

## Strong-coupling approach to the magnetization process of polymerized quantum spin chains

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(Received 15 September 1998)

Polymerized quantum spin chains (i.e., spin chains with a periodic modulation of the coupling constants) exhibit plateaus in their magnetization curves when subjected to homogeneous external magnetic fields. We argue that the strong-coupling limit yields a simple but general explanation for the appearance of plateaus as well as of the associated quantization condition on the magnetization. We then proceed to explicitly compute series for the plateau boundaries of trimerized and quadrumerized spin-1/2 chains. The picture is completed by a discussion of how the universality classes associated with the transitions at the boundaries of magnetization plateaus arise in many cases from a first-order strong-coupling effective Hamiltonian.

[S0163-1829(99)00109-5]

Quantum spin systems at low (or zero) temperatures can exhibit plateaus in their magnetization curves when subjected to strong external fields. Such phenomena in quasi-one-dimensional systems have recently been a subject of intense interest. In one dimension, there is an intriguing interplay between theoretical progress on a systematic understanding of the underlying mechanisms (see, e.g., Ref. 1) and an increasing number of experiments (see, e.g., Refs. 2 and 3) on materials which are believed to be predominantly one dimensional.

Here we study polymerized spin- $S$  quantum spin chains in a magnetic field. Their Hamiltonian is given by

$$H = \sum_x J_x \vec{S}_x \cdot \vec{S}_{x+1} - h \sum_x S_x^z, \quad (1)$$

where we assume periodicity of the coupling constants with period  $p$ , i.e.,

$$J_x = J_{x+p}. \quad (2)$$

We will mostly concentrate on spin  $S = 1/2$  and the antiferromagnetic regime  $J_x \geq 0$ . Our work is largely motivated by Ref. 4, which studied the zero-temperature magnetization process of the  $S = 1/2$  polymerized chains (1) using finite-size diagonalization and a perturbative bosonization analysis around the case of equal coupling constants  $J_x = J$ . Polymerized XY chains had been studied before in Ref. 5 using the Jordan-Wigner transformation. In addition to the well-known dimerized case, also trimerized<sup>6,7</sup> and quadrumerized<sup>8,9</sup> chains have already been investigated in more detail. It was found that the trimerized chain exhibits a plateau in the magnetization curve at one-third of the saturation magnetization,<sup>6,7</sup> while the quadrumerized chain can exhibit a plateau at half of the saturation value in addition to a spin gap.<sup>8,9</sup> Here we wish to complete the picture by discussing the ‘‘strong-coupling’’ limit where at least one coupling constant is small with respect to the others, i.e.,  $J_{x_0} \rightarrow 0$ .

As is known, e.g., from studies of spin ladders,<sup>10,11</sup> the magnetization process is easy to understand if some  $J_{x_0} = 0$ . In this limit, the chain (1) decouples into clusters of  $p$  spins. These ‘‘strongly coupled’’ clusters magnetize independently

such that at zero temperature the magnetization  $\langle M \rangle$  can only take finitely many values. For spin  $S$  they are subject to the quantization condition

$$pS(1 - \langle M \rangle) \in \mathbb{Z}, \quad (3)$$

with a normalization such that the magnetization has saturation values  $\langle M \rangle = \pm 1$ . This quantization condition was obtained (for  $S = 1/2$ ) in Ref. 4. Similar quantization conditions were found in Refs. 1 and 11. In particular the latter was also motivated by considering a limit in which the system decouples into clusters of finitely many spins. In fact, this counting argument is completely independent of the internal coupling inside the cluster of the  $p$  spins. The quantization condition (3) is therefore insensitive to details of the model. However, not only the transition values of the magnetic field but also the question if a possible plateau is realized even in this limit depends on the precise coupling inside the cluster. For the linear arrangement (1) and antiferromagnetic  $J_x > 0$  ( $x \neq x_0$ ), all values of  $\langle M \rangle$  permitted by Eq. (3) are indeed realized at  $J_{x_0} = 0$ .

