Extreme type-II superconductors in a magnetic field: A theory of critical fluctuations

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A theory of critical fluctuations in extreme type-II superconductors subjected to a finite but weak external magnetic field is presented. It is shown that the standard Ginzburg-Landau representation of this problem can be recast, with help of a mapping, as a theory of a new ''superconductor,'' in an effective magnetic field whose overall value is zero, consisting of the original uniform field and a set of neutralizing unit fluxes attached to N_Φ fluctuating vortex lines. The long-distance behavior of this theory is governed by a phase transition line in the (H,T) plane, $T_\Phi(H)$, along which the new ''superconducting'' order parameter $\Phi(\mathbf{r})$ attains long-range order. Physically, this phase transition arises through the proliferation, or ''expansion,'' of thermally generated infinite vortex loops in the background of field-induced vortex lines. Simultaneously, the field-induced vortex lines lose their effective line tension relative to the field direction. It is suggested that the critical behavior at $T_\Phi(H)$ belongs to the universality class of the anisotropic Higgs-Abelian gauge theory, with the original magnetic field playing the role of ''charge'' in this fictitious ''electrodynamics'' and with the absence of reflection symmetry along $\mathbf{H}$ giving rise to dangerously irrelevant terms. At zero field, $\Phi(\mathbf{r})$ and the familiar superconducting order parameter $\Psi(\mathbf{r})$ are equivalent, and the effective line tension of large loops and the helicity modulus vanish simultaneously, at $T=T_{c0}$. In a finite field, however, these two forms of ''superconducting'' order are not the same and the ''superconducting'' transition is generally split into two branches: the helicity modulus typically vanishes at the vortex lattice melting line $T_m(H)$, while the line tension and associated Φ order disappear only at $T_\Phi(H)$. We expect $T_\Phi(H)>T_m(H)$ at lower fields and $T_\Phi(H)=T_m(H)$ for higher fields. Both Φ and Ψ order are present in the Abrikosov vortex lattice $[T<T_m(H)]$ while both are absent in the true normal state $[T>T_\Phi(H)]$. The intermediate Φ-ordered phase, between $T_m(H)$ and $T_\Phi(H)$, contains precisely N_Φ field-induced vortices having a finite line tension relative to $\mathbf{H}$ and could be viewed as a ''line liquid'' in the long-wavelength limit. The consequences of this ''gauge theory'' scenario for the critical behavior in high-temperature and other extreme type-II superconductors are explored in detail, with particular emphasis on the questions of three-dimensional XY versus Landau level scaling, physical nature of the vortex ''line liquid'' and the true normal state (or vortex ''gas''), and fluctuation thermodynamics and transport. It is suggested that the empirically established ''decoupling transition'' may be associated with the loss of integrity of field-induced vortex lines as their effective line tension disappears at $T_\Phi(H)$. A ''minimal'' set of requirements for the theory of vortex lattice melting in the critical region is also proposed and discussed. The mean-field-based description of the melting transition, containing only field-induced London vortices, is shown to be in violation of such requirements. [S0163-1829(98)06441-8]

I. INTRODUCTION

Recent intense activity in the area of superconducting fluctuations has brought into sharp focus the following fundamental questions: What is the relationship between the Landau-level-based[1–3] and the three-dimensional (3D) XY-based[4–6] descriptions of superconducting fluctuations in a magnetic field? Can the mean-field-based London model containing only magnetic field-induced vortices[7] describe the vortex lattice melting transition in the region of strong (critical) fluctuations? What is the nature of the normal phase and can it be usefully represented as a ''line liquid''[8] of field-induced vortices? What role is played at finite fields by thermally generated vortex loops,[9,10] which are responsible for the zero-field transition in extreme type-II superconductors? Particular importance and urgency has been attached to these questions following the ground-breaking experiments[11–13] on the thermodynamics of vortex lattice melting transition[14] which clearly indicate that the low-field end of the melting line is entering the critical regime of high-temperature superconductors.

In this paper, precise answers to these questions are provided within a theoretical framework which allows for a systematic solution to the problem of critical fluctuations in an extreme type-II superconductor subjected to a finite, but weak magnetic field. This framework is built around the ''gauge theory'' scenario proposed earlier.[9] Two main predictions follow from this scenario: first, there is a new transition line in the H-T phase diagram, $T_\Phi(H)$, along which a thermally generated vortex loop ''expansion'' takes place, reminiscent of the zero-field transition. At $T_\Phi(H)$, well defined field-induced vortex lines are formed, having a *finite* line tension *relative* to the field direction. Initially, these lines are in a liquid state and solidify only at some lower temperature $T_m(H)$ (Fig. 1). This is different from Abrikosov's theory, where vortices and their (Abrikosov) lattice are formed simultaneously, at $H_{c2}(T)$; second, in contrast to the 3D XY behavior at zero field, the description of the critical behavior along $T_\Phi(H)$ requires the combination of a complex ''superconducting'' order parameter Φ associated with vortex loops *and* a fictitious gauge field $\mathbf{S}$, describing fluctuations in the background system of field-induced vortex

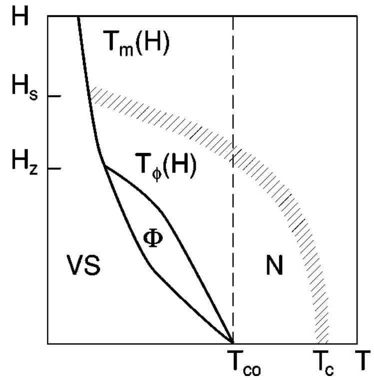

FIG. 1. Proposed H-T phase diagram for the critical region of extreme type-II superconductors. The dashed region denotes a crossover from the Gaussian regime, where amplitude fluctuations are strong, to the critical 3D XY-like regime, where amplitude fluctuations are suppressed. Within the 3D XY-like critical regime, the London-type vortex loops and lines with tight cores are well defined excitations. Along the temperature axis, this critical region is bounded by the *mean-field* T_c. Along the field axis, the critical region is bounded by H_s ($\sim H_b$) Eq. (6). Above H_s, the physics of the GL theory (1) is dominated by the formation of Landau levels for Cooper pairs (Refs. 16 and 19). The Φ transition, or the vortex loop ''expansion'' transition $T_\Phi(H)$ and the vortex lattice melting $T_m(H)$, occur *simultaneously* for $H>H_Z\sim H_s$. This is a *first-order* transition from the Abrikosov vortex lattice (VS) directly to the normal state (N). Below H_Z, $T_\Phi(H)$ and $T_m(H)$ split into two *separate* transitions and merge again only at the true zero-field superconducting transition T_{c0} as $H\to 0$. For $H\ll H_s$, the transition at $T_\Phi(H)$ is likely *continuous*, while the vortex lattice melting transition remains first order. The intermediate phase (Φ), below $T_\Phi(H)$ but above $T_m(H)$, is not a superconductor ($\langle\Psi\rangle=0$), but it differs from the true normal state (N) by a new type of long-range order, characterized by the ''superconducting'' order parameter $\Phi(\mathbf{r})$, Eqs. (2) and (4). Only N_Φ field-induced vortex lines traverse the system along the field direction in this Φ-ordered state, while the average size of thermally generated vortex loops is finite. In the true normal state (N), the Φ order is destroyed as numerous *additional* vortex paths ''expand'' across the system in all directions.

lines. The magnetic field determines the ''charge'' which couples Φ and $\mathbf{S}$. The physical picture arising from the ''gauge theory'' is remarkably detailed and compelling, and so entirely distinct from the ''standard'' approach[15] that a concentrated effort should be directed at exploring its consequences. The main purpose of this paper is to provide an explicit model for critical fluctuations, to examine its main ramifications in some detail, and to advance a set of specific predictions which can help establish the value of the ''gauge theory'' description though experiments and numerical simulations.

The essential feature of our description is that it contains, on equal footing, *both* the field-induced *vortex lines and* thermally generated critical fluctuations of the superconducting order parameter Ψ, associated primarily with *vortex loops*. It is the latter that dominate the entropy in the critical region.[9] This is in fundamental contrast to other approaches which include *only* the field-induced vortices: the Landau level (LL) description[16–19] (where other fluctuations become irrelevant at sufficiently high fields) and the mean-field-based picture of London vortices[15] (where other fluctuations can be ignored at sufficiently low temperatures, far below the critical regime). Following the prescription proposed earlier,[9] which seeks to conveniently isolate the background of field-induced from thermally generated degrees of freedom, we derive the following results: First, it is shown in Sec. II that the familiar and frequently used ''helium'' or ''London model'' of extreme type-II superconductors, in which the amplitude fluctuations are suppressed, allows for a direct mapping of the original problem to that of a new ''superconductor,'' whose order parameter Φ experiences an overall magnetic field composed of the uniform external field $\mathbf{H}$ and the set of N_Φ neutralizing ''fluxes'' attached to fluctuating vortex lines. This mapping constitutes an explicit and transparent realization of the general connection proposed in Ref. 9. The ''helium model'' is then a candidate to, in addition to the familiar vortex lattice melting line $T_m(H)$, exhibit the conjectured ''Φ transition'' (or the vortex loop ''expansion'' transition in a *finite* field), the universality class of which is defined by an anisotropic Higgs-Abelian gauge theory.[9,20] Physically, this Φ transition corresponds to the vanishing of the effective line tension for very large thermally generated vortex loops: at $T_\Phi(H)$, the energy-entropy balance in the free energy shifts in favor of large loops and spontaneously created infinite vortex-antivortex paths proliferate across the system. The ensuing change in the topology of the vortex paths results in a thermodynamic liquid-gas phase transition, associated with a change in the U(1) symmetry of a vortex system (Appendix A). Simultaneously, the field-induced vortex lines lose their line tension relative to the field direction and the ''line liquid'' description breaks down. As our second result, it is shown that the fictitious gauge theory passes a crucial test, allowing us to connect its ''charge'' to the original magnetic field. Third, we use this connection in Sec. IV to construct scaling functions for the critical thermodynamics of extreme type-II superconductors. Furthermore, the much-debated difference between the 3D XY-like description at low fields and the LL description appropriate at high fields is closely linked here to the difference between the extreme ''type-II'' and the extreme ''type-I'' behavior of the gauge theory (Sec. III). Fourth, it is shown in Sec. V that a vortex loop ''expansion'' leads to an abrupt drop in the coefficient of the q^2 term in the helicity modulus, from which one can extract the thermodynamic exponent (ν) of the Φ transition. Related criteria are also proposed which test for the presence or absence of an effective ''diffusion'' of vortex lines along the field and demonstrate the close relation between the ''Φ order''[9] and viability of the vortex ''line liquid''[8] description. These predictions, based only on global topological properties of loops and lines, can be used to efficiently identify the vortex loop ''expansion'' line $T_\Phi(H)$ in numerical simulations of the weakly frustrated 3D XY and related models. Fifth, assuming a widely used form of dynamical scaling, the explicit expression for the fluctuation conductivity $\sigma(T,H)$ is derived in Sec. VI in the vicinity of the $T_\Phi(H)$ line. At $T_\Phi(H)$ there is an experimentally detectable rapid onset of additional dissipation, caused by thermally expanding vortex loops whose size is reaching sample boundaries. It is tempting to associate this onset at $T_\Phi(H)$ with what is empirically known as the ''decoupling'' transition,[21] although the physical origin of such additional dissipation in our theory is entirely unrelated to any ''decoupling'' of any ''layers.''[22] Instead, it signifies the loss of integrity of field-induced vortex lines as their effective line tension disappears at $T_\Phi(H)$. At this point, one also expects a distinct change in the pinning properties of the liquid state: there is no pinning in the true normal state above $T_\Phi(H)$ just as there is no pinning above T_{c0}. In this sense $T_\Phi(H)$ represents an upper boundary for pinning and could be viewed loosely as ''renormalized'' $H_{c2}(T)$. Finally, in Sec. VII, it is demonstrated that the vortex lattice melting transition in the critical region involves *simultaneous* ordering of the field-induced *and* thermally generated degrees of freedom and thus cannot be faithfully represented by a mean-field-based London model,[7] which includes only the former. Actually, as the melting line tends toward T_{c0} in the limit of vanishing magnetic field, the entropy change involved in ordering of thermally generated loops overwhelms the configurational entropy of the field-induced vortex lines. This provides direct theoretical support for the fundamental nature and significance of the experiments by Zeldov *et al.*,[11] Schilling *et al.*,[12] and Roulin *et al.*,[13] and new numerical simulations of Nguyen and Sudbø.[23,24]

II. FROM GINZBURG-LANDAU THEORY TO GAUGE THEORY

The starting point is the anisotropic Ginzburg-Landau (GL) theory $Z=\int \mathcal{D}\Psi \exp\{-\int d^3r \mathcal{F}/T\}$, where

$$\mathcal{F}=\alpha|\Psi|^2+\sum_{\mu=\parallel,\perp}\gamma_\mu\left|\left(\nabla_\mu+\frac{2ei}{c}\mathbf{A}_\mu\right)\Psi\right|^2+\frac{\beta}{2}|\Psi|^4, \tag{1}$$

and $\alpha=a_0(T-T_c)$, γ_μ, and β are the GL coefficients. Free (periodic) boundary conditions are imposed in the $\parallel$ ($\perp$) direction. The limit $\kappa\to\infty$ is considered, which is particularly appropriate for high-temperature superconductors (HTS's). In this limit, the external magnetic field $\mathbf{H}=\nabla\times\mathbf{A}_\perp$ acts as a *constraint*, forcing every allowed configuration of the system to have the overall vorticity N_Φ along $\mathbf{H}$ ($\parallel\hat{\mathbf{z}}$). The overall vorticity along $\mathbf{H}$ is defined as a line integral $\int d\mathbf{l}\cdot\nabla\varphi/2\pi$, where the contour of integration goes around the perimeter of the system in the xy plane and $\varphi(\mathbf{r})$ is the phase of Ψ. N_Φ, the number of elementary flux quanta ϕ_0, is given by $L_\perp^2/2\pi l^2$, where $l=\sqrt{c/2|e|H}$ is the magnetic length. It is assumed that this constraint is enforced by N_Φ vortex paths, meandering from one end of the system to another, along $\mathbf{H}$.[9]

In this paper, a method is introduced to enforce the constraint *explicitly*, by considering a *different* partition function $Z'=\int\mathcal{D}\Phi\int\Pi_{i=1}^{N_\Phi}(\mathcal{D}\mathbf{r}_i[s]/N_\Phi!)\exp\{-\int d^3r\mathcal{F}'/T\}$, with

$$\mathcal{F}'=\alpha|\Phi|^2+\gamma_\mu\left|\left(\nabla_\mu+i\mathbf{U}_\mu+\frac{2ei}{c}\mathbf{A}_\mu\right)\Phi\right|^2+\frac{\beta}{2}|\Phi|^4. \tag{2}$$

Z' describes the system of N_Φ ''shadow,'' or *s*, vortices $\{\mathbf{r}_i[s]\}$ in thermal equilibrium with a complex field $\Phi(\mathbf{r})$. These *s* vortices sample arbitrary paths that originate (terminate) at $z=0$ ($z=L_\parallel$) and differ from the ones introduced in Ref. 9 by the full inclusion of ''overhang'' configurations. The effective magnetic field $\mathbf{H}'$ experienced by Φ consists of the uniform external field $\mathbf{H}$ and the collection of unit ''fluxes'' attached to *s* vortices: $\nabla\times\mathbf{U}=2\pi\mathbf{n}_s(\mathbf{r})$, $\nabla\cdot\mathbf{U}=0$, where $\mathbf{n}_s(\mathbf{r})$ is the flux density associated with a given configuration of *s* vortices, $\{\mathbf{r}_i[s_i]\}$:[25]

$$\mathbf{n}_s(\mathbf{r})=\sum_i^{N_\Phi}\int_{\mathcal{L}}d\mathbf{r}_i\,\delta(\mathbf{r}-\mathbf{r}_i[s_i]), \tag{3}$$

with $\mathcal{L}$ denoting the line integral. The net value of $\mathbf{H}'$ averaged over the system *vanishes*.

The superconductors (1) and (2) are *equivalent* within the familiar ''helium model'' of extreme type-II behavior; they are just two different representations of the same physical problem. To show this we recall the main features of the ''helium model'':[26] the true transition temperature T_{c0} (Fig. 1) is assumed to be *sufficiently below* the mean-field T_c for amplitude fluctuations to have effectively subsided. Around T_{c0}, the relevant fluctuations are considered to be those of London-type vortex loops and lines with steric repulsion and well defined, tight cores of size $a\ll l$. Consider now a single configuration of these loops and lines. First, we extract the singular part of $\nabla\varphi(\mathbf{r})$ by solving two equations $\nabla\times\nabla\varphi=2\pi\mathbf{n}(\mathbf{r})$ and $\nabla\cdot\nabla\varphi=0$, where $\mathbf{n}(\mathbf{r})$ is defined by the same expression as $\mathbf{n}_s$, Eq. (3), but with the summation running over *all* vortex loops and lines. After this ''vortex'' part has been extracted, the rest of $\Psi(\mathbf{r})$ is assumed to take the form which minimizes $\mathcal{F}$ for a given configuration of these line singularities. We then integrate over all regular (''spin-wave'') fluctuations in $\varphi(\mathbf{r})$. Finally, all such distinct configurations of vortex loops and lines are summed over to produce Z (1). This same procedure is imposed on $\Phi(\mathbf{r})$: first we extract the part of *its* phase, $\nabla\phi(\mathbf{r})$, due to vortex loop and line singularities in $\Phi(\mathbf{r})$ and then determine the rest of $\Phi(\mathbf{r})$ by minimizing $\mathcal{F}'$, Eq. (2), for a given configuration of these line defects *and s* vortices. Again, we integrate over all ''spin-wave'' fluctuations of ϕ relative to this given configuration of loops and lines. By direct comparison of these ''helium model'' expressions obtained from Eqs. (1) and (2), it is evident that all configurations contributing to the original Z are reproduced in Z' and have the same energy. However, some of these configurations are counted more than once in Z'. This overcounting of configurations in Z' relative to Z, given by $(N_\Phi+N_a)!/N_\Phi!N_a!$ with N_a being the number of vortex lines in Φ which traverse the sample along $\mathbf{H}$, is a surface effect in 3D and should be unimportant in the thermodynamic limit $L_\perp$, $L_\parallel\to\infty$. Moreover, within the conjectured Φ-ordered phase (Fig. 1), the configurations with $N_a\neq 0$ are irrelevant in the thermodynamic limit and there is no overcounting at all. We conclude that, within the ''helium'' model, the free energy evaluated from Z' coincides with the free energy of the original problem (1) and the two superconductors have *identical* thermodynamics.[27] Consequently, Eq. (2) accomplishes a straightforward and transparent reformulation of the original problem, in the spirit of Ref. 9, while avoiding more cumbersome gauge transformation method employed there.[25] More details on the ''helium model'' are presented in Appendix A.

If we relax the above minimization condition on the amplitude of our order parameters Ψ and Φ and permit weak amplitude fluctuations, we expect that the above close relation between Z, Eq. (1), and Z', Eq. (2), still holds, as long as the parameters of GL theory keep us in the extreme type-II limit. This requires the *average* core size a to be smaller than the average spacing between vortex segments, so that vortex excitations remain well defined. It is precisely this same requirement that is invoked to justify the frequent use of the ''helium model'' to emulate fluctuation behavior of extreme type-II superconductors in zero field. It is natural to expect that, if such a requirement is satisfied at zero field, it will remain so at low fields, such that $a \ll l$. Based on this, on the equivalence of representations (1) and (2) in the ''helium model'' limit, and on our general expectation that the extreme type-II behavior with only weak amplitude fluctuations is effectively equivalent to the ''helium model,'' for the rest of this paper I consider Eq. (2) to be simply an alternative formulation of the original problem. This new reformulation (2) can now be used instead of Eq. (1) to compute various fluctuation properties and, most importantly, its *critical* behavior should *coincide* with that of the original GL theory (1).

