expanding $\binom{f}{h}$ in the orthogonal $\hat{\Psi}_0$, $\hat{\Psi}_n^{\pm}$ $(n\geqslant 1)$ and obtain $\mathcal{J}$ from Eqs. (10) and (14),

$$\mathcal{J}=\pi\sqrt{\alpha c_F}\left[\frac{\langle\hat{\Psi}_0,\hat{\Psi}_0\rangle}{i\omega+E_0}+\sum_{s=\pm}\sum_{n=1}^{\infty}\frac{\langle\hat{\Psi}_n^{(s)},\hat{\Psi}_n^{(s)}\rangle}{i\omega+sE_n}\right], \qquad (15)$$

where $\langle\,\rangle$ stands for the Hermitian scalar product. Representation (15) for the ξ-integrated Green's function is alternative to that found in Ref. 10 as a quadratic combination of the parabolic cylinder functions.

Now the summation over ω is straightforward,[29] and we obtain from Eqs. (9) and (15) the required general representation for the mass current near zero temperature,[25]

$$\vec{j}=k_F^3(8\pi^2 c_F)^{-1}\int d\Omega\,\hat{k}\sqrt{\alpha c_F}\times\left[n(E_0)\psi_0^2+\frac{\Delta}{2}\sum_{n=1}^{\infty}(\psi_{n-1}^2-\psi_n^2)\frac{\tanh(\beta E_n/2)}{E_n}\right],\tag{16}$$

where $n(E_0)$ is the Fermi weight. Due to the explicit dependence on β, Eq. (16) admits $T\to 0$, namely, one must replace $n(E_0)$ by the Heavyside function $\theta(E_0)$ and $\tanh(\beta E_n/2)$ by 1.

IV. INTEGRAL REPRESENTATIONS AND THEIR LIMITS

A. Integral representations

Practically, it is more convenient to deal not with Eq. (16) itself, but with another equivalent representation which uses $\mathcal{J}$ in the form of an integral. This representation was presented in Ref. 25 at zero temperature. Here we shall obtain it at $T\neq 0$.[26] First, it is necessary to rearrange Eq. (15) and extract the part that is "even" in ω ($\vec{j}$ is not sensitive to the "odd" one[25,26]). We have

$$\mathcal{J}_e=\frac{\pi}{2}\frac{\Delta}{\sqrt{\alpha c_F}}\sum_{n=0}^{\infty}\psi_n^2[(|\lambda|^2+n+1)^{-1}-(|\lambda|^2+n)^{-1}],\tag{17}$$

where $|\lambda|^2\equiv(\omega^2+\Delta^2)(2\alpha c_F)^{-1}$.

By Appendix C, series (17) can formally be expressed as the integral

$$\mathcal{J}_e=-\Delta\left(\frac{\pi}{\alpha c_F}\right)^{1/2}\int_0^{\infty}dt\,(\tanh t)^{1/2}\,e^{-x^2\tanh t-2|\lambda|^2 t},\tag{18}$$

and we proceed further with the frequency summation,

$$\beta^{-1}\sum_{\omega}\mathcal{J}_e=-\Delta\left(\frac{\pi}{\alpha c_F}\right)^{1/2}\int_0^{\infty}dt\,(\tanh t)^{1/2}(T\,\vartheta_2(0,i\tau))\times e^{-x^2\tanh t-(\Delta^2/\alpha c_F)\,t},\tag{19}$$

where the elliptic theta function ϑ_2[30] is defined by the series $2\sum_{m=0}^{\infty}a^{(m+1/2)^2}=\vartheta_2(0,i\tau)$, $\tau=(-1/\pi)\log a$, and $a=\exp(-4\pi^2T^2t/\alpha c_F)$. Changing the integration variable $t\mapsto\kappa t$, $\kappa=\alpha c_F(\beta/2)^2$, one can rewrite Eq. (19) in a more convenient form to study the special limit below,

$$\beta^{-1}\sum_{\omega}\mathcal{J}_e=-\Delta\,\frac{\kappa^{1/2}}{2}\int_0^{\infty}dt\left(\frac{\tanh(\kappa t)}{t}\right)^{1/2}\tilde{\Theta}(t)\times e^{-x^2\tanh(\kappa t)-(\Delta\beta/2)^2 t},\tag{20}$$

where $\tilde{\Theta}(t)=(\pi t)^{1/2}\,\vartheta_2(0,i\pi t)$.

To obtain the general integral representations for $\vec{j}$ near zero temperature, we must substitute Eqs. (19) and (20) into Eq. (9). These representations are very convenient in calculating higher corrections to Eq. (1). Before proceeding to this in Sec. V, we consider certain specific limits to gain a certain confidence to the representations obtained.

B. Limiting cases

It follows from Eq. (A1) that the square of the absolute value of the gap parameter $\Delta_0\equiv\Delta(\hat{k},\rho=0)$ has the following simple form:

$$|\Delta_0|^2\equiv|\Delta|^2=\Delta^2+\alpha c_F x_0^2=\delta^2\sin^2\theta.\tag{21}$$

Due to $\lim_{\tau\to 0}\vartheta_2(0,i\tau)=\tau^{-1/2}$ (Ref. 30) in Eq. (19), the current can be written at $T=0$ as follows:

$$\vec{j}=-3\rho\,(8\pi c_F)^{-1}\int d\Omega\,\hat{k}\,\Delta\times\int_0^{\infty}dt\left(\frac{\tanh t}{t}\right)^{1/2}e^{-x^2\tanh t-(\Delta^2/\alpha c_F)\,t},\tag{22}$$

where $\rho=k_F^3/3\pi^2$ (two spin projections are taken into account). To get the lowest gradient approximation [see Eqs. (6), (A3)], we replace tanh t by t in Eq. (22),

$$\vec{j}=-3\rho\,(8\pi c_F)^{-1}\int d\Omega\,\hat{k}\Delta\int_0^{\infty}dt\,e^{-(|\Delta|^2/\alpha c_F)\,t}=3\rho\,(8\pi)^{-1}\int d\Omega\,\hat{k}\left((\hat{k}\cdot\vec{\partial})\arctan\left(\frac{\hat{k}\cdot\hat{\Delta}_2}{\hat{k}\cdot\hat{\Delta}_1}\right)\right),\tag{23}$$

where

$$\hat{k}\cdot\vec{\partial}=\sqrt{\frac{\alpha}{c_F}}\frac{d}{dx},\quad\frac{\hat{k}\cdot\hat{\Delta}_2}{\hat{k}\cdot\hat{\Delta}_1}=\frac{\Delta}{x\sqrt{\alpha c_F}}$$

and $x=x_0$. Equation (23) results in Eq. (1) with $\hat{l}=\hat{\Delta}_1\times\hat{\Delta}_2$ and $(v_s)_i=\frac{1}{2}\hat{\Delta}_1\cdot\partial_i\hat{\Delta}_2$,[19,20,22] and, therefore, Eq. (1) is the lowest London approximation to Eq. (22). This estimate fails when $\Delta^2/\alpha c_F\ll 1$, i.e., practically, near the nodes at $\theta=0,\pi$. However, it can be verified that these regions are irrelevant since we are interested only in the second order corrections to Eq. (1) and also in the third order corrections that contain the logarithm of the London parameter. However, the resulting equation (23) converges on the entire sphere.