Clearly, it remains also to be shown that the quantization condition (3) is indeed valid at generic points in the parameter space, not only for the special points where some  $J_{x_0} = 0$ . This can be supported by series expansions around the decoupling point, an issue to which we shall return below.

A first property which one can derive for the full interacting spin-1/2 system is the upper critical field  $h_{uc}$  at which the transition to a fully polarized ferromagnetic state takes place. For antiferromagnetic  $J_x \geq 0$  it is simply given by the vanishing of the gap for the one-spin-wave dispersion above the ferromagnetic background. The value of  $h_{uc}$  is therefore given by the maximal eigenvalue of the following  $p \times p$  matrix:

$$\frac{1}{2} \begin{pmatrix} J_p + J_1 & -J_1 & 0 & \cdots & 0 & -e^{ik} J_p \\ -J_1 & J_1 + J_2 & -J_2 & 0 & \cdots & 0 \\ 0 & -J_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & -J_{p-1} \\ -e^{-ik} J_p & 0 & \cdots & 0 & -J_{p-1} & J_{p-1} + J_p \end{pmatrix}, \quad (4)$$

where  $k$  arises from a Fourier transform using the periodicity (2) (see also Refs. 6 and 9 for a detailed analysis of special cases). For antiferromagnetic  $J_x > 0$ , the lowest-energy excitations occur at  $k=0$  for  $p$  even and at  $k=\pi$  for  $p$  odd if we introduce the momentum by a translation of  $p$  sites [as in Eq. (4)].

In order not to get lost in too many parameters, we restrict ourselves to the same subspace that was also considered in Ref. 4 before we proceed further. We will now concentrate on the following periodic arrangement of coupling constants:

$$J_x = \begin{cases} J' & \text{for } x \in p\mathbb{Z}, \\ J & \text{otherwise.} \end{cases} \quad (5)$$

Now we return to the computation of the largest eigenvalue of Eq. (4). The case  $p=2$  is a bit special; the correct specialization of Eq. (4) to  $p=2$  reads [with the notation (5)]

$$\frac{1}{2} \begin{pmatrix} J' + J & -J - e^{ik} J' \\ -J - e^{-ik} J' & J + J' \end{pmatrix}. \quad (6)$$

Using Eq. (6) for  $p=2$  and Eq. (4) for  $p=3$  and 4 at  $k = p\pi$  (modulo  $2\pi$ ) we find

$$h_{\text{uc}}^{(p=2)} = J + J', \quad (7)$$

$$h_{\text{uc}}^{(p=3)} = \frac{3}{4}J + \frac{1}{2}J' + \frac{1}{4}\sqrt{9J^2 - 4JJ' + 4J'^2}, \quad (8)$$

$$h_{\text{uc}}^{(p=4)} = J + \frac{1}{2}(J' + \sqrt{2J^2 - 2JJ' + J'^2}), \quad (9)$$

respectively.

Next we turn to series expansions of the plateau boundaries for  $p \leq 4$ . For the present systems, we expect that the sharp steps between the magnetization plateaus which are present for  $J'=0$  or  $J=0$  soften as soon as one turns on  $J, J' > 0$ , but that nothing further happens. This scenario was in fact confirmed by the numerical and perturbative analysis around  $J=J'$  of Ref. 4.

For  $p=2$  the only nontrivial plateau is located at  $\langle M \rangle = 0$ . Its boundary is given by the  $k=0$  spin gap  $E^{(p=2)}$ . Series expansions in  $J'/J$  for this gap have already been carried out some time ago in Ref. 12 up to third order and have recently been extended to ninth order in Ref. 13. Adding a further order to Eq. (29) of Ref. 13 [in passing we have also checked Eqs. (28) and (30) of the reference] one arrives at

$$\begin{aligned} \frac{E^{(p=2)}}{J} &= 1 - \frac{1}{2}\mathcal{J} - \frac{3}{8}\mathcal{J}^2 + \frac{1}{32}\mathcal{J}^3 - \frac{5}{384}\mathcal{J}^4 - \frac{761}{12288}\mathcal{J}^5 \\ &+ \frac{18997}{1769472}\mathcal{J}^6 + \frac{21739}{7077888}\mathcal{J}^7 - \frac{214359199}{6794772480}\mathcal{J}^8 \\ &+ \frac{11960596181}{4892236185600}\mathcal{J}^9 + \frac{83327779047}{117413668454400}\mathcal{J}^{10} \\ &+ \mathcal{O}(\mathcal{J}^{11}). \end{aligned} \quad (10)$$

Here we have used the abbreviation

$$\mathcal{J} = \frac{J'}{J} \quad (11)$$

in order to make the presentation more compact.