The advantage of Z', Eq. (2), is that, by isolating the background of field-induced degrees of freedom (s vortices), it focuses our attention on the new ''superconducting'' order parameter $\Phi(\mathbf{r})$ and its spatial correlations, measured by $\langle \Phi(\mathbf{r})\Phi^*(\mathbf{r}')\rangle$, where $\langle \cdots \rangle$ denotes thermal average over Z'. All excitations of $\Phi(\mathbf{r})$ are *thermally generated*, in the following precise sense: *every configuration of* $\Phi(\mathbf{r})$, *contributing a finite weight to* Z' *in the thermodynamic limit, has the overall vorticity along* $\mathbf{H}$ *equal to zero*. In particular, $\Phi(\mathbf{r})$ contains vortex loop excitations, whose ''expansion'' across the system is the mechanism behind the $H=0$ superconducting transition (Fig. 2). By focusing on Φ, we can fashion a theory of the strongly *interacting* Wilson-Fisher (3D XY) critical point, ''perturbed'' by a weak field.[9] This is precisely the opposite of the classic approach,[28] where the Gaussian theory in a *finite* field is perturbed by weak interaction. Such an approach starts with the LL structure from the outset and its critical behavior is always dominated by the lowest LL.[28,16] In the new formulation (2) we had built in from the start our expectation that the weak field modifies zero-field configurations only by introducing a low density of s vortex lines, the cores of which are well defined by virtue of strong amplitude correlations at the 3D XY critical point. This is a ''low-field'' approach by design and offers a better prospect of constructing the desired theory.

To extract such a theory from Eq. (2) we must resort to approximations. We construct the long wavelength ($\gg l$) limit of Eq. (2) by coarse-graining vorticity fluctuations produced by s vortices. The ''hydrodynamic'' vorticity $\mathbf{V}(\mathbf{r})$ is defined as the coarse-grained version of the ''microscopic'' flux density $\Delta\mathbf{n}_s(\mathbf{r}) = \mathbf{n}_s(\mathbf{r}) - (2\pi l^2)^{-1}\mathbf{z}$. Upon inserting $\int \mathcal{D}\mathbf{V}\delta[\mathbf{V}(\mathbf{r}) - \Delta\mathbf{n}_s(\mathbf{r})]$ in Eq. (2), integrating over $\{\mathbf{r}_i[s]\}$, and after introducing the fictitious vector potential $\nabla\times\mathbf{S} = 2\pi\mathbf{V}$, $\nabla\cdot\mathbf{S}=0$, the effective long-wavelength theory becomes $\sim \int\mathcal{D}\Phi\int\mathcal{D}\mathbf{S}\exp\{-\int d^3r\mathcal{F}_{\rm eff}/T\}$,

$$\mathcal{F}_{\rm eff} = \alpha|\Phi|^2 + \gamma_\mu|D_\mu\Phi|^2 + \frac{\beta}{2}|\Phi|^4 + \frac{K_\mu}{2}(\nabla\times\mathbf{S})^2_\mu, \quad (4)$$

where $D_\mu = \nabla_\mu + i\mathbf{S}_\mu$, $K_\perp(T,H) = c_\perp\Gamma^{-1}Tl$, and $K_\parallel(T,H) = c_\parallel\Gamma Tl$. Higher powers and derivatives of $(\nabla\times\mathbf{S})_\mu$, essential for the description of the vortex lattice melting, also appear in Eq. (4), but are unimportant at $T_\Phi(H)$. A detailed derivation of the ''gauge theory'' (4) is given in Appendix A. $c_{\perp,\parallel}\sim\mathcal{O}(1)$ are dimensionless and have a relatively weak H,T dependence in that portion of the critical region which is well described by Eq. (4).[9] A close relation between $K_{\perp,\parallel}$, $c_{\perp,\parallel}$ and the components of the helicity modulus tensor of the GL theory (1) is discussed later in the text (see Sec. V and Appendix B). Γ is the anisotropy at T_{c0}. Small H-dependent corrections to GL coefficients that also should appear in Eq. (4) are ignored, since they are not important for our present purposes.

The following assumptions have been used in going from Eq. (2) to Eq. (4) (see also Appendix A).

(i) The correlation length ξ_Φ, associated with the new order parameter Φ, is not limited by l and can be much longer than the original superconducting correlation length $\xi_{\rm sc}$ associated with Ψ. Of course, this is the basic reason why we are interested in the reformulation (2) in the first place. When $\xi_\Phi \gg l,\xi_{\rm sc}$, this assumption enables us to drop as irrelevant[9] at long distances terms containing higher derivatives and powers of $\mathbf{S}$ from Eq. (4). Note, however, that such higher-order terms in Eq. (4), particularly those reflecting the absence of up-down symmetry along $\mathbf{H}$ $[(\nabla\times\mathbf{S})^3_\parallel$ and the like], must be restored when discussing vortex lattice melting and a possibility of a first-order ϕ transition[9] (Sec. VII and Appendix A). The gauge theory (4) offers in this case ($\xi_\Phi \gg l,\xi_{\rm sc}$) a direct access to the deeply *nonperturbative* regime of the original GL theory (1), characterized by $\xi/l \gg 1$, where ξ is the $H=0$ correlation length. In the opposite case $\xi_\Phi \ll l$, we are in the *perturbative* regime, $\xi/l \ll 1$ of the original theory. The long-wavelength expansion that led from Eq. (2) to Eq. (4) is then not justified and the new reformulation (2) is not particularly useful.

(ii) The system (1) is not in its superconducting state in the vicinity of the putative Φ transition. This assumption fixes the form of the last two terms in Eq. (4) (see Sec. V and Appendix B). Physically, it means that the system of s vortices contains configurations that wind from one end of a sample to another in the $\perp$ directions (xy plane). The presence of such windings allows a complete ''screening'' of an arbitrary infinitesimal field $\mathbf{h}(\mathbf{r})$ added to $\mathbf{H}$ and the helicity modulus tensor vanishes along *all* of its principal axes (see Sec. V and Appendix B). This assumption has a strong theoretical justification.[29] It must be emphasized, however, that, to my knowledge, there is no rigorous argument which could rule out another possibility, that of the state right below the Φ transition being an extremely anisotropic ''superconducting'' liquid, containing no windings in the xy plane, with a finite helicity modulus *along* the field and zero perpendicular to it. Indeed, some numerical studies are suggestive of this second possibility.[30,24] However, other numerical simulations,[23,31,32] as well as the available experimental data, favor our original nonsuperconducting liquid assumption. *Both* alternatives can be described within the framework of the gauge theory, with $K_\perp/K_\parallel \to \infty$ and potentially finite ''mass terms'' below $T_\phi(H)$ added to Eq. (4) in the extreme anisotropy case. On general physical grounds,[29] I have chosen to explore in this paper the case of finite anisotropy ratio $K_\perp/K_\parallel$ and vanishing mass terms but the reader should be aware that the extreme anisotropy alternative remains a

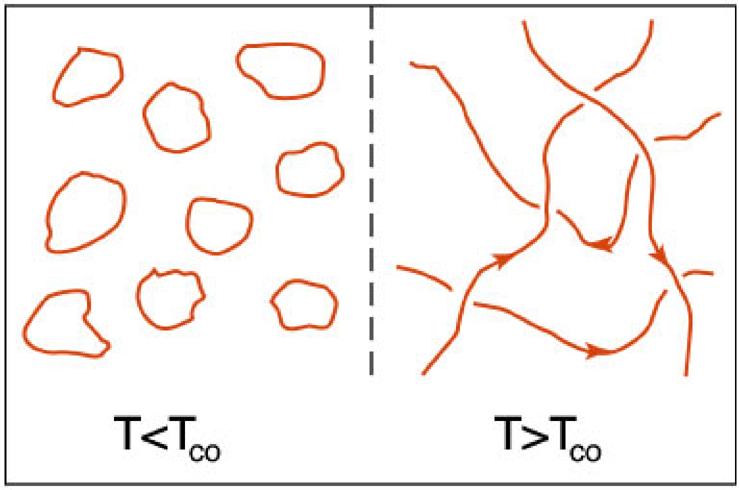

FIG. 2. (Color) A schematic representation of the $H=0$ transition in an extreme type-II superconductor. The low-temperature Meissner phase ($T<T_{c0}$) contains only finite vortex loops. In the high-temperature normal state ($T>T_{c0}$) these loops connect and ''expand'' across the system, leading to a loss of phase coherence and finite dissipation. Two forms of superconducting order, described by Ψ and Φ, are equivalent here. For clarity, the vortex paths are drawn smoother than they actually are near T_{c0}.

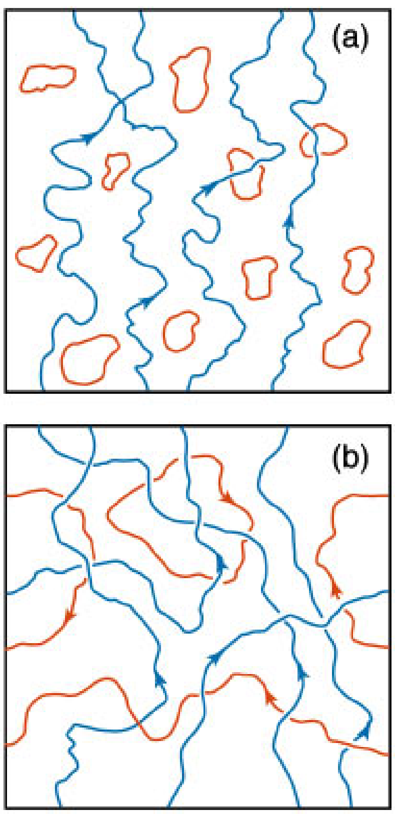

FIG. 3. (Color) Characteristic configurations of the system (a) below and (b) above $T_\Phi(H)$ [but always above $T_m(H)$]. (a) Field-induced vortices (depicted in blue) wind all the way across the system along the field direction but only undergo effective ''diffusion'' in the transverse direction. (b) After the loop ''expansion'' at $T_\Phi(H)$ this effective transverse ''diffusion'' is destroyed, as field-induced vortices can ''hitch a ride'' all the way across the system in the xy plane by connecting to thermally generated infinite loops present in the true normal state. Note the presence of ''vortex tachyons'' (depicted in red) which wind only in the xy plane. Again, for illustrative purposes, the vortex paths are drawn smoother than they actually are near $T_\Phi(H)$.

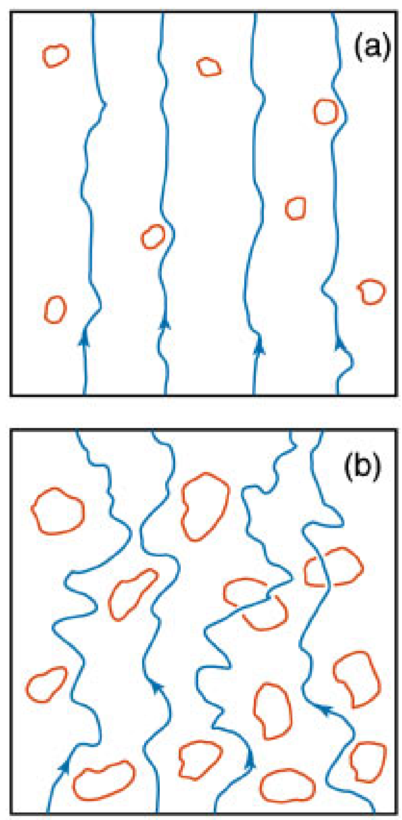

FIG. 4. (Color) Characteristic configurations of the system (a) below and (b) above $T_m(H)$ [but always below $T_\Phi(H)$]. (a) Field-induced vortices (blue) execute small oscillations around their equilibrium positions. Thermally generated vortex loops (red) are small and rare. (b) Above the melting transition thermally generated loops discontinuously grow larger and more numerous, although they still remain of finite size. This discontinuous change in the state of the loops accounts for $\langle\Phi\rangle_S \neq \langle\Phi\rangle_L$ at the simplest mean-field level.

possibility[33] and would lead to results which, while similar on a general level, differ in details.[34]

(iii) There are two relevant length scales controlling the critical behavior: ξ_Φ, which characterizes *both* the spatial correlations of Φ *and* the size of ''overhangs'' in the system of s vortices, and l, which characterizes the long-wavelength fluctuations of the background field-induced vorticity.

(iv) The core effects can be ignored. Clearly, the ''helium model'' itself is perfectly well defined in the limit $a\to 0$. For a small but finite, there is a small correction to the core line energy $E_c \to E_c + w_c \mathbf{H}\cdot\mathbf{v}$, where $\mathbf{v}=d\mathbf{r}/ds$ is the ''velocity'' of a vortex segment and $w_c H/E_c \sim a^2/l^2 \ll 1$. Similarly, there are ''velocity''-dependent corrections to the short-range repulsion between vortex cores. Such terms are irrelevant since they result in higher-order derivatives in Eq. (4). For example, it is easy to see that the correction to the core energy cancels out for any finite vortex loop and can be factored out for s vortices.

What is the physics behind gauge theory (4)? The external field has been eliminated from the gradient terms in Eq. (4) ($\langle \mathbf{H}'\rangle = 0$) but, of course, it has not vanished: it *reappears* through the H dependence of $K_{\perp,\parallel}$. The gauge theory (4) can be viewed as fictitious, anisotropic ''electrodynamics'' with ''magnetic permeability'' $\mu_0 = 1/4\pi T$. The ''vector potential'' $\mathbf{S}$ is coupled to the''matter'' field Φ via ''electrical charge''

$$\tilde{e}^2_{\perp,\parallel} = \frac{\Gamma^{1/3}}{c_{\perp,\parallel} l} \propto \sqrt{H}.$$

The above ''charge'' and $K_{\perp,\parallel}$ describe the ''polarizability'' of the medium composed of s vortices and are directly related to the long-wavelength components of the helicity modulus tensor (Sec. V and Appendix B). This picture embodies the physical idea that the dominant effect of a weak magnetic field in Eq. (1), once $\xi_{\rm sc}$ has saturated to $\sim l$, arises through the mutual ''screening'' of large thermally generated loops and the background of field-induced vorticity, at distances $\gg l$. Such ''screening'' reduces the effective line tension of these large loops relative to its value at the $H=0$ ($\tilde{e}_{\perp,\parallel}=0$) transition. The strength of the ''screening'' is measured by the fictitious ''Ginzburg parameter'' of Eq. (4),

$$\kappa_s^2 \sim c \frac{\beta l}{2a_0^2 T_c \xi_{GL}^4} = \frac{b}{2q_0^2} \propto \frac{1}{\sqrt{H}}, \tag{5}$$

where $c=(c_\perp^2 c_\parallel)^{1/3}$ and $\xi_{GL}=(\xi_{GL\perp}^2 \xi_{GL\parallel})^{1/3}$, with $\xi_{GL\perp,\parallel} = \sqrt{\gamma_{\perp,\parallel}/a_0 T_c}$ being the GL coherence lengths. $b = \beta/a_0^2 T_c \xi_{GL}^3$ and $q_0^2 = \tilde{e}^2 \xi_{GL}$ are the dimensionless quartic coupling and ''charge,'' respectively. As $H\to 0$, the fictitious ''charge'' vanishes and we recover the zero-field 3D XY critical point. For H finite but weak, the ''screening'' is weak ($\kappa_s \gg 1$), indicating that the effects of finite $\tilde{e}$ are small compared to strong amplitude correlations produced by the quartic term in the GL theory (1). We have therefore manufactured a critical theory (4) describing the strongly interacting Wilson-Fisher (3D XY) critical point weakly ''perturbed'' by a finite magnetic field (finite ''charge'' $\tilde{e}_{\perp,\parallel}$).[35]

The gauge theory scenario is clearly different from what takes place in spin systems, where the external field couples *paramagnetically* to the order parameter. In an extreme type-II superconductor (1), the *diamagnetic* coupling of $\mathbf{H}$ to Ψ does not explicitly break the U(1) symmetry, which was spontaneously broken at the zero-field 3D XY critical point. The high-temperature phase (true normal state) still retains the full U(1) symmetry. This symmetry can be broken at low temperatures, either in a ''simple'' way, with Ψ acting as the order parameter, as is the case in the ''vortex solid'' state, or in a more subtle fashion, with Φ assuming the role of the new order parameter. Similarly, the gauge theory (4) differs from frequently used ''dimensional reduction'' approaches,[36] where the behavior of Eq. (1) at finite fields is related to that at *zero* field but in a *finite* system, the size of which is set by the magnetic length l. A typical dimensional reduction ($D \to D-2$) approach leads to the superconducting correlation length which is limited by l, i.e., the ''system size.'' This *agrees* with the gauge theory scenario,[9] since ''electrodynamics'' (4) also predicts $\xi_{\rm sc}(H)\sim l$ in the critical region (see Sec. V). However, a dimensional reduction approach also predicts that *all other* correlations are limited by l and, consequently, eliminates the possibility of any true thermodynamic phase transition in the GL theory (1). This is in contradiction with the overwhelming experimental and numerical evidence indicating some form of a ''vortex liquid'' to ''vortex solid'' transition at low temperatures. In sharp contrast, gauge theory (4) and the reformulation (2) are fully three-dimensional theories, just like Eq. (1). They naturally lead to two basic types of correlations that can extend over distances $\gg l$ and produce phase transition(s) at low temperatures (Fig. 1): those associated with positional order of s vortices and the familiar superconducting order parameter $\Psi(\mathbf{r})$ and those associated with the new ''superconducting'' order parameter $\Phi(\mathbf{r})$.

The conjecture[9] that connects the critical behavior of an extreme type-II superconductor (1) to a fictitious superconductor in *zero* field (4), the ''charge'' of which is set by the original external field H, must pass the following test: the way H enters in $\mathcal{F}_{\rm eff}$ must be consistent with its being a relevant operator of scaling dimension 2 in the renormalization group (RG) sense at the 3D XY critical point. This scaling dimension is suggested by dimensional analysis[37] [relevant effects of H enter through the dimensionless ratio $\xi_{\rm sc}^2(H=0)/l^2 \propto H\xi_{\rm sc}^2(H=0)$], is correct to two-loop order,[38] and is likely an exact property of the original GL theory (1) by virtue of gauge invariance. In addition, the scaling dimension appears independent of the nature of the zero-field critical point (i.e., whether it is 3D XY or Gaussian). On the other hand, as the ''charge'' $\tilde{e}$ is turned on in the gauge theory (4), the RG analysis indicates that, first, the finite charge anisotropy ($\tilde{e}_\perp \neq \tilde{e}_\parallel$) is *marginally irrelevant*,[39,40] and second, the scaling dimension of ''charge'' at the 3D XY critical point is 1/2; i.e., the relevant dimensionless operator is $\tilde{e}\sqrt{\xi_{\rm sc}(\tilde{e}=0)}$. The second statement is *exact* to all orders in perturbative RG and is also independent on the nature of the *neutral* critical point.[41] Since, in $\mathcal{F}_{\rm eff}$, Eq. (4), $\tilde{e}^2 \propto 1/l \propto \sqrt{H}$, this translates immediately to the scaling dimension of H being 2, as required. More generally, for dimension $D<4$, $\tilde{e}^2 \propto l^{D-4}$, while the scaling dimension of $\tilde{e}$ is $2-(D/2)$, again consistent with the scaling dimension of H

being 2. Note that in both formulations, Eqs. (1) and (4), the corresponding relevant operators H and $\tilde{e}$ ($\propto H^{1/4}$) are protected against acquiring anomalous dimensions by the same symmetry, the gauge invariance. These results demonstrate the internal consistency of the coarse-graining procedure leading to Eq. (4) and strongly support the conjecture[9] that $\mathcal{F}_{\rm eff}$ captures the long-wavelength (critical) behavior of Eqs. (2) and (1). In what follows, I promote this conjecture to a fact and examine its consequences.