In the opposite way, one should replace $\tanh(\kappa t)$ by κt in Eq. (20) since the steepest descent is possible at $|\Delta|\neq 0$ for large β:

$$\beta^{-1}\sum_{\omega}\mathcal{J}_e=-\frac{\alpha c_F\Delta}{8}\beta^2\int_0^{\infty}dt\,\tilde{\Theta}(t)\,e^{-(|\Delta|\beta/2)^2\,t}.\tag{24}$$

The RHS of Eq. (24), which is the Laplace transform of $\tilde{\Theta}(t)$, can be expressed in terms of the so-called Yosida function Y:[31]

$$a^2\int_0^{\infty}dt\,\tilde{\Theta}(t)\,e^{-a^2 t}=1-\int_0^{\infty}\frac{dy}{\cosh^2\sqrt{y^2+a^2}}\equiv 1-\mathrm{Y}(a).\tag{25}$$

The proof of (25) can be found in Ref. 26. By (25), we have

$$\vec{j}=-3\rho\,(8\pi)^{-1}\int d\Omega\,\hat{k}\,\frac{\alpha\Delta}{|\Delta|^2}\left(1-\Upsilon\left(\frac{|\Delta|\beta}{2}\right)\right). \quad (26)$$

Equation (26), found by Cross in Ref. 8, gives the leading $\vec{j}_0$ ''dressed'' by thermal corrections. At zero temperature, we have $\Upsilon(\infty)=0$, and we recover (23).

V. EXPLICIT CALCULATIONS

This section is devoted to the main calculations of the present paper, i.e., it is concerned with the asymptotic expansion of $\vec{j}$ (22) to deduce the London limit corrections to Eq. (1). At fixed $\hat{k}$, the overall phase of the order parameter $\Delta(\hat{k},\vec{r})$ can always be changed to make $\alpha c_{_F}$ a real positive. Thus, Eq. (5) can be regarded as

$$\exp(-i\psi)\,\Delta(\hat{k},\vec{r})\equiv\Delta_0+\alpha\,\rho, \quad (27)$$

where

$$\alpha=\delta\,\mathcal{M}\exp(i(\pi/2-\psi)), \quad \Delta_0=\delta\sin\theta\exp(i(\phi-\psi)) \quad (28)$$

(see Appendix A). We require $\alpha c_{_F}$ to be positive in order to assign a meaning to Eqs. (7), (8) as well as to all other resulting formulas. As noted above, a weakly varying texture of the order parameter is considered, and all the information about it (i.e., about the order parameter gradients) is enclosed in the function $\mathcal{M}$ (A2). Here it is appropriate to put several explanatory remarks about $\mathcal{M}$. Without loss of generality, $\hat{\Delta}_2(\mathcal{O})$ can be chosen along $\hat{l}\times\mathrm{rot}\,\hat{l}$ so that $\partial_3\hat{l}_1$, $\partial_3\hat{l}_3$ become zero and, thus, $\mathrm{div}\,\hat{l}=\partial_1\hat{l}_1+\partial_2\hat{l}_2$. Besides, it is useful to recall that $\hat{l}\cdot\mathrm{rot}\,\hat{l}$ is simply $\partial_1\hat{l}_2-\partial_2\hat{l}_1$ once the third axis is chosen along $\hat{l}(\mathcal{O})$. Moreover, $\partial_1\hat{l}_1,\partial_2\hat{l}_2$ can be excluded from the consideration.[10] Therefore, apart from the velocity components v_1 and v_2, only three gradient combinations $\partial_1\hat{l}_2+\partial_2\hat{l}_1$, $2mv_3-(1/2)\,\hat{l}\cdot\mathrm{rot}\,\hat{l}$, and $\partial_3\hat{l}_2=\mathrm{rot}\,\hat{l}\times\hat{l}$ are relevant in $\mathcal{M}$. Besides, no technical difference is expected if $\mathcal{M}$ is regarded as dependent either on $\partial_1\hat{l}_2+\partial_2\hat{l}_1$ or $2mv_3-(1/2)\,\hat{l}\cdot\mathrm{rot}\,\hat{l}$ separately.

The approach of Ref. 10 uses positivity of α but does not specify the auxiliary phase ψ. It is crucial that we determine the phase ψ in (28) by explicit calculation. To make the calculation manageable, it is convenient to put a part of gradients in $\mathcal{M}$ equal to zero, and so to consider the dependence of $\vec{j}$ on the remaining ones. Clearly, it is not necessary to enumerate all the possibilities, but it is sufficient to point out only characteristic combinations. To this end, we take $\mathcal{M}$ in the following reduced form:

$$\mathcal{M}=-\,\partial_3\hat{l}_2\cos^2\theta+(2mv-\partial_1\hat{l}_2)\sin\theta\cos\theta\,e^{i\phi}$$
$$\equiv-\frac{1}{\xi_0\chi_1^2}\cos^2\theta+\frac{1}{\xi_0\chi_2^2}\sin\theta\cos\theta\,e^{i\phi}, \quad (29)$$

where $v\equiv v_3$ and $2mv-\partial_1\hat{l}_2>0$.

A convenience is apparent after Refs. 26,27 to integrate by parts in Eq. (22) so that

$$\vec{j}=\vec{j}_0+\vec{j}_{\mathrm{corr}},$$

$$\vec{j}_{\mathrm{corr}}=-\,3\rho\,(8\pi c_{_F})^{-1}\int d\Omega\,\hat{k}\,\frac{\Delta}{Q}\,\Phi\,(x^2,Q), \quad (30)$$

with

$$\Phi(x^2,Q)=\int_0^\infty e^{-\,Q\,t}\left(\sqrt{\frac{\tanh t}{t}}\,e^{x^2(t-\tanh t)}\right)'dt,$$

$$Q=\frac{|\Delta|^2}{\alpha c_{_F}}. \quad (31)$$

The angle integration in $\vec{j}_{\mathrm{corr}}$ (30) is extended on the entire sphere owing to the convention that we are interested in the two types of corrections, and thus it is just responsible for the correcting contribution. Once ψ is found so that α is real positive with $\mathcal{M}$ (29), we get the following formulas for the parameters:

$$x_0^2=\left(\frac{\sin\phi}{\chi_1^2\tan^2\theta}\right)^2Q^3, \quad \delta^{-1}\frac{\Delta}{Q}=\cos\theta\left(\frac{1}{\chi_1^2}\,\frac{\cos\phi}{\tan\theta}-\frac{1}{\chi_2^2}\right),$$

$$\frac{1}{Q^2}=\left(\frac{1}{\chi_2^2}\,\frac{\sin\phi}{\tan\theta}\right)^2+\left(\frac{1}{\chi_2^2}\,\frac{\cos\phi}{\tan\theta}-\frac{1}{\chi_1^2}\,\frac{1}{\tan^2\theta}\right)^2, \quad (32)$$

while $\vec{j}_{\mathrm{corr}}$ has the form:

$$\vec{j}_{\mathrm{corr}}=3\rho\,(8\pi\xi_0)^{-1}\int d\Omega\,\hat{k}\cos\theta$$
$$\times\left(\frac{1}{\chi_2^2}-\frac{1}{\chi_1^2}\,\frac{\cos\phi}{\tan\theta}\right)\Phi\,(x_0^2,Q). \quad (33)$$