A few remarks may be in place regarding the method used here which is summarized, e.g., in Sec. 3 of Ref. 14. Like the method of Ref. 13 it exploits the fact that the leading coefficients of the series can be obtained on a finite lattice. However, we use recurrence relations for the coefficients and an exact symbolic representation throughout the computation while in Ref. 13 a symbolic result was reconstructed from a high-precision numerical computation. Presumably, cluster expansion algorithms (see, e.g., Ref. 15) would be more efficient than the two aforementioned methods, but we prefer a simpleminded approach because of the ease with which it can be applied to  $p > 2$  as well.

For  $p=3$ , there is a plateau at  $\langle M \rangle = 1/3$ , as one infers from the above inspection of the case  $J'=0$ . Its lower and upper boundaries ( $h_{c_1}^{(p=3)}$  and  $h_{c_2}^{(p=3)}$ , respectively) are determined by the  $k=\pi$  gap of the single-spin excitations. Up to fifth order in  $J'$ , one finds the following series:

$$\begin{aligned} \frac{h_{c_1}^{(p=3)}}{J} &= \frac{8}{9}\mathcal{J} + \frac{211}{810}\mathcal{J}^2 - \frac{77437}{1312200}\mathcal{J}^3 + \frac{7606883}{188956800}\mathcal{J}^4 \\ &+ \frac{7188324510751}{269989034112000}\mathcal{J}^5 + \mathcal{O}(\mathcal{J}^6), \\ \frac{h_{c_2}^{(p=3)}}{J} &= \frac{3}{2} - \frac{1}{18}\mathcal{J} - \frac{521}{6480}\mathcal{J}^2 - \frac{394169}{6998400}\mathcal{J}^3 \\ &- \frac{2260895171}{79361856000}\mathcal{J}^4 - \frac{535736196039221}{43198245457920000}\mathcal{J}^5 \\ &+ \mathcal{O}(\mathcal{J}^6). \end{aligned} \quad (12)$$

Last, for  $p=4$  all relevant excitations have  $k=0$  in the antiferromagnetic regime  $J, J' > 0$ . For reasons that should be obvious from the results we content ourselves with second-order series for the gap  $E^{(p=4)}$  and the lower and upper boundaries of the  $\langle M \rangle = 1/2$  plateau ( $h_{c_1}^{(p=4)}$  and  $h_{c_2}^{(p=4)}$ , respectively):

$$\begin{aligned} \frac{E^{(p=4)}}{J} &= \frac{1}{2}(1 + \sqrt{3} - \sqrt{2}) - \frac{1}{24}(4 + \sqrt{6} + \sqrt{2})\mathcal{J} \\ &- \frac{1}{132480}(3079\sqrt{6} - 163960\sqrt{3} + 28775\sqrt{2} \\ &+ 276026)\mathcal{J}^2 + \mathcal{O}(\mathcal{J}^3), \\ \frac{h_{c_1}^{(p=4)}}{J} &= \frac{1}{2}(1 + \sqrt{3} - \sqrt{2}) + \frac{2\sqrt{6} + 8\sqrt{2} + 17}{48}\mathcal{J} + \frac{1}{14837760} \\ &\times (-2385712\sqrt{6} + 4730320\sqrt{3} - 4947835\sqrt{2} \\ &+ 7747472)\mathcal{J}^2 + \mathcal{O}(\mathcal{J}^3), \\ \frac{h_{c_2}^{(p=4)}}{J} &= 1 + \frac{1}{\sqrt{2}} + \frac{2\sqrt{2} - 3}{16}\mathcal{J} + \frac{21691\sqrt{2} - 31648}{43008}\mathcal{J}^2 \\ &+ \mathcal{O}(\mathcal{J}^3). \end{aligned} \quad (13)$$