III. HIGH FIELDS VERSUS LOW FIELDS

Two key consequences for the physics of the present problem follow from $\mathcal{F}_{\rm eff}$. First, the gauge theory Eq. (4) has two distinct regimes of behavior: the weak ''screening'' limit ($\kappa_s \gg 1$) corresponding to the extreme ''type-II'' limit of the fictitious ''electrodynamics'' and the strong ''screening'' limit ($\kappa_s \ll 1$) corresponding to the extreme ''type-I'' behavior. The extreme ''type-II'' behavior of Eq. (4) is precisely the low-field regime of the original theory (1) which exhibits the 3D XY-like critical fluctuations. In this low-field regime, the ''screening'' provided by the background of field-induced vorticity is weak and the dominant fluctuations are still London-type vortex loops and lines. The core size a remains small and well-defined, kept in check by strong amplitude correlations coming from the quartic term in Eq. (1), just as was the case at the zero-field 3D XY critical point. It is in this sense ($\kappa_s \gg 1$) that we can think of a 3D XY critical point weakly ''perturbed'' by a finite field.[35] In the extreme ''type-I'' limit, the situation is entirely different. There, the ''screening'' is strong ($\kappa_s \ll 1$) and the amplitude fluctuations ran rampant. It is not possible any longer to think of relevant fluctuations in the gauge theory (4), nor in Eqs. (2) and (1), as being London-like vortices. Rather, amplitude fluctuations are now of essential importance and individual vortex cores are ill defined. In the gauge theory (4), the two regimes are separated by the condition $\kappa_s \sim 1$. However, a word of caution must be inserted here since, once we are in the ''type-I'' regime of Eq. (4), our original line of reasoning that led from Eq. (1) to the gauge theory (4), via reformulation (2), is itself compromised and it is not clear whether there is a useful connection between the extreme ''type-I'' limit of (4) and our original problem (1). Instead, we must return back to the beginning (1) and start from scratch. It is natural to identify this extreme ''type-I'' behavior at high fields, characterized by strong amplitude fluctuations, as the regime in which the Landau level structure of the original GL theory (1) becomes important. The condition $\kappa_s \sim 0.4/\sqrt{2}$,[41] separating ''type-II'' from ''type-I'' ''electrodynamics'' in Eq. (4), translates to the criterion for the external magnetic field, $H \sim H_s$, telling us whether H is ''low'' or ''high.'' From Eq. (5) one gets

$$H_s \cong \left(\frac{c}{0.16}\right)^2 b^2 H_{c2}^{GL}(0). \tag{6}$$

If $H \ll H_s$, then the field is ''low'' and the use of a 3D XY-like description is justified. In the opposite limit $H \gg H_s$, the field is ''high'' and a 3D XY-like description falls apart (Fig. 1). If this is the case, we must abandon our zero- and low-field imagery of the ''helium model'' and use as a starting point an approach that is explicitly designed to deal with a high-field behavior, an example being the GL-LLL theory.[16,19] Note that H_s, within factors of order unity, *coincides* with H_b, the field below which the high-field, Landau-level-based description breaks down, due to strong LL mixing.[16] Since this criterion[16] is derived from entirely different arguments, we briefly reproduce it here for completeness. Going back to the GL theory (1), we expand $\Psi(\mathbf{r}) = \Sigma_{j=0}\Psi_j(\mathbf{r})$ in the set of LL manifolds, $\Psi_j(\mathbf{r})$. Recast in terms of dimensionless variables, the GL free energy functional becomes

$$\int d^3r \left\{ \sum_{j=0} [t+(2j+1)h]|\Psi_j|^2 + |\nabla_\parallel \Psi_j|^2 + \frac{b}{2}|\Psi|^4 \right\}, \tag{7}$$

where $t=(T/T_c)-1$, $h=H/H_{c2}^{GL}(0)$, and b is defined below Eq. (5). After rescaling Ψ and $\mathbf{r}$ by b in such a way that the coefficients of quartic and gradient terms in Eq. (7) are set to 1/2 and 1, respectively, the ''mass term'' for Ψ_j becomes

$$\frac{t}{b^2} + (2j+1)\frac{h}{b^2}.$$

As we reduce the field H, for T in the critical region ($T \approx T_c$), the mixing of LL's becomes strong when h/b^2 becomes some number of order unity. We can view this as a ''Ginzburg criterion'' along the H axis. It implies that the high-field, Landau level description becomes inadequate for fields less than

$$H_b \sim b^2 H_{c2}^{GL}(0) \sim \mathrm{Gi} H_{c2}^{GL}(0), \tag{8}$$

where we have used a close relation between b and a conventional Ginzburg fluctuation parameter Gi.[16] The definition and meaning of Gi exhibit wide variations in the literature, but typically $\mathrm{Gi} \sim b^2$.[15] As advertised, $H_b \sim H_s$. The same situation is encountered in 2D, except now $H_s \sim H_b \sim bH_{c2}^{GL}(0)$. The fact that the criterion for the breakdown of the Landau-level-based theory derived from the high-field side agrees with the region of validity of our 3D XY-like approach derived from the opposite, low-field side is another argument in favor of the gauge theory scenario.

While it is the GL theory (1) that provides a realistic description of fluctuation behavior in extreme type-II superconductors, many numerical studies are performed on the 3D XY model. The ultimate low-field critical behavior should be the same and computational effort is much reduced. It is therefore useful to discuss here the physical meaning of the ''high''- and ''low''-field regimes in the context of the frustrated 3D XY model. There is an immediate difference between this model and the GL theory (1) regarding the high-field behavior. In the GL theory this regime is dominated by Landau levels and is characterized by strong amplitude fluctuations. In contrast, in the 3D XY model, the amplitude fluctuations are frozen at the ''microscopic'' level of a single XY spin. As a result, there is no Landau level formation in this model. Instead, the high-field behavior of a uniformly frustrated 3D XY model, as one approaches the ''mean-field'' $H_{c2}(0)$, is entirely determined by the pinning of field-induced vortices by the underlying lattice. We can think of this situation, to some extent, as having the LL structure of

Eq. (7) thoroughly ''mixed'' by a very strong external periodic potential. There is, however, a relationship between the low-field critical behavior of the 3D XY model and GL theory. It derives from our concept of ''screening'' of large thermally generated vortex loops by the background of field-induced vorticity. In the weakly frustrated 3D XY model such ''screening'' is measured by a parameter κ_{sXY}, which is the XY model counterpart of κ_s in the GL theory (5):

$$\kappa_{sXY}^2 \sim \sqrt{\frac{f^T(T,H)}{f_\Phi}} \propto \frac{1}{\sqrt{H}}, \tag{9}$$

where f_Φ measures the uniform frustration and is the fraction of the elementary flux quantum ϕ_0 per plaquette, while $f^T(T,H)$ is the average density per plaquette of vortex and antivortex segments $\parallel$ **H** piercing the xy plane. Note that $f^T(T,H)$ includes *all* such vortex segments, not just those connected to infinite vortex loops. In order for the system to be in the low-field critical regime of a weakly frustrated 3D XY model we need $\kappa_{sXY} \gg 1$ or $f_\Phi \ll f^T(T,H)$.

The actual value of H_s (or H_b) in high-temperature superconductors is of considerable importance. There are numerous estimates in the literature, based both on Eq. (8) and on the analysis of various experimentally measured quantities in terms of either the GL-LLL theory or the so-called ''3D XY scaling'' (see Sec. IV). A direct estimate from Eq. (8), in a moderately anisotropic HTS system like optimally doped YBCO, uses Gi≈ 0.01 and $H_{c2}^{GL}(0) \approx 160$ T, leading to $H_s \sim 1-2$ T. This estimate is subject to an irksome uncertainty, both intrinsic (due to our inability to theoretically determine H_s or H_b with a precision better than within factors of order unity) and extrinsic [due to difficulties in extracting precise values of the GL parameters entering Eq. (1), although the situation here is rapidly improving[1]]. The estimates of H_b based on the fits of fluctuation thermodynamics to the GL-LLL theory are in general agreement with the above value of 1–2 T (Ref. 1) and seem to give an upper limit $H_b < 8$ T.[42] Similar analyses, based on the fits to a low-field ''3D XY scaling,'' generally produce results which seem consistent with the 3D XY-like behavior to much higher fields, 14 T or even higher.[5,6,43] An important difference between the two approaches is that, within the GL-LLL theory, not only the scaling law but the scaling function and explicit expressions for thermodynamic quantities are known with considerable accuracy.[16,1,19] In the 3D XY approach only the scaling law itself is known but the actual scaling function and, more importantly, the physics behind it are not. The gauge theory scenario should help remedy this situation.

IV. CRITICAL THERMODYNAMICS AND Φ TRANSITION

This brings us to the second important consequence of description (4), which has bearing on the nature of critical behavior in the low-field ($H \ll H_s$), extreme ''type-II'' limit of the gauge theory. The most significant property in this regime is that, for $\tilde{e}$ small but *finite*, there is a true thermodynamic phase transition separating the high- and low-temperature phases of the theory, the ''normal'' and the ''Meissner'' state, respectively. For $\tilde{e}=0$ this is the standard $H=0$ phase transition of Ginzburg-Landau theory. This phase transition is continuous and in the universality class of the 3D XY model. The actual mechanism of the phase transition is directly tied to the expansion of thermally generated vortex loops, as depicted in Fig. 2. In the ordered state below T_{c0}, there is a *finite* average size for such loops, Λ_Φ, and configurations which contain infinite loops, ''percolating'' from one end of the system to another, do not contribute to the partition function in the thermodynamic limit. At distances much larger than Λ_Φ, there is nothing to disturb the long-range correlations in $\langle \Phi(0)\Phi^*(\mathbf{r})\rangle$: it is not possible to ''polarize'' closed loops at such large distances and they behave as bound ''dipoles.'' This is what enables the long-range phase order that characterizes the superconducting state. Above T_{c0}, as more and more vortex segments are created by thermal excitation, the loops connect, in the sense that now there is a finite contribution to the partition function from configurations having infinite loops, ''percolating'' across the system. This implies that $\Lambda_\Phi \to \infty$ and it is now possible to ''polarize'' the system of loops over arbitrary large distances. Such infinite vortex loops act as ''free charges'' and produce a ''metallic screening'' of small external magnetic fields, resulting in a vanishing of the helicity modulus, as discussed in the next section. This picture of the 3D XY phase transition as a vortex loop ''expansion'' has its origins in the works by Onsager[23,44] and Feynman,[45] in the context of superfluid helium,[20] but should equally well apply to high-temperature superconductors with their short BCS coherence lengths and extremely large κ (~ 100).[46]

As finite $\tilde{e}$ [finite H in Eq. (1)] is turned on in Eq. (4) we are facing a potentially dramatic change in this picture. In the neutral-superfluid picture described previously, vortex loops have long range London-Biot-Savart interactions. Once $\tilde{e}$ is finite, these interactions are ''screened'' by the vector potential **S** and, at distances much longer than the ''penetration depth'' $\lambda_s \propto 1/\tilde{e}$, all the interactions are short ranged. The simplest and best known example of this is just an ordinary superconductor at *zero* external field. There $\tilde{e}$ is the real electrical charge e, while **S** turns into the ordinary Maxwell vector potential **A**. This charged-superfluid problem has been studied extensively, starting with Ref. 47, and is presently thought to have the following properties:[41] as already indicated in Sec. II, the charge e is a relevant operator in the RG sense, with scaling dimension equal to 1/2. This immediately destabilizes the neutral-superfluid, zero-charge 3D XY critical point. There are, however, two new critical points, characterized by *finite* charge. The behavior of strongly type-II superconductors ($\kappa \gg 1$) is determined by the stable critical point and describes the *continuous* phase transition between the normal state and the Meissner phase in real superconductors.[41] Another critical point is *tricritical* and unstable in one RG direction, in addition to temperature. This *tricritical* point defines the transition between type-II (small charge) and type-I (large charge) behavior and takes place for $\kappa \sim 0.4/\sqrt{2}$.[41] In a type-I superconductor, the phase transition is expected to be *discontinuous*, as originally argued in Ref. 47. In the type-II regime, where the transition is continuous, the universality class for the charged-superfluid appears to be the ''inverted 3D XY,''[48,49] or very ''close'' to it[41,50] (see Appendix A for further details).

What is the connection between these general properties of the charged-superfluid model and our problem? It stems from the gauge theory (4). This theory looks just like the theory for a charged superfluid, except for the charge anisotropy, which should be irrelevant.[39,40] The underlying physics, of course, is very different. There is no fluctuating electrodynamic vector potential in our case, since we are in the $\kappa\to\infty$ limit. Instead, our fictitious vector potential $\mathbf{S}$ describes the long-wavelength vorticity fluctuations in the background s vortex system and our ''charge'' $\tilde{e}$ is the original magnetic field H in disguise ($\tilde{e}^2\propto\sqrt{H}$). Despite this difference in physical meaning, the long-distance behavior of Eq. (4) should still be closely related to the electrodynamics of a charged superfluid. In particular, we expect two different thermodynamic phases of Eq. (4): the high-temperature phase with only short-range correlations in $\langle\Phi(0)\Phi^*(\mathbf{r})\rangle$ ($\langle\Phi\rangle=0$) and the low-temperature phase, in which $\langle\Phi(0)\Phi^*(\mathbf{r})\rangle$ develops long-range order ($\langle\Phi\rangle\neq 0$). The ''Meissner phase'' (or the Φ-ordered state) of Eq. (4) corresponds to the state of the original GL theory (1) in which only N_Φ field-induced vortex lines cross the system from one end to another along $\mathbf{H}$. All other vortex excitations form either closed thermally generated loops of *finite* size or *finite* ''overhang'' configurations decorating field-induced lines as they make their way meandering from bottom to top of the sample. These field-induced vortex lines, or s vortices in reformulation (2), have a finite line tension *relative* to the field direction and undergo effective ''diffusion'' along the z axis (this is discussed in greater detail in the next section). In the high-temperature, ''normal metal'' phase of Eq. (4), the Φ order is destroyed by the expansion of thermally generated vortex loops and ''overhangs'' decorating s (field-induced) vortices. We now have new, thermally generated infinite loops ''percolating'' all the way through the system in all directions. These new infinite loops come on *top* of the always present background of N_Φ s vortices. This is the nature of the Φ transition[9] in the gauge theory (4) and in reformulation (2). The Φ transition is the *finite*-field version of the zero-field superconducting transition.[9] Its thermodynamics, however, belong to a *different* universality class: charged superfluid (''inverted 3D XY'') as opposed to neutral superfluid (3D XY) at $H=0$, with finite H playing the role of finite charge ($\tilde{e}^2\propto\sqrt{H}$) in the gauge theory (4).

The Φ-transition line $T_\Phi(H)$ plays a pivotal role in the gauge theory scenario. Since, on general grounds,[51] we do not expect any true criticality associated with the first-order vortex lattice melting line in 3D, $T_\Phi(H)$ is the *only* critical line in the H-T phase diagram of the original GL theory Eq. (1) and controls fluctuation thermodynamics and transport at weak magnetic fields. It decides the issues of relevance or irrelevance of various terms that can be added to (1) (point or columnar disorder, true electromagnetic screening with finite κ, etc.) and provides a foundation on which one can build a meaningful phenomenology of extreme type-II superconductors. In this respect, $\Phi(\mathbf{r})$, the new ''superconducting'' order parameter characterizing the ''line liquid'' state, is ''more fundamental'' than the original $\Psi(\mathbf{r})$. This will now be amply illustrated.

To start building such a phenomenology, we first need a reasonable estimate of $T_\Phi(H)$ (Fig. 1). It starts at the zero-field superconducting transition T_{c0}, where $\Phi(\mathbf{r})$ and $\Psi(\mathbf{r})$ are one and the same: the $H\to 0$ limit of Eq. (1) coincides with the $N_\Phi\to 0$ limit in Eq. (2) and with the $\tilde{e}\to 0$ limit of the gauge theory (4). At finite, but weak field, we are in the ''extreme type-II'' regime ($\kappa_s\gg 1$) of the gauge theory (4) and we expect that the Φ transition is continuous and immediately becomes ''inverted.'' The transition temperature $T_\Phi(H)$ is gradually reduced as a function of H [or, equivalently, $\tilde{e}$ in Eq. (4)], due primarily to the reduction in the effective line tension of very large ($\gg l$) vortex loops caused by ''screening'' generated by the ''medium'' of the field-induced vorticity. At these low fields, $T_\Phi(H)$ can be evaluated directly from Eq. (4). As H increases, however, numerous additional terms present in Eq. (2), but not included in the gauge theory (4) on the grounds of their RG irrelevance at long distances ($\gg l$), start affecting $T_\Phi(H)$. Among such terms none are more important than short-distance ($\sim l$) positional correlations which eventually lead to s vortex lattice formation at low temperatures.

In general, the Φ transition and vortex lattice melting are two completely different phase transitions, with two different order parameters, driven by two different mechanisms. One is a $q\to 0$, another a $q\sim 1/l$ transition. They are not entirely unrelated, however, since they arise in the same theory, Eq. (1) or (2). For instance, as $H\to 0$, we must have $T_\Phi(H)\geq T_m(H)$.[52] This is so because only in the Φ-ordered state do s vortices in Eq. (2) have finite long-range interactions, $\propto|\langle\Phi\rangle|^2$.[53] Without such long-range interactions the s vortex system would remain in a liquid state as $H\to 0$.[8] Similarly, in the solid phase, s vortices form a lattice and cannot screen large thermally generated vortex loops; i.e., $\tilde{e}$ becomes effectively zero even for $H\neq 0$. All vortex loops will then remain small and bound, just as they were at $H=0$. This is discussed in more detail in Sec. VII. The problem is that the melting transition is *always first order* and thus, in principle, we could have $T_\Phi(H)=T_m(H)$ at some or even *all* H. This would mean that melting is so strongly discontinuous that it always ''jumps'' over the intermediate, Φ-ordered phase, straight into the true normal state. Furthermore, the Φ transition itself could become first order[9] at all H, due to dangerously irrelevant terms not included in Eq. (4) but considered in Appendix A. I know of no argument to rule out this possibility.

This being said, the most likely outcome is the one depicted in Fig. 1. At higher fields, as we approach the ''type-I'' regime of Eq. (4) [$H\sim H_s$, Eq. (6)], the gauge theory suggests that the Φ transition itself *converts* to *first-order*. In this situation, it seems justified to assume that $T_\Phi(H)=T_m(H)$, as shown in Fig. 1. For low fields, $H\ll H_s$, where the melting transition becomes *weakly* first order and Eq. (4) predicts a strong ''type-II'' behavior and *continuous* Φ transition, it is natural to expect $T_\Phi(H)>T_m(H)$. At fixed low field, as we increase the temperature in Fig. 1, *both* the effective strength of the Biot-Savart interaction between s vortices (Sec. VII) and their effective ''mass'' (Sec. V) *decrease*. As interactions and line tension go down, a natural progression of thermodynamic phases follows: a solid (Abrikosov lattice), a ''massive'' liquid (Φ-ordered phase or ''line liquid''), and, finally, a ''massless'' gas of unbound loops (a true normal state). A mean-field calculation, performed in Ref. 9, indeed leads to such results. I propose here a simple criterion which summarizes the results of such calculations and can be used to determine $T_\Phi(H)$ and $T_m(H)$ at low fields, $H \ll H_s$: the *s* vortex lattice melts when the average size of thermally generated loops, $\Lambda_\Phi(T,H)$, reaches a fraction d_m of the average distance between field-induced *(s)* vortices: $\Lambda_\Phi(T,H) = d_m\sqrt{2\pi}(\xi_\parallel/\xi_\perp)^{1/3}l$. $d_m \sim 0.2$–0.3 and $\Lambda_\Phi(T,H) \approx \Lambda_\Phi(T,0) \sim (\xi_\perp^2\xi_\parallel)^{1/3}$ seem a reasonable estimate. Here $\xi_{\perp,\parallel} = \xi_{0\perp,0\parallel}|t|^{-\nu}$ are the true superconducting correlation lengths at $H=0$. This results in an expression for the vortex lattice transition temperature in the critical region:

$$t_m(h) = -\frac{1}{(d_m\sqrt{2\pi})^{3/2}}\left(\frac{\xi_{0\perp}}{\xi_{GL,\perp}}\right)^{3/2} h^{3/4}, \tag{10}$$

where the temperature is measured relative to the *true* zero-field superconducting transition, $t = (T/T_{c0}) - 1$, h is defined below Eq. (7) and ν_{xy} was set to 2/3. The ratio $\xi_{0\perp}/\xi_{GL,\perp}$ should be ~ 1. As argued above, we expect $d_m \sim 0.2$–0.3. The Φ transition, on the other hand, takes place when the size of thermally generated loops, at *finite H*, reaches the sample dimensions, $\Lambda_\Phi(T,H) \to \infty$. This should take place along the line where the average loop size, for $H=0$, $\Lambda_\Phi(T,0)$, becomes of the order of average distance between *s* vortices, i.e., $\Lambda_\Phi(T,0) = d_\Phi\sqrt{2\pi}(\xi_\parallel/\xi_\perp)^{1/3}l$, with $d_\Phi \sim 1$. This determines the vortex loop ''expansion'' line or $T_\Phi(H)$:

$$t_\Phi(h) = -\frac{1}{(d_\Phi\sqrt{2\pi})^{3/2}}\left(\frac{\xi_{0\perp}}{\xi_{GL,\perp}}\right)^{3/2} h^{3/4}, \tag{11}$$

with $d_\Phi \sim 1$.[54] Obviously, $T_\Phi(H) > T_m(H)$ since $d_m < d_\Phi$. Equations (10) and (11) are valid only in the limit of low fields, $H \ll H_s$, Eq. (6). At higher fields $H \sim H_s$, $T_\Phi(H)$ and $T_m(H)$ merge together and both vortex loop ''expansion'' and vortex lattice melting occur *simultaneously* when $\Lambda_\Phi(T,H)$ reaches $\sim \sqrt{2\pi}l$ from within the solid phase (Fig. 1). The above expressions Eqs. (10) and (11), with d_m and d_Φ serving as numerical parameters, can be viewed as a ''Lindemann criterion'' for vortex loops and should provide good estimates of $T_\Phi(H)$ and $T_m(H)$.