The rest of this section is to extract from $\vec{j}_{\mathrm{corr}}$ (33) two first asymptotic contributions next to Eq. (1). Varying the parameters χ_1, χ_2 in Eq. (33) we shall specialize three cases, examples1, 2, and 3, respectively. Fixing χ_2 (or χ_1) $\gg1$ and tending χ_1 (or χ_2) to infinity we shall get example 1 (or example 2). Taking $\chi_1=\chi_2=\chi\gg1$ we shall arrive to example 3. Practically, we shall be concerned with quadratic and cubic (times logarithm) terms, i.e., with those like $(\xi_0\chi^4)^{-1}$ and $(\xi_0\chi^6)^{-1}\log(1/\chi^2)$. Clearly, the case *1* corresponds to $\mathrm{rot}\,\hat{l}$ parallel to $\hat{l}$ and the case *2* — to $\mathrm{rot}\,\hat{l}$ perpendicular to $\hat{l}$. Therefore, the case *1* implies all the three contributions in Eq. (1), while the second one corresponds only to the pure orbital content of Eq. (1). Notice the notations $\mathcal{F}$ and $\mathcal{Q}$ to be used instead of Φ and Q to express that appropriate specializations have been done.

A. Example 1: rot $\hat{l}$ is parallel to $\hat{l}$

Let only $2mv-\partial_1\hat{l}_2\neq0$ in $\mathcal{M}$ (29). As far as we deduce from Eq. (32) that $\mathcal{Q}=\chi^2\,|\tan\theta|$ and $x_0=0$ (label 2 in χ is omitted), Eq. (33) reads only the third component, say, j to be nonzero:

$$j=\frac{3\rho}{2}\,\frac{1}{\xi_0\chi^2}\int_0^\infty\mathcal{F}(\chi^2u)\,\frac{u\,du}{(u^2+1)^{5/2}}, \quad (34)$$

where

$$\mathcal{F}(\chi^2 u)=\int_0^\infty e^{-\chi^2 ut}\left(\sqrt{\frac{\tanh t}{t}}\right)' dt,$$

and $u=|\tan\theta|$. As to the function $\mathcal{F}$, it is suffice to know that $\mathcal{F}$ is a constant as $s\to 0$, and

$$\mathcal{F}(\chi^2 u)=\frac{a}{(\chi^2 u)^2}+\frac{b}{(\chi^2 u)^4}+\cdots \tag{35}$$

as $\chi^2 u \gg 1$, where $a=-1/3$.

To do the asymptotic estimation let us break the integral over u into two parts:

$$\int_0^\infty \mathcal{F}(\chi^2 u)\,\frac{u\,du}{(u^2+1)^{5/2}}=U_1+U_2, \tag{36}$$

where

$$U_2=\int_1^\infty \mathcal{F}(\chi^2 u)\,\frac{u\,du}{(u^2+1)^{5/2}}\simeq\frac{a}{\chi^4}\int_1^\infty (u^2+1)^{-5/2}\,\frac{du}{u},$$

because $\chi^2 u \gg 1$ is valid. As to U_1, let us write it as $X+Y$, where

$$X=\frac{1}{\chi^4}\int_0^{\chi^2}\mathcal{F}(u)\left(\left(1+\frac{u^2}{\chi^4}\right)^{-5/2}-1\right)u\,du,$$

$$Y=\frac{1}{\chi^4}\int_0^{\chi^2}\mathcal{F}(u)\,u\,du. \tag{37}$$

Since the integral Y is divergent logarithmically at $\chi^2\to\infty$, it can be represented approximately:

$$Y\simeq\frac{a}{\chi^4}\log\chi^2+\frac{1}{\chi^4}\int_0^\infty\left(u\,\mathcal{F}(u)-\frac{a}{u+1}\right)du.$$

Let us turn to X (37). Equation (35) tells us that the integral X is divergent at $\chi\to\infty$, but can be regularized with the help of the whole asymptotics (35). Expanding the inner brackets in X into power series and using Eq. (35) it is not difficult to understand that the total contribution of the order χ^{-4} appears as that counter-term which results from X where $\mathcal{F}$ is replaced by a/u^2. Therefore, we combine

$$U_2+X\simeq\frac{a}{\chi^4}\left(-\frac{4}{3}+\log 2\right),$$

and, using $a=-1/3$, obtain

$$U_1+U_2\simeq\frac{1}{3\chi^4}\left(\frac{4}{3}+\int_0^\infty\left(3u\,\mathcal{F}(u)+\frac{1}{u+1}\right)du+\log\frac{1}{2\chi^2}\right). \tag{38}$$

Finally, the use of Eqs. (34), (36) and (38) enables the third component of $\vec{j}$ to be completely written as follows:

$$j_3=\frac{\rho}{2}\,\frac{1}{\xi_0\chi^2}\left(1+\frac{1}{\chi^4}\log\frac{\mathcal{B}}{\chi^2}\right), \tag{39}$$

where

$$\log 2\mathcal{B}=\frac{4}{3}+\int_0^\infty\left(3u\,\mathcal{F}(u)+\frac{1}{u+1}\right)du,$$

and $\vec{j}_0$ is restored. All the quadratic corrections predicted in Ref. 10 are zero in the present texture, and the lowest one turns out to be cubic with the logarithm of the London parameter.

B. Example 2: rot $\hat{l}$ is perpendicular to $\hat{l}$

In this case Eqs. (32) imply $\mathcal{Q}=\chi^2\tan^2\theta$ and $x_0^2=\chi^2\tan^2\theta\sin^2\phi$, where $(\xi_0\chi^2)^{-1}\equiv\partial_3\hat{l}_2>0$. The second and third components of $\vec{j}_{\rm corr}$ are zero after Eq. (33), while the first one acquires the form:

$$j=-\frac{3\rho}{4}\,\frac{1}{\xi_0\chi^2}\int_0^\infty \mathcal{F}(\chi^2u^2)\,\frac{u\,du}{(u^2+1)^{5/2}},$$

where

$$\mathcal{F}(\chi^2u^2)=\int_0^\infty e^{-\chi^2u^2t}\left(\sqrt{\frac{\tanh t}{t}}\right.$$
$$\left.\times\,{}_1F_1\left(\frac{1}{2};2;\chi^2u^2(t-\tanh t)\right)\right)' dt,$$

$u=|\tan\theta|$ and the angle integration over $v=\sin\phi$ is expressed in terms of the Kummer function ${}_1F_1$ (Ref. 30) as follows:

$$\int_0^1 dv\,\sqrt{1-v^2}\,e^{v^2p}=\frac{\pi}{4}\,{}_1F_1\left(\frac{1}{2};2;p\right),$$

$$p=\chi^2u^2(t-\tanh t)\geq 0. \tag{40}$$

The relevant analytical properties of the auxiliary function $\mathcal{F}$ are the following: $\mathcal{F}(0)$ is constant and

$$\mathcal{F}(\chi^2u^2)=\frac{a}{(\chi u)^4}+\frac{b}{(\chi u)^8}+\cdots \tag{41}$$

as $\chi^2u^2\gg 1$, where $a=1/6$.