With the choice of coupling constants (5) there is a second decoupling limit, namely  $J \rightarrow 0$ , which for  $p \geq 3$  is not equivalent to the case discussed before. This limit is special in that several of the coupling constants (2) vanish at the same time. This leads to  $p-2$  free spins in zeroth order in  $J$ . These free spins are immediately polarized once a magnetic field is applied. Only the two spins coupled by  $J'$  require a finite magnetic field to polarize. This leads to an  $\langle M \rangle = 1 - 2/p$  plateau whose upper boundary is given by  $h_{c_2}^{(p)} = J' - \mathcal{O}(J)$ .

At first order in  $J$ , one now has to perform degenerate perturbation theory for the free spins. It turns out that at this order they behave as isolated clusters of  $p-2$  spins. The corresponding transition fields have been tabulated in Ref. 11 and are indeed a reasonable first approximation to the plateau boundaries for  $3 \leq p \leq 6$  of Ref. 4 at large  $J'$ .

It is actually not difficult to obtain expansions in  $J$  for some plateau boundaries. Poor convergence is, however, to be anticipated. In the present case, internal properties of the decoupled clusters are computed perturbatively (which were already taken care of exactly at zeroth order in the expansions around  $J' = 0$ ). This is reflected, e.g., in the fact that the fundamental excitations start to disperse (i.e., depend on  $k$ ) only in the second order in  $J$ .

At  $p=3$  we find the following 11th-order series for the boundaries of the  $\langle M \rangle = 1/3$  plateau:

$$\begin{aligned} \frac{h_{c_1}^{(p=3)}}{J} &= \mathcal{J}^{-1} + \frac{3}{2}\mathcal{J}^{-2} - \frac{107}{32}\mathcal{J}^{-4} - \frac{1185}{256}\mathcal{J}^{-5} + \frac{845}{256}\mathcal{J}^{-6} \\ &+ \frac{537329}{24576}\mathcal{J}^{-7} + \frac{834121}{32768}\mathcal{J}^{-8} - \frac{310154551}{7077888}\mathcal{J}^{-9} \\ &- \frac{15865989569}{84934656}\mathcal{J}^{-10} + \mathcal{O}(\mathcal{J}^{-11}), \\ \frac{h_{c_2}^{(p=3)}}{J} &= \mathcal{J} + \frac{1}{2} - \frac{1}{4}\mathcal{J}^{-2} - \frac{5}{64}\mathcal{J}^{-3} + \frac{19}{64}\mathcal{J}^{-4} - \frac{1317}{8192}\mathcal{J}^{-6} \\ &+ \frac{4199}{196608}\mathcal{J}^{-7} + \frac{96157}{589824}\mathcal{J}^{-8} - \frac{3539135}{28311552}\mathcal{J}^{-9} \\ &- \frac{133012373}{679477248}\mathcal{J}^{-10} + \mathcal{O}(\mathcal{J}^{-11}). \end{aligned} \quad (14)$$

This result is valid irrespective of the sign of  $J$ . Indeed, we find agreement with the second-order result of Ref. 6 for ferromagnetic coupling  $J < 0$ . For  $J < 0$  and  $J' > 0$ , there is an experimental realization of a trimerized system:  $3\text{CuCl}_2 \cdot 2\text{dioxane}$ . However, since the coupling constants of this material are roughly given by  $J/J' \approx -5$ , the experimental magnetization curve<sup>16</sup> is far outside the range of validity of our series (14); actually no magnetization plateau is observed experimentally.