I now proceed to further investigate the phase diagram represented by Fig. 1 and Eqs. (10) and (11). As the field is turned on in Eq. (1) we can immediately write down the scaling expression for the dimensionless singular part of the free energy, f, associated with critical fluctuations:

$$f = |t|^{2-\alpha}\phi_\pm\left(\frac{H}{H_k|t|^\Delta}\right), \tag{12}$$

where $t = (T/T_{c0}) - 1$ and H_k depends on material parameters. This expression is completely general and as such conveys little information. It is based only on the existence of the *zero-field* critical point. The same expression can be written for spin systems or any other system exhibiting a critical point which is then perturbed by a generalized ''field.''[55] In our case, the $H=0$ critical point is in the 3D *XY* universality class and we should have $\alpha = \alpha_{xy}$. Furthermore, based on dimensional analysis and general physical arguments, it was proposed in Ref. 37 that $\Delta = 2\nu_{xy}$. This result holds to two-loop order in the RG (Ref. 38) and is likely exact, as emphasized in Sec. II. $\phi_\pm(H/H_k|t|^\Delta)$ is a universal function of its argument inside the 3D *XY* critical region of GL theory. At present, its form is not known. Note that Eq. (12), while completely general, is written in a form which implicitly suggests that the finite-field critical behavior is governed by the zero-field critical *point*, as is frequently the case in spin systems.

In the gauge theory scenario, the situation is different and we can be more specific. First, as already emphasized, within this scenario the critical fluctuations are governed by a transition *line* and not a critical *point*.[9] This means that we immediately learn something about the function $\phi_\pm$ defined in Eq. (12): $\phi_\pm$ is *nonanalytic* along the Φ-transition line $T_\Phi(H)$, Eq. (11). This line singularity should be explicitly incorporated into the expression for the free energy. To devise such a new scaling function, based on the gauge theory scenario, we start by observing that we can eliminate the ''trivial'' part of the charge anisotropy from Eq. (4) by rescaling all lengths and fictitious vector potential **S** with an appropriate superconducting correlation length $\xi_{\perp,\parallel}$ in a way that makes the Φ-dependent part of Eq. (4) isotropic. This rescaling procedure is a variation on the familiar rescaling of anisotropy at the $H=0$ transition. After the rescaling, the Φ-dependent part of Eq. (4) describes an isotropic superconductor with a correlation length $\xi = (\xi_\perp^2\xi_\parallel)^{1/3}$, while the coupling constants in the last two terms become

$$K_{\perp,\parallel} \to K'_{\perp,\parallel} = c_{\perp,\parallel}(\xi_\parallel/\xi_\perp)^{1/3}Tl. \tag{13}$$

The following quantities appear in Eq. (13): $\xi_{\perp,\parallel} = \xi_{0\perp,\parallel}|t|^{-\nu}$ and $\xi = (\xi_\perp^2\xi_\parallel)^{1/3} = \xi_0|t|^{-\nu}$ are the true diverging superconducting correlation lengths at T_{c0}, defined by the eigenvalues of the helicity modulus tensor (see Appendix B). Accordingly, $\Gamma = \xi_\perp/\xi_\parallel$ is the *true* anisotropy ratio at the $H=0$ critical point (T_{c0}) and *not* the GL anisotropy, $\Gamma_{GL} = \xi_{GL,\perp}/\xi_{GL,\parallel}$. It now becomes clear why $K_{\perp,\parallel}$ have been defined in Eq. (4) with the anisotropy Γ explicitly factored out: new rescaled coupling constants can simply be written as: $K'_{\perp,\parallel} = c_{\perp,\parallel}T\bar{l}$, where

$$\bar{l} = \frac{l}{\Gamma^{1/3}} \text{ corresponds to } \bar{H} = \Gamma^{2/3}H. \tag{14}$$

$\bar{H}$ is just the rescaled magnetic field appearing in the original GL theory (1) *after* the anisotropy at the $H=0$ critical point has been rescaled out. Consequently, $c_{\perp,\parallel}$ describe the *fundamental* anisotropy of the gauge theory (4), which is inherent to the $H \neq 0$ problem and is not associated with a ''trivial'' anisotropy at T_{c0}.[56,57] The corresponding fictitious ''charges'' associated with $K'_{\perp,\parallel}$ are

$$\tilde{e}^2_{\perp,\parallel}(T,H) = \frac{1}{c_{\perp,\parallel}(T,H)\bar{l}}. \tag{15}$$

The product of the above rescaling procedure is a fictitious anisotropic electrodynamics with two ''charges'' $\tilde{e}_\perp$ and $\tilde{e}_\parallel$. As discussed in Sec. II, the charge is a relevant operator at the $H=0$ ($\tilde{e}=0$) critical point, with scaling dimension equal to 1/2. We thus define two dimensionless scaling variables

$$\tilde{e}^2_{\perp,\|}\xi = \frac{\xi}{c_{\perp,\|}\bar{l}} = \sqrt{\frac{2\pi}{\phi_0}}\frac{\xi_0\Gamma^{1/3}}{c_{\perp,\|}}\frac{H^{1/2}}{|t|^{\nu}}. \tag{16}$$

Note that dimensionless ratio $\xi/\bar{l}$ is common to both charges and (nontrivial) anisotropy is stored in $c_\perp$ and $c_\|$. $c_\perp$ and $c_\|$, however, are also functions of $\xi/\bar{l}$ *only*. Therefore, there is only a *single* relevant scaling variable, the dimensionless charge

$$q_0^2 = \frac{\xi}{\bar{l}} = \sqrt{\frac{2\pi}{\phi_0}}\xi_0\Gamma^{1/3}\frac{H^{1/2}}{|t|^{\nu}}, \tag{17}$$

which is precisely the original scaling variable[37] of the GL theory (1), since $\xi/\bar{l} = \xi_\perp/l$. The functions $c_{\perp,\|}(\xi/\bar{l})$ are discussed further in the next section and Appendix B.

We are now in a position to write down the scaling function for the free energy within the gauge theory scenario, with the nonanalytic part associated with the Φ transition explicitly factored out:

$$f_s = |t - t_\Phi(h)|^{2-\alpha}\Omega^{L,S}_{\pm}\left(\frac{t - t_\Phi(h)}{|t_\Phi(h)|}\right). \tag{18}$$

Note that $t_\Phi(h)$, defined in Eq. (11), also follows from $q_0^2 = \sqrt{2\pi}d_\Phi$. d_Φ is therefore a universal number of the GL theory (1), as is d_m.[58] $\Omega^{L,S}_{\pm}(x)$ is a universal and *regular* function of its argument. The subscripts $\pm$ refer to $x>0$ ($x<0$), while the superscripts L and S indicate the ''vortex liquid'' and ''vortex solid'' branches of Ω, respectively. For example, below $T_m(H)$, we should use $\Omega^S_-(x)$. In writing down Eq. (18) I have assumed that the correlation length exponent of the gauge theory $\nu_{GT}\sim\nu_{xy}\sim 2/3$ and that the hyperscaling relation holds, resulting in $\alpha\sim\alpha_{xy}$.

How do we evaluate the crossover function $\Omega(x)$? I alert the reader to the following important point: the gauge theory scenario explored in this paper allows one to, *in principle*, determine all the branches of the crossover function $\Omega(x)$ by using a combination of perturbation theory and RG techniques. Such an analytic calculation is extremely laborious and far beyond the scope of this paper. A well-informed reader will immediately realize that many aspects of this calculation are computationally extremely demanding, and actually have *not* been accomplished in the published literature even for the ordinary $H=0$ situation. Indeed, the technical difficulties involved are of the same general nature. This, however, does not detract from the main message of this section: the underlying physical picture of the gauge theory scenario provides a systematic, conceptually straightforward way to compute the $H\neq 0$ 3D XY critical thermodynamics at the *same level* of analytical accuracy as is presently feasible for the $H=0$ case.

Faced with such odds, I assume, for the purposes of this paper, that $\Omega(x)$ in Eq. (18) is some unknown universal crossover function, to be determined either from numerical simulations[23] or directly from experiments. With the free energy thus specified, we can proceed to evaluate the *singular* part of all thermodynamic functions, simply by taking requisite derivatives.[43]

V. HELICITY MODULUS, LINE ''DIFFUSION,'' TOPOLOGICAL WINDINGS, AND PHYSICAL NATURE OF Φ ORDER

I now turn to physical properties which allow a more direct look at actual configurations of loops and lines that characterize the state of a superconductor above and below $T_\Phi(H)$. A useful measure of a degree of superconducting order is a $\mathbf{q}$-dependent helicity modulus tensor $\Upsilon(\mathbf{q})$ whose components are defined as[30]

$$\Upsilon_{\mu\nu}(\mathbf{q}) = V\frac{\delta^2 F}{\delta\mathbf{a}_\nu(\mathbf{q})\,\delta\mathbf{a}_\mu(-\mathbf{q})}, \tag{19}$$

where $\mu,\nu = x,y,z$, V is the total volume, F is the free energy of the GL theory (1), and $\mathbf{a}(\mathbf{r})$ is a small (infinitesimal) vector potential added to the external $\mathbf{A}$. The uniform component of the associated magnetic field, $\mathbf{h}(\mathbf{r}) = \nabla\times\mathbf{a}$, vanishes. The above second derivative is evaluated in the $\mathbf{a}\to 0$ limit.

$\Upsilon(\mathbf{q})$ measures the ability of a system to ''screen'' out tiny external fields. In the *superconducting* phase $\lim_{\mathbf{q}\to 0}\Upsilon(\mathbf{q})$ is finite and the system is said to exhibit a differential Meissner effect. In the *normal* phase, $\Upsilon(\mathbf{q})\sim q^2$ and vanishes in the $q\to 0$ limit. Within our ''helium model,'' the way Υ is reduced to zero in the long-wavelength limit is through proliferation of *infinite* vortex loops and lines which go all the way across the system and can act as ''free charges,'' screening a weak external perturbation. In this intuitive sense, we can think of a normal state as a vortex ''metal,'' while the superconducting state is a vortex ''dielectric,'' with only vortex loops of *finite* size present as thermal excitations.

To compute the helicity modulus of our original GL theory one adds an infinitesimal $\mathbf{a}_\mu$ to $\mathbf{A}_\mu$ in Eq. (1). If we now go to reformulation (2) and finally, through the coarse-graining procedure of Appendix A, end up with our fictitious gauge theory, $\mathbf{a}_\mu$ appears as a small addition to the ''vector potential'' $\mathbf{S}_\mu$ in the second (gradient) term of Eq. (4). This implies that the long-wavelength ($q\ll 1/l$) form of the helicity modulus of the gauge theory (4) coincides with that of the original GL theory (19). Using the gauge theory (4) and ignoring the anisotropy, we obtain that, *below* $T_\Phi(H)$ (see Appendix B for details),

$$\Upsilon(\mathbf{q}) = Kq^2 - \frac{K^2q^4}{T}\langle\mathbf{S}(\mathbf{q})\cdot\mathbf{S}(-\mathbf{q})\rangle = Kq^2 + \mathcal{O}(q^4). \tag{20}$$

In the Φ-ordered state our fictitious gauge field is ''massive,'' i.e., exhibits a Meissner effect, and $\lim_{q\to 0}\langle\mathbf{S}(\mathbf{q})\cdot\mathbf{S}(-\mathbf{q})\rangle\propto|\langle\Phi\rangle|^{-2}$ goes to a *finite* value. The simple physics behind this is that thermally generated vortex loops in $\Phi(\mathbf{r})$ have an average size that is *finite* and do not contribute at all to $\Upsilon(\mathbf{q})$ in the $q\to 0$ limit. Furthermore, Eq. (20) tells us that, if $K_{\perp,\|}$ are *finite*, the Φ-ordered phase is *not* a superconductor and has a *finite* superconducting correlation length, both perpendicular and parallel to the external field (see Appendix B for details),

$$\xi_\|\sim K_\perp/T\sim c_\perp l,\qquad \xi_\perp^2/\xi_\|\sim K_\|/T\sim c_\| l. \tag{21}$$

This result is easily understood: with H finite, there now must be N_Φ field-induced vortex lines moving about in the sample. Unless these (s) vortices are pinned down, as happens in the vortex-solid phase, they will be available to ''screen'' weak (infinitesimal) external fields and the system is always a vortex ''metal'' with the ''screening length'' K/T. In particular, according to our assumption (ii), the system of field-induced vortex lines also contains windings in the xy plane and such ''screening,'' although anisotropic, is finite in all directions. Below $T_\Phi(H)$, where s vortices are exclusively responsible for the vanishing $\lim_{q\to 0}\Upsilon_\mu(\mathbf{q})$, $c_{\perp,\parallel}(q_0^2)$ determine the $\perp$ and $\parallel$ ''screening lengths'' in the s vortex system, in units of magnetic length l [Eq. (B9) in Appendix B].

Above $T_\Phi(H)$ the situation changes and thermally generated vortex loops ''expand'' across the system. Obviously, the helicity modulus still vanishes, but now there is an abrupt drop in the coefficient of the q^2 term:

$$\Upsilon(\mathbf{q})=\left(K-G\frac{K^2}{T\xi_\Phi}\right)q^2=K\left(1-C\frac{\tau^{\nu_{GT}}}{\sqrt{H}}\right)q^2, \qquad (22)$$

where $\tau(T,H)=[T-T_\Phi(H)]/T_{c0}$ and C and G are numerical constants. ν_{GT} is the thermodynamic exponent of the Meissner transition in our fictitious electrodynamics (4), and $\nu_{GT}\sim\nu_{xy}\sim 2/3$, as argued in Sec. IV. The second term in the above equation arises from $\lim_{q\to 0}\langle\mathbf{S}(\mathbf{q})\cdot\mathbf{S}(-\mathbf{q})\rangle=G/\xi_\Phi q^2$ in Eq. (20), right above $T_\Phi(H)$. This implies that the original superconducting correlations, measured by $\langle\Psi(0)\Psi^*(\mathbf{r})\rangle$, remain finite in all directions on *both* sides of $T_\Phi(H)$, but there is a *nonanalytic* drop in the superconducting correlation length at the Φ transition, as thermally generated loops proliferate through the system and additional infinite vortices become ''free charges'' and available to screen. The new order parameter $\Phi(\mathbf{r})$, however, *does* attain a true long-range order below $T_\Phi(H)$; i.e., $\xi_\Phi\to\infty$ as $T\to T_\Phi(H)$ from above. It is unfortunately rather difficult to measure the Φ correlations directly, by probing some suitably defined ''helicity modulus'' associated with the Φ order. This would require defining quantities which are configuration dependent and highly nonlocal, a rather time-consuming proposition in a typical numerical simulation of a 3D XY or related model.

Still, the situation is far from hopeless. We can devise another set of criteria that are relatively easy to implement in numerical simulations and yet allow for a rather intimate look at the Φ order and what precisely takes place as we cross the Φ-transition line. Below $T_\Phi(H)$, thermally generated vortex loops are bound and field-induced vortices execute an effective ''diffusive'' motion along the field direction. An average transverse displacement of a single field-induced vortex line from the point where it starts at $z=0$ to its ending point at $z=L_z$ goes as

$$\sqrt{\langle r_\perp^2\rangle}\sim\sqrt{D_s}L_z^p, \quad p\cong\frac{1}{2}, \qquad (23)$$

where D_s is the effective ''diffusion'' constant. This is shown in Fig. 3. The cutting and reconnecting of vortex lines does not affect this diffusion process except by renormalizing D_s, as long as we are in the Φ-ordered phase.[59] For example, in the 3D XY model, where the identification of an individual field-induced line is not unique, we should simply look at the *distribution* of distinguishable vortex paths, obtained by *randomly* resolving all the crossings, and average over all distinct configurations. Such a distribution will be ''diffusive,'' with the average rms displacement given by Eq. (23). We can use this effective ''diffusion'' constant D_s, Eq. (23), to define an *effective* ''mass'' in the elegant nonrelativistic boson analogy of Nelson.[8] Note that the world lines of such flux bosons do *not* correspond to (s) vortex lines in individual configurations of an extreme type-II superconductor. This is clear since nonrelativistic bosons describe *strictly directed* lines, i.e., contain no ''overhang'' configurations as they advance from bottom to top of the system along the z axis (the ''time'' axis in the boson analogy). Such ''overhang'' configurations, plus numerous vortex loop excitations floating around, describe world lines of ''particle-antiparticle'' creation processes and cannot be accommodated within the nonrelativistic quantum boson analogy. Still, as long as we are in the Φ-ordered phase, it is only the N_Φ field-induced vortex lines that go all the way across the system. We can then define an *effective* system of N_Φ flux bosons in the boson analogy, with suitably adjusted bare mass m_s and effective interactions, so that its *long-distance* [$\gg l$ and $\gg\Lambda_\Phi(T,H)$] behavior faithfully emulates an extreme type-II superconductor (Appendix A). Above $T_\Phi(H)$, as infinite tangles of field-induced and thermally generated vortices proliferate across the sample in *all* directions, $D_s\to\infty$ ($m_s\to 0$) and we get *hyperdiffusion*

$$\sqrt{\langle r_\perp^2\rangle}\sim L_z^{p'}, \quad p'\sim 1. \qquad (24)$$

This hyperdiffusion arises through processes depicted in Fig. 3, where a vortex line winding along the field direction *simultaneously* winds all the way in the xy plane by connecting itself to thermally generated tangles, which are naturally present in the Φ-disordered phase. In the 3D XY model, this implies that the distribution of transverse displacements of individual field-induced vortex paths is no longer ''diffusive'' and has rms displacement or higher moments limited only by the system size (24); more precisely, the distribution of $r_\perp^2$ acquires a power-law tail above $T_\Phi(H)$. Such windings in the xy plane are plainly in evidence in the recent numerical simulations of Nguyen and Sudbø,[23] somewhat above their melting line. With such additional xy windings present with a finite weight in the partition function, the ''effective mass'' m_s of nonrelativistic flux bosons *vanishes* since the vortex line tension *relative* to the field direction has gone to zero. Above $T_\Phi(H)$, infinite vortex paths longer than L (say $\sim L^2$, assuming $L_\perp=L_\parallel=L$ and a simple random walk) crossing the system in all directions contain a *finite* fraction of all vortex segments: these are the ''massless excitations.'' In this respect, the Φ transition corresponds to the restoration of ''relativistic invariance'' in a dual system of quantum particles whose world lines are our original vortex loops and lines. To wit, the ground state of such a quantum system, containing only ''vortex matter''[60] below $T_\Phi(H)$, explodes with ''vortex matter,'' ''vortex antimatter,'' and ''vortex tachyons''[61] (Fig. 3), as the ''vacuum'' becomes unstable at $T_\Phi(H)$ to the spontaneous creation of ''particles'' (Appendix A).