Again, let us represent the integral over u as the sum of U_1 and U_2, where

$$U_2=\int_1^\infty \mathcal{F}(\chi^2u^2)\,\frac{u\,du}{(u^2+1)^{5/2}}\simeq\frac{a}{\chi^4}\int_1^\infty (u^2+1)^{-5/2}\,\frac{du}{u^3}.$$

Now two subtractions are needed to estimate U_1. The first one is related to the finite counterterm which is responsible for the nontrivial quadratic contribution. The next one implies the logarithmically divergent integral analogous to that in the previous section. The corresponding calculation can be found in Appendix D. Eventually,

$$j_1=-\frac{\rho}{4}\,\frac{1}{\xi_0\chi^2}\left(1+\frac{\mathcal{A}}{\chi^2}+\frac{5}{8}\,\frac{1}{\chi^4}\log\frac{\mathcal{B}}{\chi^2}\right), \tag{42}$$

where

$$\mathcal{A}=3\int_0^\infty \mathcal{F}(u^2)\,u\,du\;\approx -2\times 10^{-1}, \tag{43}$$

$$\log 4\mathcal{B}=\frac{31}{15}-12\int_0^\infty\left(u^3\mathcal{F}(u^2)-\frac{1}{6(u+1)}\right)du.$$

In this case there are two corrections, and the lowest one is of the type $(\mathrm{rot}\,\hat{l})_\perp|\,\hat{l}\times\mathrm{rot}\,\hat{l}\,|$, i.e., of the type found in Ref. 10 for the current perpendicular to $\hat{l}$ [Eqs. (53), (54) in Ref. 10). The coefficient $\mathcal{A}$ (43) was estimated numerically in Ref. 25. The next term is the new cubic one, and it includes the logarithm of the London parameter.

C. Example 3

In this case, we take into account the whole (29) to obtain the quadratic correction of the type $|\,\hat{l}\times\mathrm{rot}\,\hat{l}\,|(\vec{v}-(1/4m)\mathrm{rot}\hat{l})_{||}$ [or, of the type $|\,\hat{l}\times\mathrm{rot}\,\hat{l}\,|(\partial_1\hat{l}_2+\partial_2\hat{l}_1)$, see Eq. (A2)]. Therefore, we consider the third component $j_{\mathrm{corr},3}$ which is along $\hat{l}$. However, we put here $\chi_1=\chi_2$ for simplicity, and, thus, the answer expected will get a more formal appearance since it will be expressed in terms of $1/\chi^2$. We obtain from Eq. (33):

$$j=\frac{3\rho}{2\pi}\,\frac{1}{\xi_0\chi^2}\int\int_\Pi\frac{du\,dv}{\sqrt{(u^2+1)^5\,(1-v^2)}}\times((u-v)\mathcal{F}_-+(u+v)\mathcal{F}_+), \tag{44}$$

$$\mathcal{F}_\mp=\int_0^\infty \exp(-t\,(\chi u)^2\,\mathcal{Q})\left(\sqrt{\frac{\tanh t}{t}}\times\exp((t-\tanh t)(\chi\,u)^2(1-v^2)\,\mathcal{Q}^3)\right)'dt, \tag{45}$$

where $\mathcal{Q}^{-2}$ stands for

$$\mathcal{Q}_\mp^{-2}=1+u^2\mp 2\,u\,v=1-v^2+(u\mp v)^2,$$

the domain Π is given by $\{(u,v)\colon u\in[0,\infty[,v\in[0,1]\}$, and $u=\tan\theta,\ v=\cos\phi$.

The function (29) is still rather complicated and so the present consideration becomes less elegant than the two previous. The estimations we are interested in will be obtained without providing the integral formulas for the coefficients. Let us proceed estimating Φ (31) in general situation. By steepest descent we get

$$\Phi\,(x^2,Q)\simeq -\frac{1}{3}\,\frac{1}{Q^2}+2\,\frac{x^2}{Q^3}, \tag{46}$$

at $Q\gtrsim 1$, i.e., either $\Delta^2/\alpha c_F$ or x^2 must be $\gtrsim 1$ (provided that $\Delta^2/\alpha c_F \ll 1$ does not occur). In the opposite case $Q<1$, we adopt

$$\Phi\,(x^2,Q)\simeq -1+\pi^{1/2}x^2\left(\frac{\alpha c_F}{\Delta^2}\right)^{1/2}, \tag{47}$$

as the leading approximation.

First, let us consider the contribution to Eq. (44) which is due to $u\in[1,\infty[$. Here the function $\mathcal{F}$ can be expanded by steepest descent because $(\chi u)^2\mathcal{Q}\gg 1$. This expansion will begin with the third order term const$\times(\xi_0\chi^6)^{-1}$ which is not of interest for us. So, in what follows we shall take $0\leqslant u\leqslant 1$ in j.

Now let us consider the domain $0\leqslant u\leqslant 1/\chi$, where we replace approximately $1+u^2$ by 1:

$$\frac{3\rho}{2\pi}\,\frac{1}{\xi_0\chi^2}\int_0^{1/\chi}du\int_0^1\frac{dv}{(1-v^2)^{1/2}}\,((u-v)\mathcal{F}_-+(u+v)\mathcal{F}_+).$$

In this case $\Delta^2/\alpha c_F\approx(\chi u)^2\,(u\mp v)^2,\ x^2\approx(\chi u)^2(1-v^2)$ and the use of Eq. (47) implies that, basically, the estimate is due to the region where $\Delta^2/\alpha c_F$ and x^2 are not close to zero or 1, but strictly less than 1. Thus, we obtain

$$\mathcal{F}_\mp\simeq -1+\pi^{1/2}\,\frac{\chi u\,(1-v^2)}{|u\mp v|},$$

and the total contribution below $u=1/\chi$ is

$$\frac{\rho}{2}\,\frac{1}{\xi_0\chi^4}\left(\frac{2}{\sqrt{\pi}}-\frac{3}{2}\right). \tag{48}$$

Eventually, let us consider the rectangle $\{(u,v)\colon 1/\chi\leqslant u\leqslant 1, v\in[0,1]\}$. Here $\mathcal{F}_+$ can safely be expanded by the Laplace method. As to $\mathcal{F}_-$, the integral diverges when $\Delta^2/\alpha c_F$ is calculated inside the strip $|u-v|\leqslant 1/\chi$ along the diagonal $u=v$ but the resulting singularity is integrable. One can check that here there are no interesting terms as far as $x^2\gtrsim 1$ and the strip's width is $2/\chi$. Outside the strip, the use of (46) allows to obtain the logarithmic third order term. The experience of the numerical calculations behind[25] (see also the next section) shows us that the coefficient at $(\xi_0\chi^4)^{-1}$ is, mainly, due to $0\leqslant u\leqslant 1/\chi$, and therefore its order of magnitude is given by (48). So, we take into account both $\mathcal{F}_+$ and $\mathcal{F}_-$, and obtain from (44)–(46),

$$-\frac{\rho}{\pi}\,\frac{1}{\xi_0\chi^6}\int_{1/\chi}^1\frac{du}{u^3}\,\frac{1}{(u^2+1)^{5/2}}\times\int_0^1\frac{dv}{(1-v^2)^{1/2}}\,(u^2+8\,v^2-5).$$

Calculating the integral and using Eq. (48), we obtain the final formula:

$$j_3=\frac{\rho}{2}\,\frac{1}{\xi_0\chi^2}\left(1+\frac{\mathcal{A}}{\chi^2}+\frac{7}{4}\,\frac{1}{\chi^4}\log\frac{\mathcal{B}}{\chi^2}\right), \tag{49}$$

where $\mathcal{A}$ is of the order of $2/\sqrt{\pi}-3/2\approx -3.7\times 10^{-1}$.