For  $p=4$  and antiferromagnetic  $J, J' > 0$ , one finds the following counterpart of Eqs. (13):

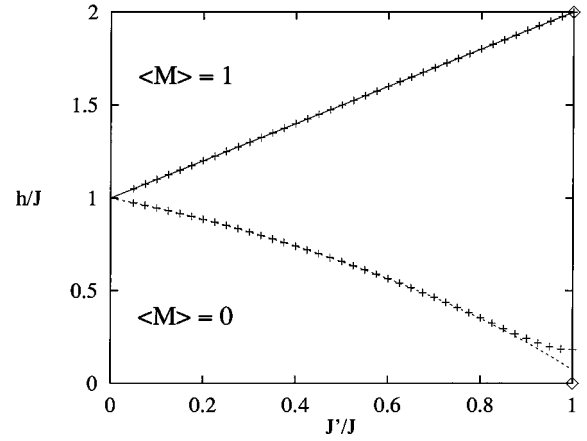


FIG. 1. Gap and transition fields for  $p=2$ . The solid line shows the upper critical field (7). The dotted line is our tenth-order series expansion (10) for the spin gap. Crosses show the  $L=24$  numerical data of Cabra and Grynberg (Ref. 4) and the diamonds denote the magnetic fields  $h$  at  $J'=J$  associated to  $\langle M \rangle = 0$  and  $\langle M \rangle = 1$ , respectively.

$$\begin{aligned} \frac{E^{(p=4)}}{J} &= 1 - \frac{1}{4}\mathcal{J}^{-1} + \mathcal{O}(\mathcal{J}^{-2}), \\ \frac{h_{c_1}^{(p=4)}}{J} &= 1 + \frac{1}{2}\mathcal{J}^{-1} + \frac{1}{4}\mathcal{J}^{-2} + \frac{1}{4}\mathcal{J}^{-3} - \frac{5}{32}\mathcal{J}^{-4} - \frac{41}{64}\mathcal{J}^{-5} \\ &- \frac{201}{256}\mathcal{J}^{-6} - \frac{497}{2048}\mathcal{J}^{-7} + \frac{11887}{8192}\mathcal{J}^{-8} + \frac{52929}{16384}\mathcal{J}^{-9} \\ &+ \frac{180845}{65536}\mathcal{J}^{-10} + \mathcal{O}(\mathcal{J}^{-11}), \\ \frac{h_{c_2}^{(p=4)}}{J} &= \mathcal{J} + \frac{1}{2} + \frac{1}{4}\mathcal{J}^{-1} - \frac{1}{16}\mathcal{J}^{-3} - \frac{3}{16}\mathcal{J}^{-4} - \frac{7}{512}\mathcal{J}^{-6} \\ &+ \frac{449}{4096}\mathcal{J}^{-7} + \frac{715}{4096}\mathcal{J}^{-8} + \frac{6555}{65536}\mathcal{J}^{-9} \\ &- \frac{62051}{524288}\mathcal{J}^{-10} + \mathcal{O}(\mathcal{J}^{-11}). \end{aligned} \quad (15)$$

For the gap  $E^{(p=4)}$  we restrict ourselves to second order only, since the high degeneracy at  $\langle M \rangle = 0$  for  $J=0$  starts to invalidate our approach at third order.

We compare our perturbative results to the  $L=24$  numerical data<sup>4</sup> in Figs. 1–3. The solid lines show our results for the upper critical fields  $h_{uc}$ , and the dotted lines denote the series expansions around  $J'=0$ , while the dash-dotted lines in Figs. 2 and 3 show the expansions around  $J=0$ . Crosses denote the  $L=24$  numerical data of Ref. 4 and the diamonds show the magnetic fields associated to the plateau values of  $\langle M \rangle$  at  $J'=J$ . For the isotropic chain they are  $h=0$  for  $\langle M \rangle = 0$  and  $h=2J$  for  $\langle M \rangle = 1$ . The fields associated with  $\langle M \rangle = 1/3$  and  $\langle M \rangle = 1/2$  are computed from the Bethe-ansatz solution of the Heisenberg chain (see, e.g., Ref. 11). Since Abelian bosonization predicts the plateaus to open for  $J' \neq J$ ,<sup>4</sup> the diamonds denote the expected ending points of the magnetization plateaus.