The above connection between the ''Φ order'' and the effective line tension of field-induced *(s)* vortex lines reveals directly the physical content of the gauge theory (4) and permits us to construct a purely geometrical picture of the Φ transition. To do so, consider once again the normal state of an extreme type-II superconductor or a 3D XY model at $H=0$. Above T_{c0}, we can find vortex paths that go all the way from one end of a sample to another, in any direction (Fig. 2). If we work with periodic boundary conditions in all directions, this statement means that we have some windings along the x, y, and z axes. To understand what is precisely meant by such windings, we wrap our system on a three-dimensional generalized torus, embedded in a four-dimensional space—this is just a geometrical way of representing periodic boundary conditions. Now, the number of windings along, say, the x axis, $\mathcal{N}_x$, is the *total* number of continuous vortex paths in the whole system that wind all around the torus in the x direction, *irrespective* of their orientation, i.e., whether they are ''vortex'' or ''antivortex'' paths relative to the x axis. Such paths are topologically distinct from finite closed vortex loops: the latter can be continuously shrunk to a point while the former cannot. $\mathcal{N}_x$ is different from the winding number W_x: W_x also counts all the windings along the x axis but with a single ''vortex'' path contributing $+1$ while an ''antivortex'' path counts as -1. In the widely recognized vocabulary of the 2D XY model, $\mathcal{N}_x$ would correspond to the total number of vortices *plus* antivortices in the yz plane, while W_x would be the total number of vortices *minus* the total number of antivortices. Back in 3D, in the superconducting state below T_{c0}, all thermally generated vortices come in the form of finite closed loops and both $\mathcal{N}_x=0$ and $W_x=0$. Above T_{c0}, W_x must remain equal to zero due to the ''vortex neutrality'' of the GL theory (1) or the 3D XY model, but $\mathcal{N}_x$ is now finite and $\mathcal{N}_x \propto L^{1+u}$, where L is the linear size of the system (we are assuming $L_\perp = L_\parallel = L$) and u is the ''anomalous'' dimension of such infinite paths. The same holds for the winding number and the total number of windings along the y and z axes, $W_{y(z)}=0$ and $\mathcal{N}_{y(z)} \propto L^{1+u}$. This can be summarized as

$$\mathcal{N}_{x,y,z}=0 \quad \text{for} \quad T<T_{c0},$$

$$\mathcal{N}_{x,y,z}=2n^T_{x,y,z}L^{1+u} \quad \text{for} \quad T>T_{c0}, \tag{25}$$

where n^T_x is the ''density'' in the yz plane of thermally generated infinite vortex-antivortex winding paths traversing the system along the x axis and so on. In the isotropic case $n^T_x = n^T_y = n^T_z$. Of course, the presence of such windings in all directions is the reason why the material is not in the superconducting state above T_{c0}: these infinite vortex paths can now move to ''screen'' weak external fields, driving the helicity modulus to zero in the long-wavelength limit and producing finite dissipation.

As we turn on a finite field in Eq. (1), we still have $W_{x(y)}=0$ but $W_z=N_\Phi$ and consequently $\mathcal{N}_z$ must be at least N_Φ in every configuration of the system.[9] Imagine now how the state of the system evolves along a small circle in the H-T phase diagram (Fig. 1), surrounding T_{c0}. Our circular path starts at $H=0$ and at some temperature T slightly *above* T_{c0}, and evolves in the counter clockwise direction toward its end point at $H=0$ and some temperature slightly *below* T_{c0}. Initially, we are very close to the $H=0$ normal state and it is safe to assume that $N_\Phi \ll \mathcal{N}_z(T,H=0)L^{2-u}$ at some temperature T slightly above T_{c0}. In this case it is natural to expect that, in a finite field,

$$\mathcal{N}_{x,y}=2n^T_{x,y}L^{1+u_\perp}, \quad \mathcal{N}_z=n_\Phi L^2+2n^T_z L^{1+u_\parallel}, \tag{26}$$

where $n_\Phi = N_\Phi/L^2 = 1/2\pi l^2$ is the density of field-induced *(s)* vortex lines and $n^T_{x,y,z}(T,H)$ are ''densities'' of thermally generated vortex-antivortex windings. Note that now $2n^T_{x,y} \neq n_\Phi + 2n^T_z$ and $n^T_{x,y} \neq n^T_z$ even in the isotropic case. There is a finite ''density'' of infinite vortex (or antivortex) winding paths in any direction. Equation (26) describes how the normal state of the system (25) has changed after the application of a finite, but weak external field.

On the opposite side of our imaginary circle, near its end point at $H=0$ and $T<T_{c0}$, the situation is completely different. Now, the zero-field state is a *superconductor* and $\mathcal{N}_{x,y,z}=0$. Any finite field, no matter how small, has a drastic effect. For very low fields, it is natural to expect that there are no thermally generated infinite vortex loops and only those windings associated with the field-induced vortex lines are present in the system:

$$\mathcal{N}_{x,y}=0, \quad \mathcal{N}_z=N_\Phi=n_\Phi L^2. \tag{27}$$

This is just the *(s)* vortex lattice state in Fig. 1. Note that in these general geometrical terms there is no difference between the *(s)* vortex lattice state and the ''anisotropic superconducting liquid'' of Feigel'man *et al.*[29] Due to the absence of windings in the xy plane, both have a superconducting response along the field direction; i.e., the $\Upsilon_{zz}(\mathbf{q})$ component is finite in the $q \to 0$ limit. Also, in both cases, $\Upsilon_{xx(yy)}(\mathbf{q})$ vanish as $q \to 0$. The only difference is that it takes arbitrary weak pinning to restore superconductivity in all directions in the vortex lattice state. On this basis, I have assumed in the phase diagram of Fig. 1 that such an ''anisotropic superconducting liquid'' phase is preempted by the first-order transition at $T_m(H)$.[62]

In our proposed phase diagram depicted in Fig. 1, the intermediate Φ-ordered phase is inserted between the true normal state (26) and the vortex lattice phase (27). What is its nature in simple geometrical terms used to describe the other two phases in Eqs. (26) and (27)? In the Φ-ordered state all thermally generated vortex loops are bound and only N_Φ field-induced *(s)* vortex lines go from one end of the sample to another, along $\mathbf{H}$, resulting in precisely N_Φ windings along the z axis. However, these *(s)* vortex lines are in a liquid state, characterized by some finite effective line tension $\mathcal{T}$ and they ''diffuse'' in the xy plane while winding along $\mathbf{H}$. This translates into a *nonvanishing* total number of windings in the xy plane:

$$\mathcal{N}_{x,y}=\mathcal{I}_{x,y}L^{2p}, \quad \mathcal{N}_z=N_\Phi=n_\Phi L^2. \tag{28}$$

We expect $1>p\cong 1/2$.[59] $\mathcal{I}_{x,y}$ are some finite quantities having dimension of $(\text{length})^{-2p}$ and we use Eq. (28) as their definition. Equation (28) describes the Φ-ordered state using a simple geometrical language of this section. The fact that $\mathcal{N}_{x,y}\neq 0$ has nothing to do with the thermal ''expansion'' of vortex loops; these are still all of finite size. Rather, it is due to the lateral ''diffusion'' of field-induced lines, as they wind along the z axis. The field-induced vortices tend to form infinite ''clusters,'' which manage to wind a *finite* number of times, $\mathcal{N}^c_{x,y}$, along $x(y)$ by winding *infinitely* many times, $\mathcal{N}^c_z \sim L$, along the field direction (I now set $p=1/2$). We can *define* a long-distance, effective ''diffusion'' constant $\mathcal{D}$ and the associated line tension $\mathcal{T}$ both relative to the field direction:

$$\mathcal{D}_{x,y}=\frac{\mathcal{N}^c_{x,y}L}{\mathcal{N}^c_z},\quad \mathcal{T}_{x,y}\sim \mathcal{D}^{-1}_{x,y}. \tag{29}$$

Defined in this fashion, $\mathcal{D}$ and $\mathcal{T}$ are truly global quantities, detached from complicated individual configurations of interacting vortex loops and lines and dependent only on thermodynamic state of the system as a whole. In the Φ-ordered state both are finite, while in the true normal state $\mathcal{D}\to\infty$ and $\mathcal{T}\to 0$ for some clusters. Infinite vortex paths inside such clusters manage to wind in the $x(y)$ direction after only a finite number of windings along the field. In this sense, $\mathcal{T}$ can serve as a probe for the presence or absence of the Φ order. Again, in the Φ-ordered state of the 3D XY model, where there is no unique identification of individual vortex paths due to their crossings, every configuration that contributes to the thermodynamic limit has precisely N_Φ vortex lines going from bottom to top in every distinguishable assignment of such paths. All other vortex paths form finite closed loops. The *distribution* of the total number of windings, obtained by randomly resolving all the crossings, still satisfies Eq. (28).

Once the system makes this phase transition to the normal state (26) and the Φ order is lost, there is only a smooth evolution. The ''densities'' of thermally generated windings $n^T_{x,y,z}$, as well as the way these windings are realized in individual configurations of vortex loops and lines, can change considerably, depending on where we are in the H-T phase diagram relative to $T_\Phi(H)$, but we do not expect to cross any additional phase boundaries.

The geometrical picture presented here, based only on global topological properties of loops and lines, gives a clear insight into two forms of ''superconducting'' order, described by two *different* order parameters $\Psi(\mathbf{r})$ and $\Phi(\mathbf{r})$. The familiar superconducting order, measured by $\Psi(\mathbf{r})$, reflects the system's ability to expel tiny external fields and is manifested by a finite helicity modulus and the absence of linear dissipation. This leads to the well-known spectacular experimental consequences and is naturally of great practical importance. A more subtle form of ''superconducting'' order, associated with the new order parameter $\Phi(\mathbf{r})$ introduced in Ref. 9, describes the presence of finite line tension at all length scales and is manifested by the suppression of large thermally generated vortex loops in the partition function. At $H=0$, or in an infinitesimal field with only a single field-induced line, Ψ and Φ are equivalent. The helicity modulus and the effective line tension vanish simultaneously at a *single* superconducting transition, $T=T_{c0}$. In a *finite* field, with a finite density of field-induced lines, the situation is different: the ''superconducting'' transition can now be viewed as *split* into two branches $T_m(H)$ and $T_\Phi(H)$. At $T_m(H)$ the standard superconducting order is lost as the vortex lattice melts into a liquid of field-induced *(s)* vortex lines. Even though the total number of windings along $\mathbf{H}$ is still locked at N_Φ, just like in the vortex lattice state (27), the effective ''diffusion'' of *(s)* vortex lines leads to windings in the xy plane, $\mathcal{N}_{x,y}\propto L^{2p}$, Eq. (28). This amount of winding *suffices* to cause vanishing of the helicity modulus (20) and drives $\Psi(\mathbf{r})$ to zero [i.e., $\langle\Psi(0)\Psi^*(\mathbf{r})\rangle$ is short ranged]. In contrast, the long-distance line tension $\mathcal{T}$, Eq. (29), is still finite below $T_\Phi(H)$, the same as in the Meissner state of the $H=0$ superconductor (25). This state *differs* from the normal state by the presence of long-range Φ order, measured by $\langle\Phi(0)\Phi^*(\mathbf{r})\rangle$. It can be considered *isomorphic*, in the *long wavelength limit*, to a liquid (or solid, as shown in Fig. 1) of N_Φ field-induced vortex lines with some *finite* effective line tension relative to the field direction and interacting via a long range, London-Biot-Savart-type interactions, whose overall strength is set by $\sim|\langle\Phi(\mathbf{r})\rangle|^2$.[53] Only at some higher temperature $T_\Phi(H)$, does $\mathcal{T}$ vanish and infinite vortex loops proliferate across the system. This signals the destruction of the Φ order as the system finally makes transition to the true normal state. Above $T_\Phi(H)$, *both* Ψ and Φ order are absent. Consequently, the *true normal* state of GL theory in a finite magnetic field (1) should not be viewed as a ''line liquid'' in the commonly accepted sense.[15] The general geometrical arguments of this section preclude such an identification and point to a direct connection between the presence (absence) of the Φ order and our ability (inability) to emulate the long-wavelength behavior of the system (1) in terms of a conventional ''line liquid'' (or a ''line solid'') of field-induced vortices. In the language of the *nonrelativistic* ''boson analogy,'' the mass of the bosons *vanishes* at $T_\Phi(H)$ and such an analogy breaks down in the true normal state.

The above criteria, involving the helicity modulus and the number of windings for individual field-induced lines and for the whole system, allow one to clearly distinguish the high- and the low-temperature states of the system in numerical simulations and to establish whether they are separated by a true thermodynamic phase transition or a sharp crossover. Such a procedure should be superior to measurements of specific heat which possesses at most nearly logarithmic singularity and is severely limited by finite-size effects.

VI. FLUCTUATION CONDUCTIVITY

Electrical conductivity (or resistivity) is a quantity easily measured experimentally. Unfortunately, fluctuation conductivity, while in principle a very useful probe of a degree of superconducting order, is not purely thermodynamic quantity and cannot be evaluated from the GL theory (1) unless additional assumptions are made concerning the time dependence of the fluctuating superconducting order parameter. We adopt here a frequently used and empirically successful assumption of dynamical scaling,[37] which connects the decay of spatial correlations with that of time correlations: $\tau_{sc}\sim\xi^z_{sc}$, where τ_{sc} is the relaxation time associated with the superconducting order parameter and z is the dynamical critical exponent. At zero field, this assumption leads to an expression for the fluctuation conductivity:

$$\sigma\propto\xi_{sc}^{z+2-D}\sim t^{\nu_{xy}(D-2-z)}, \tag{30}$$

with $z\sim 1.5$ from numerical simulations.[63] For simplicity, we have suppressed anisotropy in the above expression. When a finite external field is turned on, and we are inside the critical region near T_{c0}, where ξ_{sc} is very long, we can still write

$$\sigma\propto[\xi_{sc}(H\neq 0)]^{z+2-D}. \tag{31}$$

Strictly speaking, the dynamical exponent could be different in the finite-field case but I will ignore such a possibility. I also concentrate on dissipative transport and do not consider Hall conductivity. Right below $T_\Phi(H)$, the gauge theory (4) suggests that ξ_{sc} is finite in all directions and $\xi_{sc}\propto K/T\sim cl$, as discussed in Appendix B. Above $T_\Phi(H)$, the screening length Λ, defined in Appendix B by Eq. (B9), drops abruptly Eq. (21) and we *assume* that the superconducting correlation length ξ_{sc} does the same,

$$\xi_{sc}\propto cl\left(1-G\frac{cl}{\xi_\Phi}\right)=cl\left(1-C\frac{\tau^{\nu_{GT}}}{\sqrt{H}}\right). \tag{32}$$

This assumption seems perfectly justified on physical grounds, since the drop in the screening length arises through appearance above $T_\Phi(H)$ of thermally generated infinite vortex loops which lead to additional screening. It is natural to expect that these same loops suddenly increase dissipation and produce a non-analytic drop in the conductivity.

After restoring the anisotropy, the relative change in the conductivity, $\delta\sigma_\mu$, as one crosses over from the Φ-ordered to the true normal state is

$$\delta\sigma_\mu=\frac{\sigma_{\mu,<}-\sigma_{\mu,>}}{\sigma_{\mu,<}}=(z+2-D)C_\mu\frac{\tau^{\nu_{GT}}}{\sqrt{H}}, \tag{33}$$

where $\mu=(\perp,\|)$, $\sigma_{\mu,<,>}$ is the fluctuation conductivity below (above) $T_\Phi(H)$, C_μ are constants depending on material parameters, and $\nu_{GT}\sim\nu_{xy}\sim 2/3$. The vortex loop ''expansion'' that takes place at T_Φ leads to a nonanalytic increase in dissipation and a corresponding drop in conductivity.

VII. VORTEX LATTICE MELTING IN THE CRITICAL REGION

A theory of the vortex lattice melting in the critical region is an elaborate subject and its detailed discussion will be presented elsewhere. There are, however, several important consequences of the present gauge theory scenario that concern the very nature of the melting transition. We therefore discuss here the ''minimal'' set of requirements that should be satisfied by any theory of melting consistent with the gauge theory scenario. We should first observe that reformulation (2) obviously has a *(s)* vortex lattice as its ground state at low temperatures, just as the original GL theory (1). The gauge theory (4), however, does *not*, for the simple reason that up to this point we were interested in the *long-wavelength*, $q\ll 1/l$, behavior. The Φ transition, for example, is clearly the $q\to 0$ transition. On this basis, we have dropped a large number of terms from Eq. (4), by arguing that they are irrelevant at very long distances. The melting transition, in contrast, is a *finite q* transition $(q\sim 1/l)$, and it requires additional terms and modifications to the coarse-graining procedure applied on the way from Eq. (2) to Eq. (4). For instance, higher powers of $\mathbf{S}$, particularly the odd ones, reflecting the up-down asymmetry along $\mathbf{H}$ manifest in Eqs. (1) and (2), where, unimportant in the long-wavelength limit, are *essential* for the transition to a nonuniform *(s)* vortex lattice state.

This being so, the Φ transition casts a long shadow on the melting transition. This is clear from Eq. (4) and, by inference, from Eq. (2). In the Φ-ordered phase, there is an effective long-range interaction between *(s)* vortex lines which, in the $q\ll 1/l$ limit, takes the London-Biot-Savart form.[15] To see this, note that, in the Φ-ordered ''Meissner'' phase, our fictitious ''photon'' is ''massive;'' i.e., the second (gradient) term in Eq. (4) becomes

$$\gamma_\perp|\langle\Phi\rangle|^2\mathbf{S}_\perp^2+\gamma_\||\langle\Phi\rangle|^2\mathbf{S}_\|^2, \tag{34}$$

where $\langle\Phi\rangle$ is the order parameter associated with the Φ order. As seen in Sec. V, we expect that, as long as $\langle\Phi\rangle$ is finite, we can write some effective description of such a state in terms of a ''line liquid'' of field-induced vortices, in the sense of Nelson.[8] Long-distance physics can be described through fluctuations in the density and ''currents'' of vortex lines, $\delta\rho(\mathbf{r})$ and $\vec{j}(\mathbf{r})=(j_x,j_y)$, respectively.[8] The connection with our fictitious ''gauge'' potential $\mathbf{S}(\mathbf{r})$, Eq. (4), is (Appendix A)

$$(\nabla\times\mathbf{S})_\|\to 2\pi\delta\rho,\quad (\nabla\times\mathbf{S})_\perp\to 2\pi\vec{j}. \tag{35}$$

If we now reexpress Eq. (34) in terms of $\delta\rho$ and $\vec{j}$ we get precisely the long-distance part of the London-Biot-Savart interaction between field-induced vortex lines:

$$4\pi^2\gamma|\langle\Phi\rangle|^2\sum_{\mathbf{q}}\frac{\delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})+\vec{j}(\mathbf{q})\cdot\vec{j}(-\mathbf{q})}{q_\perp^2+q_\|^2}, \tag{36}$$

where the continuity condition $\vec{q}\cdot\vec{j}(\mathbf{q})=-q_\|\delta\rho(\mathbf{q})$ is assumed. The anisotropy, suppressed in the above expression for simplicity, can be straightforwardly restored.[64] This expression (36) for the effective interaction at long-distances is just what is obtained in the mean-field-based approach,[7] but with one *crucial difference*. The overall strength of the interaction is not given by the mean-field *amplitude* squared of the superconducting order parameter, but by $|\langle\Phi\rangle|^2\propto n_\Phi$, where n_Φ is the Φ-superfluid density, whose physical meaning is apparent from reformulation (2) and gauge theory (4).