VI. COMPARISON OF THE TWO APPROACHES

In this section, we discuss the contradicting statements from Refs. 10 and 23 about the validity of the higher (quadratic) corrections to the mass current (1). The starting point of the present investigation coincides with that of Ref. 10 but the mathematical procedure is different. Therefore, our investigation will allow one to choose between Refs. 10 and 23, and, thus, explain the above-mentioned contradiction. The necessity of including the present section arose after Ref. 23 has been found and the main part of the present paper has already been completed. This explains why the content of this section is to some extent independent of the content of the other sections of the paper.

As noted in Sec. III, there are two equivalent representations for the real part of the ξ-integrated Green's function $\mathcal{J}_e \equiv \mathrm{Re}\int d\xi g_{11}$, which are given by Eq. (18) and by

$$\mathcal{J}_e = \frac{\sqrt{\pi}}{2}\frac{\Delta}{\sqrt{\alpha c_F}}[F(|\lambda|^2+1, x_0\sqrt{2}) - F(|\lambda|^2, x_0\sqrt{2})], \tag{50}$$

where F denotes the product of the gamma function and two parabolic cylinder functions,[32]

$$F(|\lambda|^2, x_0\sqrt{2}) \equiv \Gamma(|\lambda|^2)\, U\left(|\lambda|^2 - \frac{1}{2}, x_0\sqrt{2}\right) \times U\left(|\lambda|^2 - \frac{1}{2}, -x_0\sqrt{2}\right), \tag{51}$$

$x_0 = \mathrm{Re}\Delta_0/\sqrt{\alpha c_F}$ (in Refs. 10 and 23, x_0 differs by the multiple $\sqrt{2}$), and $|\lambda|^2$ is defined in Eq. (17). The representation (50) was obtained in Ref. 10. In Ref. 23, the same equation (50) was used to claim that the corrections found in Ref. 10 vanish, and so quadratic corrections to $\vec{j}_0$ are absent.

The equivalence of Eqs. (18) and (50) follows from the fact that the Green function obtained either in the integral form or in the series form is unique. The representation (18) is especially convenient in practice. For instance, the integration over frequencies is Gaussian and can be performed first to significantly simplify the rest:

$$\int_0^\infty \frac{d\omega}{\pi}\mathcal{J}_e = -\frac{\Delta}{2}\frac{1}{Q}(1+\Phi(x_0^2,Q)), \tag{52}$$

where $\Phi(x_0^2,Q)$ and Q are given by Eq. (31). From (52) it is seen how $\vec{j}$ is split into the sum of $\vec{j}_0$ and $\vec{j}_{\mathrm{corr}}$ in our approach. The integration by parts over t fails when Q tends to zero, though we can use (52) on the entire unit sphere provided that the pure polynomial corrections next to the quadratic ones are irrelevant for us.

If $Q = |\Delta|^2/\alpha c_F \gtrsim 1$ (this corresponds to $x_0^2 + 2|\lambda|^2 \gtrsim 1$ in Refs. 10 and 23), we can expand Φ by the Laplace method so that (52) becomes

$$-\frac{\Delta}{2}\frac{1}{Q} + \frac{\Delta}{6}\frac{1}{Q^3} - \Delta\frac{x_0^2}{Q^4}\cdots. \tag{53}$$

The same result can be obtained if we use the Darwin expansion[32] for the parabolic cylinder functions in (50) and (51) with subsequent integration over ω.

In Refs. 10 and 23, the total correcting contribution is presented, e.g., Eq. (10) in Ref. 23. In our notation, it has the form

$$\vec{j}_{\mathrm{corr}} = \frac{3\rho}{4\pi c_F}\int_0^\infty \frac{d\omega}{\pi}\int d\Omega\hat{k}\left[\mathcal{J}_e + \frac{\pi}{2}\frac{\alpha c_F \Delta}{(\omega^2+|\Delta|^2)^{3/2}}\right], \quad T=0, \tag{54}$$

where $\mathcal{J}_e$ is given by (50), (51). The second term in (54) appears because the leading $\vec{j}_0$ is added and subtracted from the total $\vec{j}$ to express $\vec{j}_{\mathrm{corr}}$. The main idea in Refs. 10 and 23 is that the dominant contribution to $\vec{j}_{\mathrm{corr}}$ comes from the region $x_0^2 + 2|\lambda|^2 \lesssim 1$, i.e., it is due to ω/δ, $|\hat{k}_1|$, $|\hat{k}_2| \lesssim 1/\chi$. Therefore, in Ref. 23, the angle integration, $\int d\Omega$, is replaced by the plane integration, $2\int d\hat{k}_1 d\hat{k}_2$. In the present paper, we use the variables $u = |\tan\theta|, v = \sin\phi$ to integrate over the unit sphere. Now, to simplify the comparison with Ref. 23, we consider another change of variables. Namely, we introduce u,v by the formulas $\sqrt{u^2+v^2} = \sin\theta$ and $u/v = \tan\phi$ so that $\int d\Omega \mapsto 2\int du dv(1-u^2-v^2)^{-1/2}$, and u,v are now inside the unit disc $\mathcal{D}$.

For definiteness, we focus our attention on the quadratic correction, example 2, which is one of the two nonanalyticities discussed in Ref. 23. Our choice in example 2 implies $e^{-i\psi} = i$, and, thus, $\Delta_0 = i\delta\sin\theta e^{i\phi}$, $\alpha c_F = (\delta/\chi)^2\cos^2\theta$ (near the poles it correlates with $\alpha = -\delta\partial_3\hat{l}_2$ (14) in Ref. 23. Moreover, it should been noted that, in our calculation, α and Δ_0 include the additional multiplier i. This should not bother us since we have a freedom in ψ), and so

$$\frac{\Delta}{\sqrt{\alpha c_F}} = \chi\frac{v}{\sqrt{1-u^2-v^2}}$$

in the new variables. Using the new u and v in Eq. (54), we replace, in the leading approximation, $1-u^2-v^2$ by 1 and ''stretch'' the integration domain as follows:

$$\frac{\chi}{\sqrt{2}}v = \mathcal{X}, \quad \frac{\chi}{\sqrt{2}}u = \mathcal{Y}, \quad \frac{\chi}{\sqrt{2}}\frac{\omega}{\delta} = \mathcal{E}.$$