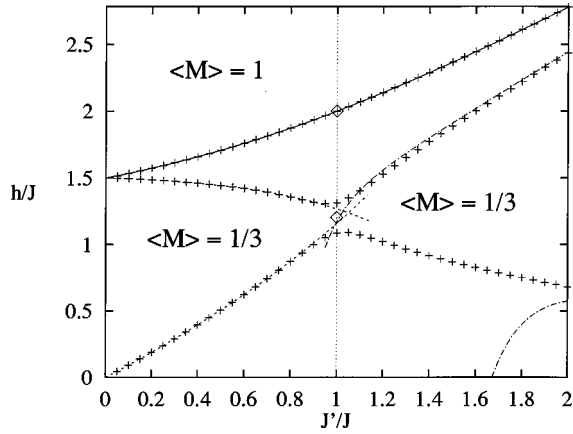


FIG. 2. Transition fields for  $p=3$ . The solid line shows the upper critical field (8), the dotted lines the series (12) for small  $J'$ , and the dash-dotted lines the series (14) for small  $J$ . Crosses show  $L=24$  numerical data of Cabra and Grynberg (Ref. 4) and the diamonds denote the magnetic fields  $h$  at  $J'=J$  associated to  $\langle M \rangle = 1/3$  and  $\langle M \rangle = 1$ , respectively.

The case  $p=2$  is shown for completeness in Fig. 1. Since here the regimes  $J' \leq J$  and  $J' \geq J$  are equivalent, we display only the former. Here the dotted line shows our tenth-order series expansion (10) for the excitation energy at  $k=0$ . The overall agreement is excellent, as has already been observed for the gap in Ref. 13.

Figure 2 shows the next case  $p=3$ . The series (12) are in excellent agreement with the  $L=24$  numerical data for  $J' < J$ . Even the ending point of the  $\langle M \rangle = 1/3$  plateau is reproduced quite well. For  $J' > J$ , the upper boundary of the plateaus is also reproduced reasonably by Eqs. (14), while the agreement for the lower boundary is poor despite the length of the series. This is not entirely surprising as we have remarked above. In fact, inspection of the expression for  $h_{c_1}^{(p=3)}$  in Eqs. (14) shows that the coefficients get larger with increasing order such that this series might actually not converge in the region shown in Fig. 2.

This comparison of perturbation theory and finite-size diagonalization is completed with the case  $p=4$  in Fig. 3. The

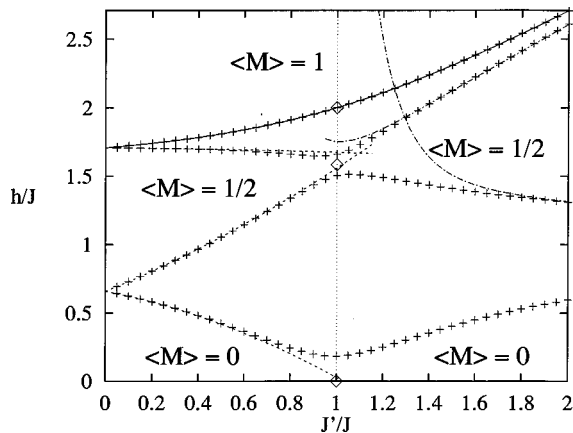


FIG. 3. Same as Fig. 2, but for  $p=4$ . The solid line shows Eq. (9), the dotted lines Eqs. (13), and the dash-dotted lines Eqs. (15). Diamonds denote magnetic fields associated with  $\langle M \rangle = 0$ ,  $1/2$ , and  $1$ , respectively.

expansions (13) around  $J'=0$  compare again favorably with the numerical data of Ref. 4 although the series are only of second order. Also the series for  $h_{c_2}^{(p=4)}$  in Eqs. (15) agrees quite well with the numerical data for  $J' > J$ , while that for  $h_{c_1}^{(p=4)}$  yields good agreement at least at the right boundary of Fig. 3. The small- $J$  series for  $E^{(p=4)}$  is not even shown, since due to its low order it cannot be expected to give sensible results in the region of interest. As in the case  $p=3$ , the limited quality of the series for  $J' > J$  can be expected on general grounds and is also indicated by inspection of the value of the coefficients in Eqs. (15).