The immediate consequence of Eq. (36) is that the melting line $T_m(H)$, goes into T_{c0}, the true zero-field superconducting transition, as $H\to 0$. This result is strongly suggested by all available numerical simulations on the 3D *XY* model and arises naturally in the gauge theory scenario. For all its apparent simplicity, this result is not trivial: the mean-field based theories of melting including only field-induced London vortices[7] naturally lead to $T_m(H)\to T_c$, the *mean-field* transition temperature, as $H\to 0$.[65,66] Furthermore, the exponent ν has its mean-field value $\nu_{mf}=1/2$ and is *not* equal to $\nu_{xy}\sim 2/3$. Therefore, the thermodynamics of the melting transition resulting from such theories cannot satisfy the 3D *XY* scaling properties of Sec. IV. This is a direct consequence of ignoring those very degrees of freedom (vortex loops) which are primarily responsible for moving the true *zero*-field superconducting transition temperature from T_c to T_{c0} and changing ν_{mf} to ν_{xy} in the first place. This is a serious flaw and must be rectified in a proper theory of vortex lattice melting in the critical region.

As a first step, we attempt to remedy the situation by simply replacing, *by hand*, the mean-field amplitude squared with the true superfluid density at *zero* field. This amounts to installing $|\langle\Phi\rangle_{H=0}|^2$ instead of $|\langle\Phi\rangle|^2$ in Eq. (36). With the interaction fixed in this fashion, we can then proceed to analyze the *same*, *finite*-field model of Ref. 7, still including only the field-induced vortices. This is equivalent to having $T_\Phi(H)=T_\Phi(H=0)\equiv T_{c0}$, as represented by the dashed vertical line in Fig. 1. This procedure, arbitrary as it is within the framework of mean-field-based theories, seems to get us out of the above difficulties, since now evidently $T_m(H)\to T_{c0}$ for $H\to 0$ and the exponent ν takes its true $H=0$ value, $\nu_{xy}\sim 2/3$.

However, things are not that simple: to understand why note that the above remedial procedure is in fact *exact*, but only for a *single* field-induced line. For a *finite* density of lines, the physical state of thermally generated vortex loops and other fluctuations described by $\Phi(\mathbf{r})$, which controls the effective interaction between field-induced vortex lines through $|\langle\Phi\rangle|^2$ in Eq. (36), is *itself* strongly affected by interactions with those same field-induced lines. The effective coupling of these two interpenetrating systems, vortex loops and lines in reformulation (2), must be solved *self-consistently* at finite H: this is precisely what is accomplished in the gauge theory scenario (4) for the long-wavelength ($q\ll 1/l$) behavior. Clearly, a ''minimal'' theory of vortex lattice melting in the critical region must involve *both* the positional order parameter of the vortex lattice, ρ_G (or the original Ψ), *and* the new ''superconducting'' order parameter Φ. The coupled equations governing the (T,H) dependence of these two order parameters must be solved simultaneously and self-consistently near $T_m(H)$.

An important physical feature is expected to emerge from such a solution: *the formation of vortex lattice is a phase transition involving simultaneous ordering of both field-induced and thermally generated degrees of freedom*. This is illustrated with two qualitative points. First, at low fields within the Φ-ordered state (Fig. 1), we can consider some effective ''line liquid'' description (35). Right above $T_m(H)$, in the liquid phase, the self-consistent solution gives $\langle\Phi\rangle=\langle\Phi\rangle_L$ to be inserted in the effective interaction (36). Similarly, just below $T_m(H)$, we have $\langle\Phi\rangle=\langle\Phi\rangle_S$. In general, however, $\langle\Phi\rangle_L\neq\langle\Phi\rangle_S$. This result follows immediately from the gauge theory scenario (4) since positional correlations of *s* vortices strongly influence $\langle\Phi\rangle$. While both $\langle\Phi\rangle_L$ and $\langle\Phi\rangle_S$ are finite, as we cross $T_m(H)$, there is a discontinuous change in the average density and size of vortex loop and ''overhang'' excitations, resulting in a different effective interaction (36) on two sides of the melting line (Fig. 4). The entropy jump at melting, ΔS, will receive a significant contribution from such excitations. In fact, while $\Delta S\to 0$ as $H\to 0$, the *critical fluctuations* greatly *enhance* ΔS over the configurational entropy of *field-induced lines*. This provides natural explanation[9] for the excess entropy at $T_m(H)$ observed in low-field thermodynamics.[11,12] It should be stressed that this effect is *different* from the ''microscopic entropy'' contribution discussed by Hu and MacDonald[19] (see also Ref. 7). Such entropy arises from the *electronic* degrees of freedom and is reflected in the T dependence of our GL coefficients (1). This contribution is important in the high-field, LL regime.[16,19] In the 3D XY critical region, however, such a T dependence is a *minor* effect since it involves T_c and not the true transition temperature T_{c0}. The entropy contribution discussed here, arising from the T dependence of $|\langle\Phi\rangle|^2$ in Eq. (36) and $\langle\Phi\rangle_L\neq\langle\Phi\rangle_S$, is due to degrees of freedom of the *superconducting order parameter* itself and must be part of any proper theory of melting.

Second, at higher fields, the self-consistent solution for ρ_G (or Ψ) and Φ leads to a rapid suppression of $T_\Phi(H)$ far below T_{c0} and its subsequent merging with $T_m(H)$ for $H>H_Z$ (Fig. 1). This exposes a large region of the phase diagram where the Abrikosov vortex lattice melts directly into the true normal state. In this case $\langle\Phi\rangle_L=0$, while $\langle\Phi\rangle_S$ might still be close to its mean-field value. Such a dramatic difference in the nature of these two phases, with the solid being rather unremarkably mean-field-like and the liquid right above the melting line exhibiting very strong fluctuations even at rather low fields, with vortex lines winding *both* along the field *and* in the perpendicular directions, is evident in recent simulations by Nguyen and Sudbø.[23,67] Note that such a situation *never* arises in mean-field-based theories with only field-induced vortices,[7] even after the application of our remedial procedure, since we would still have $|\langle\Phi\rangle_S|^2=|\langle\Phi\rangle_L|^2=|\langle\Phi\rangle_{H=0}|^2$ in Eq. (36). As argued in Sec. V, it does not appear possible to write an effective description of the true normal state in terms of field-induced degrees of freedom only. For example, we could start again with our remedial procedure and argue that, even though it fails for $q\ll 1/l$, it still describes the effective interaction of field-induced vortices for $q\sim 1/l$, which is what matters most at $T_m(H)$. However, within such a ''line liquid'' description, I do not see any simple way by which one could account for the part of ΔS associated with an abrupt change in windings $\mathcal{N}_{x,y}$ across $T_m(H)$, Eqs. (26) and (27). In this region of higher fields, where $T_\Phi(H)=T_m(H)$, our gauge theory is becoming less and less ''type II'' and amplitude fluctuations are becoming stronger in GL theory. It is likely that the low-field description discussed here and LL theories now have an equal chance of providing reliable theory of the melting transition: both routes, however, are certain to be most challenging.

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APPENDIX A: DERIVATION OF THE GAUGE THEORY

I present here a derivation of the gauge theory (4). An abbreviated version is found in Ref. 9. For simplicity, I consider the isotropic case $\gamma_\parallel=\gamma_\perp=\gamma$.

Within the ''helium model,''[26] the partition function of the superconductor (1) can be written as

$$Z=\sum \cdots \sum_{N^{(\boldsymbol{\omega})}}\prod_{\boldsymbol{\omega}}\frac{1}{N^{(\boldsymbol{\omega})}!}\prod_{l_{\boldsymbol{\omega}}=1}^{N^{(\boldsymbol{\omega})}}\int \mathcal{D}\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]\times\exp\left(-\frac{F_v}{T}\right),$$

$$F_v=\gamma\langle|\Psi|^2\rangle\int_{\mathcal{C}}d^3r\left|\nabla\varphi(\mathbf{r})-\frac{2e}{c}\mathbf{A}\right|^2+\sum_{\boldsymbol{\omega}}\sum_{l_{\boldsymbol{\omega}}=1}^{N(\boldsymbol{\omega})}\int ds_{l_{\boldsymbol{\omega}}}E^{(1)}(\{\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]\})+\frac{1}{2}\sum_{\boldsymbol{\omega}}\sum_{\boldsymbol{\omega}'}\sum_{l_{\boldsymbol{\omega}}=1}^{N(\boldsymbol{\omega})}\sum_{l_{\boldsymbol{\omega}'}=1}^{N(\boldsymbol{\omega}')}\int ds_{l_{\boldsymbol{\omega}}}\int ds_{l_{\boldsymbol{\omega}'}}\times\left\{V_0^{(2)}(|\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]-\mathbf{x}_{l_{\boldsymbol{\omega}'}}[s_{l_{\boldsymbol{\omega}'}}]|)+\frac{d\mathbf{x}_{l_{\boldsymbol{\omega}}}}{ds_{l_{\boldsymbol{\omega}}}}\cdot\frac{d\mathbf{x}_{l_{\boldsymbol{\omega}'}}}{ds_{l_{\boldsymbol{\omega}'}}}V_1^{(2)}(|\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]-\mathbf{x}_{l_{\boldsymbol{\omega}'}}[s_{l_{\boldsymbol{\omega}'}}]|)\right\},\tag{A1}$$

where

$$\nabla\times\mathbf{A}=\mathbf{H},\quad \nabla\cdot\nabla\varphi(\mathbf{r})=0,$$

$$\nabla\times\nabla\varphi(\mathbf{r})=2\pi\sum_{\boldsymbol{\omega}}\sum_{l_{\boldsymbol{\omega}}=1}^{N(\boldsymbol{\omega})}\int_{\mathcal{L}}d\mathbf{x}_{l_{\boldsymbol{\omega}}}\delta(\mathbf{r}-\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]).\tag{A2}$$

The partition function (A1) is a 3D counterpart of the familiar representation of the (continuum) 2D XY model in terms of its pointlike topological excitations, vortices and antivortices. In 3D, the relevant excitations are loops and lines of vortices, classified by their global topology. Only vortex paths of unit vortex ''charge'' are considered since they are the important excitations in the critical region. The lattice regularization of Eq. (A1) is a gas of nonintersecting oriented paths on a lattice which are either closed or can start or end only on sample surfaces. The lattice spacing is set by the characteristic ''bending length'' of vortex lines. Each individual step along a path takes a given energy to create and can be either up or down along the x, y, or z axis. These steps represent vortex segments and have a long-range ''directional'' Coulomb interaction, operating only between the steps going along the same axis. This is the lattice version of the ''Biot-Savart'' interaction between vortex loops and lines of the continuum model. The background free energy, composed of the uniform condensation energy and ''spin-wave'' contributions, is not included explicitly.

The summation in Eq. (A1) runs over all distinct configurations of vortex line excitations of arbitrary length and shape. The index $\boldsymbol{\omega}$ denotes different classes of oriented loops and lines which are distinguished by their global topology. For example, for periodic boundary conditions, when only closed loops are present, $\boldsymbol{\omega}=(m_x^{\pm},m_y^{\pm},m_z^{\pm})$, where $m_{x,y,z}^{\pm}=0,\pm1,\pm2,\ldots$ denote the winding numbers of a given loop around x,y,z directions. *Finite* closed loops correspond to $m_x^{\pm}=m_y^{\pm}=m_z^{\pm}=0$. Similarly, for free (periodic) boundary conditions along the z (x,y) direction, $\boldsymbol{\omega}=(m_x^{\pm},m_y^{\pm},0,0)$ denotes loops that wind in the x (y) direction while $\boldsymbol{\omega}=(0,0;0,0;0,0)$ again denotes finite closed loops. In addition, there are vortex (antivortex) paths that traverse the system from $z=0$ to $z=L_z$ and ''half-loops'' which originate and terminate at the *same* $z=0$ or $z=L_z$ surface. $\int\mathcal{D}\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]$ represents summation over all configurations of a given loop and line $l_{\boldsymbol{\omega}}$ consistent with its global topology. $\mathcal{C}\{\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]\}$ in the first term of F_v signifies that the integral of the gradient energy over the system excludes well-defined core regions associated with a given configuration of loops and lines. The second and third terms represent core contributions: $E^{(1)}=E_c+E_b(\{\mathbf{x}_{l_{\boldsymbol{\omega}}}[s_{l_{\boldsymbol{\omega}}}]\})$ is a ''single-particle'' term, with E_c and E_b corresponding to the core line and bending energies, respectively. $V_{0,1}^{(2)}(|\mathbf{r}-\mathbf{r}'|)$ denotes ''two-particle'' effects of core overlap. These ''two-particle'' terms describe both the energy cost of core overlap and the entropic effects of keeping vortex and antivortex segments from annihilating each other. ''Multiparticle'' terms, arising from simultaneous overlap of more than two cores, can be neglected in the extreme type-II regime, where the average core size a is small compared to the average separation between vortex segments. Core contributions, like E_c, E_b, and $V_{0,1}^{(2)}(|\mathbf{r}-\mathbf{r}'|)$, can be computed in a specific microscopic model of vortex lines.[68,69] Their precise form is not needed for our present purposes since we will return to the GL representation at the end of this appendix; it suffices to know that E_c and E_b are finite and the ''interaction'' $V_{0,1}^{(2)}(|\mathbf{r}-\mathbf{r}'|)$ is short ranged, of order a, and repulsive on average. Without loss of generality, we could set $V_{0,1}^{(2)}(|\mathbf{r}-\mathbf{r}'|)\to V_{0,1}\delta(\mathbf{r}-\mathbf{r}')$.

With the uniform magnetic field present, the overall vortex ''charge'' neutrality demands that every configuration contain N_Φ field-induced vortex paths going from $z=0$ to $z=L_z$. This fact is used to observe that the low-temperature expansion of Z', Eq. (2), in terms of topological defects of the new order parameter $\Phi(\mathbf{r})$ and s vortices, is the same as Eq. (A1) except for different prefactors: $1/(N_\Phi+N_a)!$ in Z versus $1/N_\Phi!N_a!$ in Z', where N_a is the number of thermally generated infinite vortex-antivortex paths which extend from $z=0$ to $z=L_z$. This leads to a difference in entropy between two representations $\Delta S\sim T\ln[(N_\Phi+N_a)!/N_\Phi!N_a!]$. ΔS scales at most as L_xL_y, in contrast to the full entropy which goes as $L_xL_yL_z$. Consequently, in the thermodynamic limit, ΔS does not affect the free energy per unit volume. In particular, within the Φ-ordered state, Z, Eq. (1), and Z', Eq. (2), have identical expansions:

$$Z=\sum_{N^{(0)}=0}^{\infty}\frac{1}{N^{(0)}!}\frac{1}{N_\Phi!}\prod_{l_0=1}^{N^{(0)}}\oint\mathcal{D}_b\mathbf{x}_{l_0}[s_{l_0}]\prod_{i=1}^{N_\Phi}\int\mathcal{D}_b\mathbf{r}_i[s_i]\exp\left(-\frac{F_v}{T}\right),$$

$$F_v=\gamma\langle|\Psi|^2\rangle\int_{\mathcal{C}}d^3r\left|\nabla\phi(\mathbf{r})+\mathbf{U}-\frac{2e}{c}\mathbf{A}\right|^2+\sum_{l_0=1}^{N^{(0)}}E_c\int ds_{l_0}+\sum_{i=1}^{N_\Phi}E_c\int ds_i+\frac{1}{2}\int d^3r\int d^3r'\{[d_0(\mathbf{r})+d_s(\mathbf{r})]\times[d_0(\mathbf{r}')+d_s(\mathbf{r}')]V_0^{(2)}(|\mathbf{r}-\mathbf{r}'|)+[\mathbf{n}_0(\mathbf{r})+\mathbf{n}_s(\mathbf{r})]\cdot[\mathbf{n}_0(\mathbf{r}')+\mathbf{n}_s(\mathbf{r}')]V_1^{(2)}(|\mathbf{r}-\mathbf{r}'|)\}.\tag{A3}$$

Here $\{\mathbf{x}_{l_0}[s_{l_0}]\}$ denote *finite* closed loops of arbitrary length and shape. Half-loops attached to $z=0$ and $z=L_z$ surfaces are not included since they do not matter for the bulk thermodynamics. The symbol $\mathcal{D}_b$ indicates that the ''single-particle'' bending energy $[E_b(\{\mathbf{x}_{l_\omega}[s_{l_\omega}]\})]$ has been absorbed into the measure of the path integral. ''Densities'' and ''currents'' $d_{0,s}$ and $\mathbf{n}_{0,s}$ are defined as

$$\{d_0,\mathbf{n}_0\}(\mathbf{r})=\sum_{l_0=1}^{N^{(0)}}\int ds_{l_0}\left\{1,\frac{d\mathbf{x}_{l_0}}{ds_{l_0}}\right\}\delta(\mathbf{r}-\mathbf{x}_{l_0}[s_{l_0}]),$$

$$\{d_s,\mathbf{n}_s\}(\mathbf{r})=\sum_{i=1}^{N_\Phi}\int ds_i\left\{1,\frac{d\mathbf{r}_i}{ds_i}\right\}\delta(\mathbf{r}-\mathbf{r}_i[s_i]), \quad \text{(A4)}$$

and

$$\nabla\cdot\nabla\phi(\mathbf{r})=0, \quad \nabla\times\nabla\phi(\mathbf{r})=2\pi\mathbf{n}_0(\mathbf{r}),$$

$$\nabla\cdot\mathbf{U}(\mathbf{r})=0, \quad \nabla\times\mathbf{U}(\mathbf{r})=2\pi\mathbf{n}_s(\mathbf{r}). \quad \text{(A5)}$$

To understand the problem at low fields, we first consider the $H=0$ situation. The superconducting (Meissner) state is described by the $\mathbf{H}\to 0$, $N_\Phi\to 0$ limit of Eq. (A3):

$$Z(H=0)=\sum_{N^{(0)}=0}^{\infty}\frac{1}{N^{(0)}!}\times\prod_{l_0=1}^{N^{(0)}}\oint\mathcal{D}_b\mathbf{x}_{l_0}[s_{l_0}]\left(-\frac{F_v(0)}{T}\right),$$

$$F_v(0)=\gamma\langle|\Psi|^2\rangle\int_C d^3r|\nabla\phi(\mathbf{r})|^2+\sum_{l_0=1}^{N^{(0)}}E_c\int ds_{l_0}+\frac{1}{2}\int d^3r\int d^3r'\{d_0(\mathbf{r})d_0(\mathbf{r}')V_0^{(2)}(|\mathbf{r}-\mathbf{r}'|)+\mathbf{n}_0(\mathbf{r})\cdot\mathbf{n}_0(\mathbf{r}')V_1^{(2)}(|\mathbf{r}-\mathbf{r}'|)\}. \quad \text{(A6)}$$

The gradient term in $F_v(0)$ can be decoupled by a *dual* gauge field $\mathbf{A}_d(\mathbf{r})$, in the Coulomb gauge $\nabla\cdot\mathbf{A}_d=0$:

$$\exp\left[-\frac{\gamma\langle|\Psi|^2\rangle}{T}\int_C d^3r|\nabla\phi(\mathbf{r})|^2\right]\to\int\mathcal{D}\mathbf{A}_d(\mathbf{r})\exp\left[\int d^3r\left(-i\mathbf{n}_0\cdot\mathbf{A}_d-\frac{1}{2e_d^2}(\nabla\times\mathbf{A}_d)^2\right)\right], \quad \text{(A7)}$$

where $e_d^2=8\pi^2\gamma\langle|\Psi|^2\rangle/T$ is a dual charge. Equations (A7) and (A6) have the appearance of a path integral over trajectories of *relativistic* charged quantum particles in a (2+1)-dimensional Euclidean space (z being the imaginary time) coupled to the ''electromagnetic'' gauge potential $\mathbf{A}_d$ and interacting via short-range ''density-density'' and ''current-current'' interactions. $Z(H=0)$ describes the *vacuum* structure of such ''electrodynamics,'' with our vortex loops corresponding to world lines of relativistic quantum bosons and describing *virtual* particle-antiparticle creation and annihilation processes in the vacuum. This similarity can be exploited further by using the particle-field duality to define the field theory version[70] of $Z(H=0)$, Eqs. (A7) and (A6):

$$\int\mathcal{D}\Psi_d(\mathbf{r})\int\mathcal{D}\mathbf{A}_d(\mathbf{r})\exp\left\{-\int d^3r\left[m_\Psi^2|\Psi_d|^2+|(\nabla-i\mathbf{A}_d)\Psi_d|^2+\frac{1}{2}g_0|\Psi_d|^4+\frac{1}{2q_d^2}(\nabla\times\mathbf{A}_d)^2+\frac{M_d^2}{2}\mathbf{A}_d^2\right]\right\}. \quad \text{(A8)}$$

$\Psi_d(\mathbf{r})$ is a field operator of these relativistic bosons. The gradient term in the action has been rescaled into dimensionless form so that short-range repulsion and dual charge assume their canonical dimensions: $V_0\delta(\mathbf{r}-\mathbf{r}')\to g_0\delta(\mathbf{r}-\mathbf{r}')$, $[g_0]=$ (length)$^{-1}$, $e_d^2\to q_d^2$, $[q_d^2]=$ (length)$^{-1}$. V_1 has been dropped since it is irrelevant in the long-wavelength limit. Its effect on critical behavior can be incorporated into the bare values of g_0, q_d^2, and m_Ψ^2. Finally, the ''bare mass'' of $\mathbf{A}_d$, M_d, is *absent* in our problem ($M_d=0$). Finite M_d reflects the presence of a gauge field minimally coupled to the original, superconducting order parameter. For example, if the condition $\kappa\to\infty$ is relaxed and the real electromagnetic screening is restored in Eq. (1), $M_d^2\to\mu_0e^2/\pi$, where e and μ_0 are the real charge and magnetic permeability, respectively.