Then

$$j_{\mathrm{corr},1} = \frac{3}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\frac{\rho}{\xi_0\chi^4}\int_0^\infty d\mathcal{E}\int_{\tilde{\mathcal{D}}} d\mathcal{X}d\mathcal{Y}\mathcal{X}^2 H(\mathcal{E}^2+\mathcal{X}^2,\mathcal{Y}), \tag{55}$$

$$H(\mathcal{E}^2+\mathcal{X}^2,\mathcal{Y}) \equiv F(\mathcal{E}^2+\mathcal{X}^2+1,-2\mathcal{Y}) - F(\mathcal{E}^2+\mathcal{X}^2,-2\mathcal{Y}) + \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{1}{(\mathcal{E}^2+\mathcal{X}^2+\mathcal{Y}^2)^{3/2}}, \tag{56}$$

where $\tilde{\mathcal{D}}$ is the disc of radius $\chi/\sqrt{2} \gg 1$. Up to the sign at $\mathcal{Y}$, Eq. (55) is just the correction, Eq. (16), in Ref. 23 (with $\tilde{\mathcal{D}}$ replaced by the plane). Generally speaking, in Ref. 23, two contributions are considered, $j_{\mathrm{corr},1}$ and $j_{\mathrm{corr},3}$. Both contributions contain the integral $\int_{-\infty}^{\infty} d\mathcal{Y} H$, which is claimed to be, mainly, zero and, therefore, ''to leading order the transverse and longitudinal nonanalytic contributions to the current vanish. . . .''[23]

Since the origin of Eq. (55) is understood, we now regard the quadratic correction, example 2, using Eqs. (9) and (18) but with the new choice of the parameters u,v. We obtain

$$j_1=-\frac{3\rho}{2\pi c_F}\frac{\chi}{\sqrt{\pi}}\int_{\mathcal{D}}\frac{du\,dv\,v^2}{1-u^2-v^2}\int_0^\infty d\omega\int_0^\infty dt\,(\tanh t)^{1/2}\times\exp\left(-\frac{\chi^2}{1-u^2-v^2}\left[u^2\tanh t+\left(v^2+\frac{\omega^2}{\delta^2}\right)t\right]\right). \tag{57}$$

We neglect u^2, v^2 to 1, then "stretch" the variables $\chi v=\mathcal{X}$, $\chi u=\mathcal{Y}$ (thus repeating[23]) and integrate over ω. We have

$$j_{\mathrm{corr},1}=-\frac{3\rho}{4\pi}\frac{1}{\xi_0\chi^4}\int_{\tilde{\mathcal{D}}}d\mathcal{X}d\mathcal{Y}\mathcal{X}^2\left[\int_0^\infty dt\left(\frac{\tanh t}{t}\right)^{1/2}\times e^{-\mathcal{Y}^2\tanh t-\mathcal{X}^2 t}-(\mathcal{Y}^2+\mathcal{X}^2)^{-1}\right], \tag{58}$$

where the counterterm is included since $\vec{j}_0$ is added and subtracted from Eq. (57). Up to the multiple $\sqrt{2}$ in $\mathcal{Y}$ and $\mathcal{X}$, Eq. (58) is nothing but an equivalent form of Eq. (55).

Further, following the strategy of the present paper, we pass in Eq. (58) to the polar coordinates in the $\mathcal{X}$–$\mathcal{Y}$-plane as follows:

$$j_{\mathrm{corr},1}=-\frac{3\rho}{4}\frac{1}{\xi_0\chi^4}\int_0^\chi dr\,r^3\left[\int_0^\infty dt\left(\frac{\tanh t}{t}\right)^{1/2}\times{}_1F_1\left(\frac{1}{2};2;r^2(t-\tanh t)\right)e^{-r^2t}-\frac{1}{r^2}\right], \tag{59}$$

where the hypergeometric function ${}_1F_1$ (40) accounts for the angle integration. It is not difficult to see that the radial variable r is simply the renormalized angle θ, i.e., $\chi\,\theta$. The inner integral is estimated at $r\gtrsim 1$ by the Laplace method,

$$\frac{1}{r^2}+\frac{1}{6r^6}+\cdots, \tag{60}$$

and $\chi=\infty$ can be taken as the upper bound in Eq. (59) (i.e., $\tilde{\mathcal{D}}$ can be replaced by the plane). Eventually, the r-integral, Eq. (59), demonstrates $\mathcal{A}/3$ (43). Evaluating the bracket in Eq. (59) numerically, we find that $\mathcal{A}$ is not zero, and the dominant contribution to it is due to $0<r\lesssim 1$ (i.e., $0<\theta\lesssim 1/\chi$), where the series (60) is not valid. Notice that the coefficient at $(\xi_0\chi^4)^{-1}$ is of order $1/\chi^2$, if we restrict the integration only with the asymptotic region $r\gtrsim\chi\gg 1$, and can be made arbitrarily small. The latter would correlate with the statement in Ref. 23 that, to leading order, the coefficient is zero. On the other hand, it is seen from Eqs. (59) and (60) that the region $0<r\lesssim 1$ comes to play thus spoiling the conclusion of Ref. 23.

Recall that, in Ref. 23, the fact that the integral $\int_{-\infty}^{\infty}d\mathcal{Y}H$ vanishes generally was deduced from the equation

$$\int_{-\chi}^{\chi}d\mathcal{Y}\left(F(|\lambda|^2+1,-2\mathcal{Y})-F(|\lambda|^2,-2\mathcal{Y})\right)=-\frac{\Gamma(|\lambda|^2)}{2}U\left(|\lambda|^2-\frac{1}{2},2\mathcal{Y}\right)U\left(|\lambda|^2+\frac{1}{2},-2\mathcal{Y}\right)\Big|_{-\chi}^{\chi}. \tag{61}$$

Namely, as $\chi\to\infty$, the RHS of Eq. (61) is estimated as

$$-\left(\frac{\pi}{2}\right)^{1/2}|\lambda|^{-2}=-\left(\frac{\pi}{2}\right)^{1/2}(\mathcal{E}^2+\mathcal{X}^2)^{-1}, \tag{62}$$

and so the integral in question vanishes. Although Ref. 23 does not explain this exhaustively, it seems that Eq. (62) is obtained with the help of the Darwin expansion[32] for $U(|\lambda|^2\pm 1/2,2\mathcal{Y})$. It is important to observe that the latter is valid when the combination $\mathcal{Y}^2+|\lambda|^2$ is large and, thus, $\mathcal{Y}^2$ and/or $|\lambda|^2$ is large. In Ref. 23, the $\mathcal{E}$-integration has not been performed, whereas the approach of the present work allows us to integrate over frequencies easily. Therefore, discussing the estimate, we forget about $\mathcal{E}^2$ in $\mathcal{Y}^2+|\lambda|^2$. If so, then the statement that the integral is governed by large $\mathcal{Y}^2+|\lambda|^2$ implies that the domain $r\gg 1$ (which is responsible for the negligible contribution to the coefficient) in integral Eq. (59) is considered as the most important.