A comparison of the series (14) and (15) with the numerical data of Ref. 4 is complicated by the fact that the latter does not extend into the region of small  $J$ —only the region  $J' \ll J$  is covered well. We have therefore performed some further numerical computations for  $p=3$  and  $p=4$ . At sufficiently small  $J$  one can then nicely verify the series order by order—much in the spirit of Ref. 13. In this manner we have verified the lowest five to six orders of all series in Eqs. (14) and (15) (a standard numerical accuracy is not sensitive to the highest orders).

It has been pointed out recently by several authors (see, e.g., Refs. 17–22) that the strong-coupling approach can be extended to describe the transitions between plateaus by an effective Hamiltonian. For the case discussed here, one will in general have to retain two states per site in first order in  $J'$ . These two states correspond to the two plateau ground states at  $J'=0$  between which we wish to describe the transition. If the coupling constants are chosen to preserve parity [as is the case, e.g., for Eq. (5)], symmetry arguments imply that the effective Hamiltonian is an XXZ chain. Hence one can immediately carry over some well-known universal properties of the XXZ chain (see, e.g., Sec. II of Ref. 11 for a review) to polymerized spin chains. First, the mapping to the XXZ chain in the strong-coupling limits implies that the exponents of the correlation functions at the plateau boundaries are given by

$$\eta_z = 2, \quad \eta_{xy} = \frac{1}{2}. \quad (16)$$

Second, the transitions at the plateau boundaries are predicted to be of the Dzhaparidze-Nersesyan-Pokrovsky-Talapov (DN-PT) type,<sup>23,24</sup> i.e., the magnetization as a function of applied field  $h$  has a square-root behavior close to the plateau boundaries. The same conclusions are obtained by the Abelian bosonization analysis of the limit  $J' \rightarrow J$ .<sup>4</sup> This follows from results in the theory of commensurate-incommensurate transitions<sup>25</sup> which in addition imply that the exponents (16) as well as the DN-PT square-root behavior should be universal. The fact that identical conclusions are reached by considering two different limiting cases is in agreement with such a universal scenario.

Before concluding, it should be mentioned that the quantization condition (3) may have to be relaxed in certain cases. For example, it has been shown (see, e.g., Refs. 17 and 26) that an  $\langle M \rangle = 1/2$  plateau can appear if a next-nearest neighbor interaction is added to the dimerized chain, Eq. (1) with  $p=2$  (for generalizations of this situation see Ref. 27). This phenomenon can be also understood within the strong-

coupling analysis<sup>17–20</sup> if one goes to first order, i.e., beyond the naive decoupling limit  $J'=0$ . The crucial role is played by the  $XXZ$  anisotropy appearing in the first-order effective  $XXZ$  chain. If this effective  $XXZ$  anisotropy turns out to be sufficiently large ( $\Delta > 1$ ), a gap opens and translational symmetry is spontaneously broken. In this manner one finds a further plateau precisely in the middle between the two values of  $\langle M \rangle$  predicted by considering just the limit  $J'=0$ . This illustrates that  $p$  in Eq. (3) should be taken as the period of the ground state which in general can be different (i.e., an integer multiple) of the period of the Hamiltonian.

To summarize, we have shown that the study of the strong-coupling limit not only provides a simple way to understand the magnetization process of polymerized spin chains qualitatively, but that also quantitatively competitive results can be obtained with moderate effort. In some respects, the situation is even nicer than for spin ladders:<sup>10,11</sup> The bare series in  $J'$  yield good results in the entire region

$J' < J$ , including the ending points of the plateaus. Such favorable conditions are probably a special feature of polymerized spin-1/2 chains, as is the observation<sup>4</sup> that here the condition (3) is necessary *and* sufficient for the appearance of a plateau at positive  $J' \neq J$ .

It is straightforward to extend the approach of this paper to more general interactions or to the computation of other quantities. We are confident that further explicit strong-coupling computations will provide a useful tool, e.g., if new experimental data are to be explained in terms of model Hamiltonians.

I would like to thank D.C. Cabra and M.D. Grynberg for providing me with their numerical data. I am grateful to them and P. Pujol for useful discussions and comments. This work was carried under under financial support from TMR Grant No. FMRX-CT96-0012.

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