Expression (A8) forms the basis for our ''dual'' picture of the 3D XY critical behavior. In this picture, we are viewing vortices as primary objects and their field operator $\Psi_d(\mathbf{r})$ as our order parameter, instead of the original $\Psi(\mathbf{r})$. We can think of $\Psi(\mathbf{r})$ in Eq. (1) as being the field operator describing creation and annihilation of Cooper pairs. In the GL theory, with $H=0$ (1), Cooper pairs have only short-range interactions and it suffices to keep only the quartic term, describing the pointlike repulsion, since the rest is irrelevant for the critical behavior. In contrast, the *vortex* excitations of Ψ interact via long-range London-Biot-Savart forces, mediated by massless ($M_d=0$) $\mathbf{A}_d$, Eq. (A8). Next, we can convert our neutral GL theory (1) into one with a finite real charge e by introducing the fluctuating vector potential $\mathbf{A}$, as well as the electromagnetic field energy $(1/8\pi\mu_0e^2)(\nabla\times\mathbf{A})^2$. Now, it is the Cooper pairs that have long-range interactions, mediated by $\mathbf{A}$, but the vortices in Ψ interact only through short-range forces, due to the electromagnetic screening inducing a finite $M_d=\mu_0e^2/\pi$ in Eq. (A8). Precisely in 3D, there is an exact duality, at least on a lattice,[48] between the form of this interaction for vortices and for Cooper pairs. This is what ''inverted'' stands for in the ''inverted 3D XY'' model. In the dual language of Ψ_d, it is the low-temperature Meissner phase of the original superconductor that is symmetric ($\langle\Psi_d\rangle=0$), while the high-temperature normal metal is the ''broken symmetry'' state of the dual theory ($\langle\Psi_d\rangle\neq 0$). So there is an inversion of the temperature axis. However, it is still the same symmetry U(1) that is being broken and this implies that the *thermodynamic* exponent should be the same, $\nu=\nu_{xy}$, *both* for $e=0$ *and* for $e\neq 0$. It is important to stress that the ''inverted 3D

XY'' behavior of a charged superfluid remains *different* from the 3D *XY* critical behavior of a neutral superfluid, described by Eq. (1) with $H=0$, since they are associated with two different critical points. For example, the anomalous dimension exponent η_Ψ of $\langle\Psi(0)\Psi^*(\mathbf{r})\rangle$ will be *different* in two cases, $e=0$ and $e\neq 0$.

Equation (A8) describes the ''true vacuum'' state $|0\rangle$ of a Euclidean relativistic field theory. Particle (antiparticle) excitations are massive, $m_\Psi \sim 1/\xi_d \sim 1/\Lambda_\Psi$, where ξ_d is the dual correlation length associated with Ψ_d and Λ_Ψ measures the typical loop size. The actual value of m_Ψ reflects both the cost in energy and gain in entropy arising from large thermally generated vortex loops in the original problem. As we approach T_{c0} from below, $m_\Psi \to 0$, and we enter the ''false vacuum'' state $|f\rangle$ of the theory (A8). Particle (antiparticle) excitations are now massless and infinite vortex paths proliferate across the system, as depicted in Fig. 2. This ''false vacuum'' is just the normal metallic state. If we introduce the particle (antiparticle) number operators, $\hat{N}_{P,A}$, $|0\rangle$ is an eigenstate of $\hat{N}_{P,A}$, $\hat{N}_P|0\rangle=0|0\rangle$ and $\hat{N}_A|0\rangle=0|0\rangle$. In contrast, $|f\rangle$ is *not* an eigenstate of $\hat{N}_{P,A}$, and contains a *finite* average number of (anti)particles, $\langle f|\hat{N}_{P,A}|f\rangle = N_{P,A}\neq 0$. Both $|0\rangle$ and $|f\rangle$, however, are eigenstates of the total vorticity operator $\hat{N}_P-\hat{N}_A$ with eigenvalue 0, which ensures $N_P=N_A$. For H small but finite, the ground state of Eq. (A8) must still be an eigenstate of $\hat{N}_P-\hat{N}_A$ but now the eigenvalue is N_Φ. Starting from $|0\rangle$, such a ground state $|\Phi\rangle$ is naturally constructed by introducing N_Φ *massive* particles into the true, stable vacuum. We then have $\hat{N}_P|\Phi\rangle=N_\Phi|\Phi\rangle$ and $\hat{N}_A|\Phi\rangle=0|\Phi\rangle$. On the other hand, starting from the ''false vacuum'' $|f\rangle$, the ground state $|n\rangle$ is formed by having additional N_Φ *massless* particles added to an already present finite average number of particle-antiparticle pairs. $|n\rangle$ is not an eigenstate of $\hat{N}_{P,A}$ and satisfies $\langle n|\hat{N}_{P,A}|n\rangle = N_{P,A}\neq 0$, where now $N_P-N_A=N_\Phi$. At $T_\Phi(H)$, a phase transition takes place between these two different types of ground state, $|\Phi\rangle$ and $|n\rangle$, driven by the change in U(1) symmetry of the vortex system.

We now return to Eq. (A3). A finite density of *s* vortices produces two main effects on the loop ''expansion'' as we approach $T_\Phi(H)$ from below. First, there is a long-range interaction between loops and *s* vortices which will influence the long-range correlations among the loops themselves. Second, there is a short-range effect of *s* vortices suppressing certain configurations of large loops, through mutual contact interactions (an ''excluded volume'' effect). Intuitively, one expects the long-range effect to be essential for the critical behavior at low fields. Furthermore, the short-range effect should be weak since the total number of vortex segments connected to *s* vortices forms a tiny minority of all vortex segments. Based on these observations, we devise the following strategy: according to our basic assumption (ii) and results of Sec. V, the system of *s* vortices below $T_\Phi(H)$ can be viewed as equivalent, in the long-wavelength limit, to an effective system of *nonrelativistic* 2D quantum bosons in its *superfluid* state, in the sense of Nelson.[8] The collective modes of such a system are its ''density'' and ''current'' fluctuations, which is precisely how the loops couple to *s* vortices. The long-wavelength effective action of these collective modes is described by two coupling constants m_s/n_s and c_s^2, where, in this boson analogy, n_s, m_s, and c_s are the superfluid density, mass, and speed of ''sound,'' respectively. We could compute these quantities explicitly, by starting sufficiently below $T_\Phi(H)$. Here, however, we treat them as general parameters which characterize the long-wavelength fluctuations of *s* vortices and whose ultimate values can be determined through their direct connection with the components of the helicity modulus tensor (Sec. V and Appendix B). We assume that the effect of mutual contact interaction between loop and *s* vortex subsystems, measured by some interaction strength V_ρ, can be fully included into these ultimate values of m_s/n_s and c_s^2, as well as into a renormalized loop-loop contact interaction $\tilde{V}_0$ and the loop core line energy $\tilde{E}_c$. This amounts to reexpressing the short-range interaction in Eq. (A3) as

$$\frac{1}{2}\int d^3r\{\tilde{V}_0 d_0^2(\mathbf{r})+2V_\rho\delta d_0(\mathbf{r})\delta d_s(\mathbf{r})+V_{\rho\rho}d_s^2(\mathbf{r})\},$$

$$\tilde{E}_c=E_c+V_0\langle d_s(\mathbf{r})\rangle,\quad E_s=E_c+V_0\langle d_0(\mathbf{r})\rangle, \tag{A9}$$

while the core line energies of loops and *s* vortices are also renormalized to include the average ''excluded volume'' effect. E_s and $V_{\rho\rho}$ will shortly be subsumed into m_s/n_s and c_s^2. We set $V_\rho\to 0$ in Eq. (A9) and proceed to derive the long-distance description of Eq. (A3). Indeed, we find that the long-range interactions lead to a major change in the behavior of the system once $\mathbf{H}$ is finite: most importantly, the long-range interactions between vortex loops are ''screened'' by fluctuations of *s* vortex lines. The amount of ''screening'' is determined by m_s/n_s and c_s^2. Once the critical behavior associated with this mutual ''screening'' has been understood, we reinstate the residual contact interaction between loop and *s* vortex subsystems and test for consistency. We find, both within the ϵ expansion and perturbative RG in fixed dimension $D=3$, that such residual coupling is irrelevant for $3\leqslant D<4$; i.e., it does not lead to any new relevant terms in the effective action, apart from those already present. Our procedure is therefore ''exact'' for long range and self-consistent for short-range interactions between loop and line subsystems.

The first step is to decouple the gradient term in Eq. (A3) by using $\mathbf{A}_d$, except that now $\mathbf{n}_0\to\mathbf{n}_0+\Delta\mathbf{n}_s$ in Eq. (A7) [$\Delta\mathbf{n}_s$ is defined above Eq. (4)]. The *s* vortex part of Eq. (A3) becomes

$$\frac{1}{N_\Phi!}\prod_{i=1}^{N_\Phi}\int\mathcal{D}_b\mathbf{r}_i[s_i]$$
$$\times\exp\left\{-\sum_{i=1}^{N_\Phi}\frac{E_s}{T}\int ds_i-\frac{V_{\rho\rho}}{2T}\int d^3r\,d_s^2(\mathbf{r})\right.$$
$$\left.-i\int d^3r\,\Delta\mathbf{n}_s(\mathbf{r})\cdot\mathbf{A}_d(\mathbf{r})\right\}. \tag{A10}$$

This part is now reexpressed in terms of *s* vortex ''density'' and ''current'' fluctuations. Sufficiently below $T_\Phi(H)$, the overhangs are small and *s* vortex lines are almost fully directed. We can use a nonrelativistic boson analogy of Nelson[8] and replace $\int ds_i \approx \int_0^{L_z} dz[1+\frac{1}{2}(d\vec{r}_i/dz)^2]$, where $\vec{r}_i(z)=(x_i(z),y_i(z))$, while absorbing the effect of small overhangs into the effective mass, $m_s \sim E_s/T$. Similarly, $d_s(\mathbf{r}) \rightarrow \mathbf{n}_{s\parallel}(\mathbf{r}) \rightarrow \Sigma_i \delta(\vec{r}-\vec{r}_i(z))$ and $\mathbf{n}_{s\perp}(\mathbf{r}) \rightarrow \Sigma_i (d\vec{r}_i/dz)\delta(\vec{r}-\vec{r}_i(z))$. We then introduce a field operator of these nonrelativistic bosons, $\Psi_s(\vec{r},z)$, replace the path integral in Eq. (A10) with the functional integral over Ψ_s (periodic or free boundary conditions give the same result in the thermodynamic limit), and follow the standard procedure[26] for deriving the ''hydrodynamic'' action of s vortices:[9] the phase $\varphi_s(\vec{r},z)$ and amplitude $\pi_s(\vec{r},z)$ are introduced via $\Psi_s = |\Psi_s|\exp(i\varphi_s)$ and $\pi_s = |\Psi_s|^2 - \langle|\Psi_s|^2\rangle$, where $\langle|\Psi_s|^2\rangle$ is evaluated self-consistently. After integration over π_s, the ''hydrodynamic'' action for the phase becomes

$$\int d^3r \left\{ \frac{n_s}{2m_s c_s^2}(\nabla_\parallel \varphi_s - \mathbf{A}_{d\parallel})^2 + \frac{n_s}{2m_s}(\nabla_\perp \varphi_s - \mathbf{A}_{d\perp})^2 + (\cdots) \right\}, \tag{A11}$$

where $(\cdots)$ denotes higher powers of $\nabla\varphi_s - \mathbf{A}_d$ and higher order derivatives. The ''mean-field'' part of the action is absorbed into the background. The action (A11) is decoupled using real fields $\delta\rho(\mathbf{r})$ and $\vec{j}(\mathbf{r})=(j_x,j_y)$:

$$\rightarrow \int \mathcal{D}\delta\rho(\mathbf{r}) \int \mathcal{D}\vec{j}(\mathbf{r}) \exp\left\{ \int d^3r \left[-i\,\delta\rho \mathbf{A}_{d\parallel} - i\vec{j}\cdot\mathbf{A}_{d\perp} + i\,\delta\rho\nabla_\parallel\varphi_s + i\vec{j}\cdot\nabla_\perp\varphi_s - \frac{m_s c_s^2}{2n_s}\delta\rho^2 - \frac{m_s}{2n_s}\vec{j}^2 \right]\right\}, \tag{A12}$$

which finally results in Eq. (A3) expressed in terms of the true collective modes of the superfluid s vortex system, its ''density'' $\delta\rho$ and ''current'' $\vec{j}$ fluctuations:

$$Z \rightarrow \sum_{N^{(0)}=0}^{\infty} \frac{1}{N^{(0)}!} \prod_{l_0=1}^{N^{(0)}} \oint \mathcal{D}_b \mathbf{x}_{l_0}[s_{l_0}] \int \mathcal{D}\delta\rho(\mathbf{r}) \int \mathcal{D}\vec{j}(\mathbf{r}) \times \int \mathcal{D}\mathbf{A}_d(\mathbf{r}) \exp(-S),$$

$$S = \sum_{l_0=1}^{N^{(0)}} \frac{\tilde{E}_c}{T} \int ds_{l_0} + \int d^3r \left\{ \frac{\tilde{V}_0}{2T} d_0^2(\mathbf{r}) + i\mathbf{n}_0\cdot\mathbf{A}_d + i\,\delta\rho\mathbf{A}_{d\parallel} + i\vec{j}\cdot\mathbf{A}_{d\perp} + \frac{m_s c_s^2}{2n_s}\delta\rho^2 + \frac{m_s}{2n_s}\vec{j}^2 + \mathcal{W}(\delta\rho,\vec{j}) + \frac{1}{2e_d^2}(\nabla\times\mathbf{A}_d)^2 \right\}, \tag{A13}$$

where the functional integral must be appended by the continuity condition $\nabla_\parallel \delta\rho + \nabla_\perp\cdot\vec{j}=0$, which follows from the integration over φ_s in Eq. (A12). $\mathcal{W}$ arises from the $(\cdots)$ terms in Eq. (A11) and contains powers higher than quadratic in $\delta\rho$ and $\vec{j}$, as well as assorted derivatives. In particular, *odd* powers of $\delta\rho$ are present, like $\sim\delta\rho^3$, reflecting the $\Delta\mathbf{n}_{s\parallel} \rightarrow -\Delta\mathbf{n}_{s\parallel}$ asymmetry of the original problem (A3). Such higher-order asymmetric terms are essential for a description of the melting transition but, as shown below, are irrelevant at the Φ transition, provided the latter is continuous. Note that the above relatively simple dependence of Eq. (A13) on $\delta\rho$ and $\vec{j}$ holds only at distances $>l$. This is precisely what we are interested in as we approach $T_\Phi(H)$. Equation (A13) captures an essential effect of a finite field on the loop ''expansion'': fluctuations of field-induced s vortex lines result in the ''screening'' of the ''Biot-Savart'' interaction between the loops. This ''screening'' is manifested by $\mathbf{A}_d$ gaining a finite ''mass,'' $M_\parallel^2 \sim n_s/m_s c_s^2$, $M_\perp^2 \sim n_s/m_s$, after integration over $\delta\rho$ and $\vec{j}$. The effect of a finite magnetic field on the original problem (2) is now stored in the finite values of $M_\parallel^2$ and $M_\perp^2$. As one attempts to create a small number of very large loops in Eq. (A13), upon approaching $T_\Phi(H)$ from below, their effective line tension and mutual interactions will be essentially influenced by such ''screening.''

The importance of this mutual ''screening'' mechanism is particularly apparent in the dual representation (A8). The action (A13) becomes

$$\int d^3r \left[m_\Phi^2|\Phi_d|^2 + |(\nabla - i\mathbf{A}_d)\Phi_d|^2 + \frac{\tilde{g}_0}{2}|\Phi_d|^4 + i\,\delta\rho A_{dz} + i\vec{j}\cdot\vec{A}_d + \frac{m_s c_s^2}{2n_s}\delta\rho^2 + \frac{m_s}{2n_s}\vec{j}^2 + \mathcal{W} + \frac{1}{2q_d^2}(\nabla\times\mathbf{A}_d)^2 \right], \tag{A14}$$

where the meaning of the loop field operator $\Phi_d(\mathbf{r})$, m_Φ, and $\tilde{g}_0$ is evident in light of the discussion surrounding Eq. (A8). For simplicity, I suppress the anisotropy in the second (gradient) term of Eq. (A14) which generically arises [even if $\gamma_\parallel = \gamma_\perp$ in Eq. (1)] from the interaction of loops with s vortices. The finite mass of $\mathbf{A}_d$, generated by the integration over $\delta\rho$ and $\vec{j}$, ''cuts off'' the long-range ''Biot-Savart'' interactions present in zero field (A8). This ''screening'' causes a decoupling of the dual gauge field and transforms the critical behavior from a ''charged,'' Eq. (A8), to a ''neutral,'' Eq. (A14), dual superfluid. This is the inverted (anisotropic) 3D XY behavior of the dual theory, hinting at the presence of a *massless* gauge field in the *original* ''superconducting'' formulation (4), as discussed below Eq. (A8).