Strictly speaking, Eq. (61) does not suggest that it is natural to estimate its RHS in terms of the combination $\mathcal{Y}^2+|\lambda|^2$ as $|\mathcal{Y}|=\chi\to\infty$. For instance, the same Ref. 32 suggests another way in the case where $2|\mathcal{Y}|$ is large and $||\lambda|^2\pm 1/2|$ is moderate. Let us recall the origin of $\mathcal{X}$ and $\mathcal{Y}$:

$$\frac{\Delta}{\sqrt{\alpha c_F}}=\chi|\tan\theta|\cos\phi=\frac{\chi v}{\sqrt{1-u^2-v^2}}\simeq\mathcal{X},$$

$$x_0=-\chi|\tan\theta|\sin\phi=-\frac{\chi u}{\sqrt{1-u^2-v^2}}\simeq\mathcal{Y}.$$

Therefore, we do not see any reasons for $\Delta/\sqrt{\alpha c_F}$ and x_0 be large simultaneously at $\theta\gtrsim 1/\chi$. More likely the situation is such that the ϕ-symmetry should be incorporated separately, so that the relevant θ-dependence remains to be discussed. Moreover, if we accept, for a moment, the viewpoint of Ref. 23, then the coefficient in question is governed by the boundary of $\tilde{\mathcal{D}}$, where the main assumption $d\Omega\mapsto d\hat{k}_1 d\hat{k}_2$ is not valid. On the contrary, Eqs. (59) and (60) show that the region $0<\theta\lesssim 1/\chi$ is crucial in demonstrating that the r-integral is nonzero. To conclude, Eq. (61) should not be approximated but requires the subsequent integrations before letting $\chi\to\infty$.

We restrict ourselves by the demonstration of how the arguments of Ref. 23 about $j_{\mathrm{corr},1}$ are spoiled. Let us only note that there are no considerable simplifications if we reconsider our example 3 by means of the newly chosen u,v, though the estimates get confirmation. The present paper itself should convince that the results presented stem from the procedure reliable enough. For instance, the results of Refs.

10, 13, and 22 about the quadratic corrections are independently confirmed and the objection[23] is removed.

VII. DISCUSSION

The present paper completes the papers[24–26] concerned with the following two main problems: calculation of the mass current $\vec{j}$ in slightly inhomogeneous ^{3}He-A by means of the thermal Green's functions and determination of its asymptotic expansions at $T=0$ in the London limit. Here the following two basic assumptions are of importance: the static order parameter can be linearized since its spatial variation is slow, and only the first order differentiations that arise due to the kinetic energy are kept in the mixed representation. Since the texture of the orbital vector is weak, three initial dimensions are reduced to the one-dimensional situation so that the Dyson-Gorkov governing equation can be solved exactly by means of the eigenfunctions of the Landau problem. Thus, a collection of exact formulas arises, which allows us to systematically derive the higher corrections to the dominant $\vec{j}_0$ (1). Our approach provides a correct procedure for determination of the order of magnitude for the corrections in question. The given approach is manifestly advantageous because the Laplace method is appropriate in the London limit.

We are concerned with the normal Green's function, which is, first, ξ-integrated and, then, ω-summated, and which results in two representations for $\vec{j}$, in the series and integral forms. The integral form seems to be more attractive because it can satisfactory be studied by steepest descent. Particular limits (the zero temperature limit following the limit of the lowest order in gradients, and *vice versa*) confirm the choice of our strategy, demonstrating Eq. (1) as the lowest contribution in gradients of the order parameter.

Three orientations of $\mathrm{rot}\,\hat{l}$ are considered in Sec. V to obtain the correcting terms explicitly: $\mathrm{rot}\,\hat{l}$ is parallel to $\hat{l}$ (example 1), and perpendicular to $\hat{l}$ (example 2), while in example 3 we consider an intermediate orientation of $\mathrm{rot}\,\hat{l}$ with respect to $\hat{l}$. Corrections are considered up to the third order in the gradients of $\hat{l}$: we estimate the numerical coefficients at the second order terms, and provide new cubic corrections that contain the logarithm of the London parameter.

In the first case, there is only the third order logarithmic correction. In the second case, both corrections arise: the quadratic and the cubic one. As is clear from the analysis in Ref. 10, the quadratic corrections must be proportional to $|\hat{l}\times\mathrm{rot}\,\hat{l}|$, and this correlates with the absence of the quadratic term in example 1 (Ref. 22 also explains $|\hat{l}\times\mathrm{rot}\,\hat{l}|$ in quadratic terms). In example 2, $\vec{j}_{\mathrm{corr}}$ is orthogonal to $\hat{l}$. Thus, the second order term in (42) corresponds to that given in Ref. 10 in the form $(\mathrm{rot}\,\hat{l})_\perp|\hat{l}\times\mathrm{rot}\,\hat{l}|$, and the corresponding numerical coefficient is $\mathcal{A}$ (43).[25] Example 3 also results in the corrections of both types. As to the quadratic correction along $\hat{l}$, the answer contained in Ref. 10 is as follows:

$$\rho\,\xi_0|\hat{l}\times\mathrm{rot}\,\hat{l}|\left(A\,(v_3-\hat{l}\cdot\mathrm{rot}\,\hat{l}/4m)+\frac{B}{m}(\partial_1\hat{l}_2+\partial_2\hat{l}_1)\right).$$

The quadratic term in (49) must be compared with the last expression at $A\neq 0$, $B=0$. It is clear from (A2) that the result obtained here is also applicable to establish the contribution at $A=0$, $B\neq 0$.

In agreement with the statements by previous investigators, the discs of radius $1/\chi$ near the topologically stable nodes of the order parameter at $\theta=0,\pi$ are responsible for the integrals at the quadratic corrections. Moreover, our situation is more rich than in Ref. 10 since the logarithmic terms are demonstrated. In this respect, we recall the correction found in Ref. 13. As is seen from (A2), the components of the superfluid velocity $2m\vec{v}$ and the gradients of $\hat{l}$ enter the parameter α on equal terms. Therefore, $\chi_{\mathrm{orb}}(\vec{v}_s\cdot\vec{\partial})\,\hat{l}_a\vec{\partial}\,\hat{l}_a$, where χ_{orb} is logarithmically large,[9] should be regarded as the logarithmic third order term that constitutes a part of the third order contribution in (49).

The expansion procedure suggested here differs from that in Ref. 22, though the use of the large London parameter χ to ''stretch'' the θ-variable resembles the scaling theory in Ref. 22 and also the special rescaling of variables in Ref. 10. The procedure presented here provides much more exhausting unravelling of the situation in Ref. 10 since it enables us to explain the negative result of Ref. 23. It should be noted that to claim the both corrections (the quadratic and the cubic one with the logarithm), we need the following. First, the functions $\mathcal{F}(u)$ (which enter the ω- and ξ-integrated Green's function) are constant at $u=0$ whereas their first asymptotic term at large u is known. Secondly, their argument itself is not restricted from below. The latter is due to the properties of the gap function of the A-phase, i.e., due to its nodes on the Fermi surface.

Reference 22 tells us that the gauge transformation strategy[13] fails when investigating the nonanalytic terms in the mass current. Therefore, only in Ref. 22 and in the present paper, the gradient expansions of nonanalyticities of the mass currents are given. Although the present paper makes evident that the coefficients at the quadratic nonanalyticities can numerically be found provided the texture is chosen, a similar estimate remains to be elaborated in the framework of Ref. 22.

To conclude, the our investigation confirms rigorously and connects the corresponding results of Refs. 10, 13 and 22, and, moreover, allows us to say that in the London limit $\xi_0\ll|\mathrm{gr}|^{-1}$, each component of the mass current of ^{3}He-A can schematically be written at $T=0$ as

$$j=\mathrm{const}\times\rho\,\mathrm{gr}\,(1+\mathcal{A}\,\xi_0|\hat{l}\times\mathrm{rot}\,\hat{l}|+\mathcal{C}\,\xi_0^2\,(\mathrm{gr})^2\log(\mathcal{B}\,\xi_0\,\mathrm{gr})).$$

Since we do not consider the vector and symmetry structures of the correcting terms, each gr here denotes an appropriate combination of the gradients of the order parameter. Our results concerning the coefficients $\mathcal{A}$ (at least the order) and $\mathcal{C}$ seem to be reliable enough, while the knowledge of $\mathcal{B}$ requires more accuracy with the contributions rejected. Moreover, the linearization of the order parameter could become insufficient. The general representations obtained for $\vec{j}$ and the procedure itself would serve for further investigations.

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APPENDIX A

In spherical coordinates, the linearized order parameter takes the form[10]

$$\begin{aligned}\Delta(\hat{k},\vec{r}) &= \delta(\hat{k}_1+i\hat{k}_2)+\rho\, i\,\delta\,(2mv_p\hat{k}_p(\hat{k}_1+i\hat{k}_2)\\ &\quad -\hat{k}_3\hat{k}_p(\partial_p\hat{l}_2-i\partial_p\hat{l}_1))\\ &\equiv \delta\sin\theta\, e^{i\phi}+\rho[\,\delta\,\mathcal{M}\,e^{i(\pi/2-\psi)}]\,e^{i\psi}, \qquad \text{(A1)}\end{aligned}$$

where the square brackets contain α, the phase ψ must be adjusted, and $\vec{v}\equiv\vec{v}_s$. The function $\mathcal{M}$ can be written as follows:

$$\begin{aligned}\mathcal{M} &= -\cos^2\theta\,\partial_3\hat{l}_2+\frac{1}{2}\sin\theta\cos\theta\, e^{-i\phi}\,(-\partial_1\hat{l}_2-\partial_2\hat{l}_1\\ &\quad +i(\partial_1\hat{l}_1-\partial_2\hat{l}_2))+2m\sin^2\theta\, e^{i\phi}\,(v_1\cos\phi+v_2\sin\phi)\\ &\quad +\sin\theta\cos\theta\, e^{i\phi}\left(2mv_3+\frac{i}{2}\operatorname{div}\hat{l}-\frac{1}{2}\hat{l}\cdot\operatorname{rot}\hat{l}\right). \qquad \text{(A2)}\end{aligned}$$

It is seen from Eq. (A1) that $\alpha c_{_F}$ is a linear form of gradients,

$$\alpha c_{_F}=\delta^2\xi_0\sum\ (\text{gradients})=\frac{\delta^2}{\chi^2}\sum\frac{\text{gradients}}{|\text{gradients}|}. \qquad \text{(A3)}$$

By (6), we consider $\alpha c_{_F}/\delta^2$ as a small parameter.

APPENDIX B

It is easy to obtain the eigenvalues $E_0, \pm E_n$ and the eigenfunctions $\hat{\Psi}_0, \hat{\Psi}_n^{\pm}$ $(n\geqslant 1)$ for $\mathcal{H}_{\text{em}}$ (12),[24]

$$\hat{\Psi}_0=\begin{pmatrix}0\\ \psi_0(x)\end{pmatrix}, \quad E_0=-\Delta,$$

$$\hat{\Psi}_n^{(s)}=\frac{1}{\sqrt{2E_n}}\begin{pmatrix}\sqrt{E_n+s\Delta}\ \ \psi_{n-1}(x)\\ -is\sqrt{E_n-s\Delta}\ \ \psi_n(x)\end{pmatrix}, \quad sE_n,$$

where $s=\pm$, $E_n=\sqrt{\Delta^2+2\alpha c_{_F}n}$ and $\psi_n(x)$ are the Hermite functions.

APPENDIX C

By the Mehler formula,[30] we have

$$\sum_{n=0}^{\infty} a^n\psi_n^2(y)=\frac{1}{\sqrt{\pi(1-a^2)}}\exp\left(-\frac{1-a}{1+a}y^2\right), \quad |a|<1,$$

where $\psi_n(y)$ are the Chebyshev-Hermite functions. Using

$$\Gamma(d)=(n+q)^d\int_0^\infty ds\, s^{d-1}\, e^{-s(n+q)}, \quad n\geqslant 0, \quad d,q>0,$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\psi_n^2(y)}{(n+q)^d} &= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(d)}\int_0^\infty ds\,\frac{s^{d-1}}{\sqrt{\sinh s}}\\ &\quad \times\exp\left(\left(\frac{1}{2}-q\right)s-y^2\tanh(s/2)\right),\\ &\quad d>1/2.\end{aligned}$$

APPENDIX D

Now two first subtractions are needed to estimate $U_1=X+Y+Z$, where

$$Z=\frac{1}{\chi^2}\int_0^\chi \mathcal{F}(u^2)\,u\,du, \quad Y=-\frac{5}{2}\frac{1}{\chi^4}\int_0^\chi \mathcal{F}(u^2)\,u^3du,$$

$$X=\frac{1}{\chi^2}\int_0^\chi \mathcal{F}(u^2)\left(\left(1+\frac{u^2}{\chi^2}\right)^{-5/2}-1+\frac{5}{2}\frac{u^2}{\chi^2}\right)u\,du.$$

Clearly, Z is convergent at large χ and approximately

$$Z\simeq\frac{1}{\chi^2}\int_0^\infty \mathcal{F}(u^2)\,u\,du-\frac{a}{2\chi^4}.$$

Further, a single counterterm is required for Y:

$$Y\simeq-\frac{5}{2}\frac{1}{\chi^4}\int_0^\infty\left(u^3\mathcal{F}(u^2)-\frac{a}{u+1}\right)du-\frac{5a}{2}\frac{\log\chi}{\chi^4}.$$

Now we consider X. The total contribution of the order χ^{-4} is given once $\mathcal{F}$ is replaced by a/u^4 in X. The net result reads

$$U_2+X\simeq\frac{a}{\chi^4}\left(\frac{37}{12}-\frac{5}{2}\log 2\right),$$

and, therefore, at $a=1/6$

$$\begin{aligned}U_1+U_2 &\simeq \frac{1}{\chi^2}\int_0^\infty \mathcal{F}(u^2)\,u\,du\\ &\quad +\frac{1}{2\chi^4}\left(\frac{31}{36}+\frac{5}{6}\log\frac{1}{2\chi}\right.\\ &\quad \left.-5\int_0^\infty\left(u^3\mathcal{F}(u^2)-\frac{1}{6(u+1)}\right)du\right).\end{aligned}$$

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