To investigate the critical behavior of Eq. (A13) in more detail, we generalize Eq. (A14) to arbitrary dimension D. Simultaneously, we restore in the action the residual part V_ρ, Eq. (A9), of the contact interaction between loops and s vortex lines, i.e., the part not already incorporated into the values of coupling constants appearing in Eq. (A14):

$$\int d^Dr \left[m_\Phi^2|\Phi_d|^2 + |(\partial_\mu - iA_\mu)\Phi_d|^2 + \frac{1}{2}g_0|\Phi_d|^4 + g_\rho|\Phi_d|^2 J_0 + iJ_\mu A^\mu + \frac{1}{2M_\mu^2}J_\mu J^\mu + \mathcal{W}(J_\mu) + \frac{1}{2q_d^2}F_{\mu\nu}F^{\mu\nu} \right]. \tag{A15}$$

Here g_ρ denotes V_ρ rescaled to its canonical dimension, $\mu=0,1,2,\ldots,D-1$, $\mathcal{W}(J_\mu)$ is the generalization of $\mathcal{W}(\delta\rho,\vec{j})$, and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. The functional integration runs over fields Φ_d, A_μ and J_μ and includes a constraint $\partial_\mu J^\mu=0$. The integration over J_μ generates a finite mass $M_{\mu=0}=M_\parallel$, $M_{\mu\neq 0}=M_\perp$ for the dual gauge field A_μ. The theory, however, retains gauge invariance implying that the combination $q_d A_\mu$ must be an RG invariant. This in turn sets the canonical dimension of g_ρ to $[g_\rho]=$ (length)$^{D-3}$. Therefore, the term $g_\rho|\Phi_d|^2 J_0$ is *irrelevant* within ϵ expansion around the upper critical dimension $D=4$. Similarly, all higher powers and derivatives of J_μ appearing in $\mathcal{W}$ are irrelevant as well. For example, the canonical dimension of the J_0^3 coupling constant is (length)$^{2D-3}$. The relevant couplings below $D=4$ are g_0, q_d^2, and $M_{\parallel,\perp}$. Due to finite $M_{\parallel,\perp}$, A_μ decouples and the β function for g_0 is the same as that of the neutral ($q_d=0$) complex Φ^4 theory. In this case the ϵ expansion is expected to hold down to, and include, $D=3$, where the critical behavior of the Φ^4 theory should be that of a (inverted) 3D XY model, in agreement with our earlier assertion. The same conclusions can also be reached within the perturbative RG in fixed dimension $D=3$. Here g_ρ is marginal at the ''engineering'' level, the gauge field again decouples due to finite $M_{\parallel,\perp}$, and we can compute the relevant β functions at the one-loop order and to the leading order in g_ρ:

$$\beta_0(\hat{g}_0,\hat{g}_\rho)\equiv\frac{d\hat{g}_0}{d\ln(p)}=-\hat{g}_0+\mathcal{C}_1\hat{g}_0^2,$$

$$\beta_\rho(\hat{g}_0,\hat{g}_\rho)\equiv\frac{d\hat{g}_\rho}{d\ln(p)}=\mathcal{C}_2\hat{g}_0\hat{g}_\rho, \tag{A16}$$

where $\hat{g}_{0,\rho}(p)$ are the dimensionless running coupling constants and $\mathcal{C}_{1,2}$ are (regularization-dependent) numerical constants which are *both* positive, $\mathcal{C}_{1,2}>0$. At the (inverted) 3D XY critical point $\hat{g}_0=1/\mathcal{C}_1$ and therefore $\beta_\rho>0$, indicating stability of our assumed $g_\rho=0$ fixed point against residual g_ρ perturbation. The above results allow us to conclude that the critical theory (A14) (with $\mathcal{W}=0$) remains valid and that the effects of V_ρ can be included by a proper choice of relevant couplings, as originally assumed. The presence of long-range interactions between vortices, mediated by $\mathbf{A}_d$, is essential for the validity of this argument. Note, however, that V_ρ and $\mathcal{W}$ terms (A13) are dangerously irrelevant operators since they break the up-down symmetry: they could change the critical behavior nonperturbatively or restore a first-order transition[9] in 3D.

One step remains: as $T\to T_\Phi(H)$, some overhangs attached to s vortex lines become very large and we might doubt the accuracy of the straightforward nonrelativistic boson analogy approximations below Eq. (A10). However, throughout the Φ-ordered state, the s vortices remain ''massive'' and there should always exist a suitably defined quantum system of nonrelativistic 2D bosons whose long ''distance'' (x,y) and ''imaginary time'' (z) behavior faithfully emulates that of s vortices. We therefore expect that the overall *symmetry* of Eq. (A13) remains preserved at $T_\Phi(H)$. This leads to a generalization of Eq. (A13):

$$Z(H)\to\sum_{N^{(0)}=0}^{\infty}\frac{1}{N^{(0)}!}\prod_{l_0=1}^{N^{(0)}}\oint\mathcal{D}_b\mathbf{x}_{l_0}[s_{l_0}]\int\mathcal{D}\mathbf{V}(\mathbf{r})\int\mathcal{D}\mathbf{A}_d(\mathbf{r})\times\exp(-S),$$

$$S=\sum_{l_0=1}^{N^{(0)}}\frac{\tilde{E}_c}{T}\int ds_{l_0}+\int d^3r\left\{\frac{\tilde{V}_0}{2T}d_0^2(\mathbf{r})+i\mathbf{n}_0\cdot\mathbf{A}_d+i\mathbf{V}\cdot\mathbf{A}_d+\frac{2\pi^2K_\parallel}{T}\mathbf{V}_\parallel^2+\frac{2\pi^2K_\perp}{T}\mathbf{V}_\perp^2+\frac{1}{2e_d^2}(\nabla\times\mathbf{A}_d)^2\right\}, \tag{A17}$$

where $\mathbf{V}(\mathbf{r})$ describes long-distance ($\gg l$) fluctuations of s vortex ''currents'' $\mathbf{n}_s(\mathbf{r})$, Eqs. (A4) and (A5), and satisfies $\nabla\cdot\mathbf{V}=0$. Here $K_{\parallel,\perp}/T$ now play the role of $m_s c_s^2/n_s$ and m_s/n_s in Eq. (A13) and fully include the effect of overhang configurations as $T\to T_\Phi(H)$. At present, we cannot compute $K_{\parallel,\perp}(T,H)$ (or $\tilde{E}_c$ and $\tilde{V}_0$) from first principles. This would require an analytic solution to the problem of large overhangs, something far beyond the scope of this paper. However, if we start with the general form (A17), we can determine various parameters appearing there by connecting them self-consistently to directly (numerically or experimentally) measurable physical quantities. For example, $K_{\parallel,\perp}(T,H)$ can be extracted from the components of the helicity modulus tensor (Appendix B) or the fluctuation conductivity (Sec. VI). We should therefore consider Eq. (A17) a self-consistent, perturbative RG description of the Φ transition.

We can now enforce the constraint $\nabla\cdot\mathbf{V}=0$ by introducing a gauge field $\mathbf{S}(\mathbf{r})$: $2\pi\mathbf{V}\to\nabla\times\mathbf{S}$, $\nabla\cdot\mathbf{S}=0$. Alternatively, we can integrate over $\mathbf{V}$, obtain the mass term for $\mathbf{A}_d$, and then decouple it by introducing $\mathbf{S}$. The final result is

$$Z(H)\to\sum_{N^{(0)}=0}^{\infty}\frac{1}{N^{(0)}!}\prod_{l_0=1}^{N^{(0)}}\oint\mathcal{D}_b\mathbf{x}_{l_0}[s_{l_0}]\times\int\mathcal{D}\mathbf{S}(\mathbf{r})\int\mathcal{D}\mathbf{A}_d(\mathbf{r})\exp(-S),$$

$$S=\sum_{l_0=1}^{N^{(0)}}\frac{\tilde{E}_c}{T}\int ds_{l_0}+\int d^3r\times\left\{\frac{\tilde{V}_0}{2T}d_0^2(\mathbf{r})+i\mathbf{n}_0\cdot\mathbf{A}_d+\frac{i}{2\pi}(\nabla\times\mathbf{S})\cdot\mathbf{A}_d+\frac{K_\parallel}{2T}(\nabla\times\mathbf{S})_\parallel^2+\frac{K_\perp}{2T}(\nabla\times\mathbf{S})_\perp^2+\frac{1}{2e_d^2}(\nabla\times\mathbf{A}_d)^2\right\}. \tag{A18}$$

This is just the vortex loop expansion (A6) and (A7) of the Meissner phase of a ''superconductor'' described by an order parameter $\Phi(\mathbf{r})$ and coupled to the gauge field $\mathbf{S}$ ($\nabla\cdot\mathbf{S}=0$). The GL functional of such a superconductor is

$$\mathcal{F}_{\rm eff}=\tilde{\alpha}|\Phi|^2+\tilde{\gamma}_\mu|(\nabla_\mu+i\mathbf{S}_\mu)\Phi|^2+\frac{\tilde{\beta}}{2}|\Phi|^4+\frac{K_\parallel}{2}(\nabla\times\mathbf{S})_\parallel^2+\frac{K_\perp}{2}(\nabla\times\mathbf{S})_\perp^2, \tag{A19}$$

which is precisely the gauge theory (4). Note that Φ is a dual of the loop field operator Φ_d (A14). Here $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}_\mu$ are some suitably renormalized GL coefficients which can be determined phenomenologically. Note that we have now restored the anisotropy in $\tilde{\gamma}_\mu$, arising both from the bare anisotropy $[\gamma_\parallel\neq\gamma_\perp$ in Eq. (1)] and the anisotropy induced by the interaction of loops with the *s* vortex background.

APPENDIX B: GAUGE THEORY AND THE HELICITY MODULUS

Here we consider the connection between $K_{\perp,\parallel}$ appearing in Eq. (4) and the helicity modulus tensor $\Upsilon(\mathbf{q})$. The conventional definition of the components of $\Upsilon(\mathbf{q})$ can be found in Ref. 30:

$$\Upsilon_{\mu\nu}(\mathbf{q})=V\frac{\delta^2 F}{\delta\mathbf{a}_\nu(\mathbf{q})\,\delta\mathbf{a}_\mu(-\mathbf{q})}. \tag{B1}$$

All quantities appearing in Eq. (B1) are defined in Sec. V, below Eq. (19). We will limit our attention to the isotropic case $[\gamma_\perp=\gamma_\parallel$ in Eq. (1)] whenever we consider the $H\neq 0$ situation. The anisotropic case with finite field requires far more extensive algebra and can be reconstructed by combining the discussion below with illuminating presentation in Ref. 30.

We first evaluate $\Upsilon(\mathbf{q})$ from the original GL theory (1) and start with the $H=0$ $(\mathbf{A}=0)$ case. We consider the situation right *above* T_{c0}; so there is no superconducting long-range order. After adding small $\mathbf{a}$ to the gradient term, we can expand the free energy up to second order in $\mathbf{a}$:

$$F[\nabla\times\mathbf{a}]=F[0]+\int d^3r\times\left[\frac{e^2}{2c^2}\chi_\perp(\nabla\times\mathbf{a})_\perp^2+\frac{e^2}{2c^2}\chi_\parallel(\nabla\times\mathbf{a})_\parallel^2\right]+\cdots. \tag{B2}$$

This equation requires a brief explanation: *e* is the real electric charge and *c* is the real speed of light, appearing in Eq. (1). Here $F[0]$ is just the original free energy of the GL theory with $H=0$ *and* $\mathbf{a}=0$. The full free energy with small $\mathbf{a}(\mathbf{r})$ is written as a functional of $\nabla\times\mathbf{a}$ *only*. This is required by the gauge invariance of Eq. (1). Two additional terms, proportional to the perpendicular and parallel components of $(\nabla\times\mathbf{a})^2$, represent the leading corrections in powers and derivatives of $\nabla\times\mathbf{a}$. The subleading contributions are denoted by the ellipsis. These subleading terms are unimportant in the long-wavelength limit.

$\chi_\perp$ and $\chi_\parallel$ are some functions of T and are different from each other in the anisotropic case, $\gamma_\parallel\neq\gamma_\perp$, while $\chi_\perp=\chi_\parallel=\chi$ if the superconductor is isotropic. As T approaches T_{c0} from above, these functions take the following form:

$$\chi_\perp=\mathcal{C}T\xi_\parallel,\quad \chi_\parallel=\mathcal{C}T\frac{\xi_\perp^2}{\xi_\parallel}. \tag{B3}$$

$\xi_{\perp,\parallel}=\xi_{0\perp,\parallel}t^{-\nu}$ are the superconducting correlation lengths and $\mathcal{C}$ is an unknown universal constant, intrinsic to the GL theory (1). Note that $e^2\chi_{\perp,\parallel}/c^2$ is just the perpendicular (parallel) magnetic susceptibility.

The components of the helicity modulus tensor in the long-wavelength limit $(q\rightarrow 0)$ are uniquely determined by $\chi_{\perp,\parallel}$. Conversely, the measurement of the long-wavelength "tilt" and "compression" helicity moduli[30] determines $\chi_{\perp,\parallel}$. In general, from definition (B1), we find

$$\frac{c^2}{e^2}\Upsilon_{\mu\nu}(\mathbf{q})=\chi\epsilon_{\rho\alpha\mu}\epsilon_{\rho\beta\nu}q_\alpha q_\beta, \tag{B4}$$

for the isotropic case, while for the anisotropic situation $\chi_\perp\neq\chi_\parallel$

$$\frac{c^2}{e^2}\Upsilon_{\mu\nu}(\mathbf{q})=(\chi_\parallel-\chi_\perp)\epsilon_{z\alpha\mu}\epsilon_{z\beta\nu}q_\alpha q_\beta+\chi_\perp\epsilon_{\rho\alpha\mu}\epsilon_{\rho\beta\nu}q_\alpha q_\beta. \tag{B5}$$

$\epsilon_{\alpha\beta\gamma}$ is the Levi-Cività symbol and summation over repeated indices is understood. e^2/c^2 is factored out for later convenience.

After this preliminary discussion, we go to the case of interest, finite H in Eq. (1), and limit our consideration to the isotropic situation $\gamma_\perp=\gamma_\parallel$. We introduce small $\mathbf{a}$ into Eq. (1) and expand to second order in $\nabla\times\mathbf{a}$:

$$F[H,\nabla\times\mathbf{a}]=F[H,0]+\int d^3r\times\left[\frac{e^2}{2c^2}\tilde{\chi}_\perp(\nabla\times\mathbf{a})_\perp^2+\frac{e^2}{2c^2}\tilde{\chi}_\parallel(\nabla\times\mathbf{a})_\parallel^2\right]+\cdots, \tag{B6}$$

where $F[H,0]$ is now the free energy of the GL theory (1) at *finite* field $\mathbf{H}$ and $\mathbf{a}=0$. We are again focusing on the "normal," i.e., not *superconducting* state, in accordance with assumption (ii) of Sec. II. The above expression looks very much like Eq. (B2) but there are the following significant differences: the expansion is anisotropic, $\tilde{\chi}_\perp\neq\tilde{\chi}_\parallel$, even though our superconductor is isotropic. The reason for this is finite field $\mathbf{H}$ along the *z* axis which reduces spherical symmetry of the $H=0$ situation down to cylindrical. The finite field also breaks the "up-down" symmetry along the *z* axis. This is manifested by the subleading corrections, denoted by the ellipsis in Eq. (B6), containing in general *odd* powers of $\nabla\times\mathbf{a}$ (the leading such term is cubic); such terms were prohibited by symmetry in the $H=0$ case. Again, by combining definition (B1) and Eq. (B6), we arrive at the expression for the long-wavelength limit of the helicity modulus:

$$\frac{c^2}{e^2}\Upsilon_{\mu\nu}(\mathbf{q})=(\tilde{\chi}_\parallel-\tilde{\chi}_\perp)\epsilon_{z\alpha\mu}\epsilon_{z\beta\nu}q_\alpha q_\beta+\tilde{\chi}_\perp\epsilon_{\rho\alpha\mu}\epsilon_{\rho\beta\nu}q_\alpha q_\beta. \tag{B7}$$

The components of the helicity modulus tensor for the finite field (isotropic) case are determined by $\tilde{\chi}_{\perp,\parallel}$ which are some functions of T and H. It is tempting to conclude that

$$\tilde{\chi}_{\perp}\to\mathcal{C}T\xi_{\parallel}(T,H),\quad \tilde{\chi}_{\parallel}\to\mathcal{C}T\frac{\xi_{\perp}^2(T,H)}{\xi_{\parallel}(T,H)},\tag{B8}$$

where $\xi_{\perp,\parallel}(T,H)$ are now superconducting correlation lengths at finite field. This result is plausible on physical grounds, expressing the fact that, with $H\neq 0$, the superconducting correlation lengths are now finite in the ''liquid'' phase, limited by magnetic length l, and consequently the helicity moduli vanish in the $q\to$ limit. We are simply making the assumption that the same length that limits the range of superconducting correlations appears in the coefficient of the q^2 term in the helicity modulus; this assumption is known to be correct for the $H=0$ case (B3). Unfortunately, I am unable to provide a mathematical proof that the conjectured result (B8) is exact. Instead, I *define* perpendicular and parallel ''screening'' lengths $\Lambda_{\perp}(T,H)$ and $\Lambda_{\parallel}(T,H)$ by

$$\tilde{\chi}_{\perp}=\mathcal{C}T\Lambda_{\parallel}(T,H),\quad \tilde{\chi}_{\parallel}=\mathcal{C}T\frac{\Lambda_{\perp}^2(T,H)}{\Lambda_{\parallel}(T,H)}.\tag{B9}$$

Note that there is a one-to-one correspondence between these ''screening'' lengths $\Lambda_{\perp,\parallel}$ and $\chi_{\perp,\parallel}$ and, in turn, between $\Lambda_{\perp,\parallel}$ and the long-wavelength helicity moduli (B7). Since $\Lambda_{\perp,\parallel}$ are purely thermodynamic quantities, they satisfy scaling laws of Sec. IV, just like the superconducting correlation lengths:

$$\Lambda_{\perp,\parallel}(T,H)=l\mathcal{R}^{\Lambda}_{\perp,\parallel}(q_0^2),\quad \xi_{\perp,\parallel}(T,H)=l\mathcal{R}^{\xi}_{\perp,\parallel}(q_0^2),\tag{B10}$$

where the ''dimensionless charge'' $q_0^2=\xi(T,H=0)/l\propto H^{1/2}/|t|^{\nu}$is our scaling variable of Eq. (17) and $\mathcal{R}^{\Lambda}_{\perp,\parallel}$ and $\mathcal{R}^{\xi}_{\perp,\parallel}$ are the screening length and correlation length scaling functions, respectively. In the $H\to 0$ ($q_0^2\to 0$) limit, all these scaling functions $\mathcal{R}^{\Lambda}_{\perp,\parallel}(q_0^2)$ and $\mathcal{R}^{\xi}_{\perp,\parallel}(q_0^2)$ go as q_0^2. We now make the following *assumption*: around $T_{\Phi}(H)$, the ratios

$$\frac{\Lambda_{\perp}(T,H)}{\xi_{\perp}(T,H)}=\frac{\mathcal{R}^{\Lambda}_{\perp}(q_0^2)}{\mathcal{R}^{\xi}_{\perp}(q_0^2)},\quad \frac{\Lambda_{\parallel}(T,H)}{\xi_{\parallel}(T,H)}=\frac{\mathcal{R}^{\Lambda}_{\parallel}(q_0^2)}{\mathcal{R}^{\xi}_{\parallel}(q_0^2)}\tag{B11}$$

are some unremarkable *smooth* functions of the scaling variable q_0^2. In particular, the nonanalytic drop in the coefficient of the q^2 term in the helicity modulus, which takes place as we cross the $T_{\Phi}(H)$ transition line (21) and which is directly reflected as a nonanalytic decrease in the screening lengths $\Lambda_{\perp,\parallel}(T,H)$, is manifested also in the superconducting correlation lengths $\xi_{\perp,\parallel}(T,H)$. This assumption, which appears justified physical grounds, was used in the discussion of fluctuation conductivity (Sec. VI).

Finally, we are in position to discuss our anisotropic gauge theory of Eq. (4). A small $\mathbf{a}$ added to the external vector potential $\mathbf{A}$ in the original GL theory (1) translates into a small vector potential $(e/c)\mathbf{a}$ added to our fictitious gauge field $\mathbf{S}$ in Eq. (4). Since we are integrating over $\mathbf{S}$, it is useful to define new gauge field $\mathbf{S_n}=\mathbf{S}+(e/c)\mathbf{a}$ and integrate over $\mathbf{S_n}$ in the partition function. The effect of this is to move $(e/c)\mathbf{a}$ from the covariant gradient terms $|D_{\mu}\Phi|^2$ to the ''gauge field energy'' $K_{\mu}[\nabla\times(\mathbf{S}-(e/c)\mathbf{a})]^2_{\mu}$ in Eq. (4). We now expand the free energy of the gauge theory, $F_{\rm eff}$, to second order in $\nabla\times\mathbf{a}$, following the same philosophy as in Eq. (B6):

$$F_{\rm eff}[\tilde{e}_{\perp},\tilde{e}_{\parallel},\nabla\times\mathbf{a}]=F_{\rm eff}[\tilde{e}_{\perp},\tilde{e}_{\parallel},0]+\int d^3r\left[\frac{e^2}{2c^2}\mathcal{K}_{\perp}(\nabla\times\mathbf{a})^2_{\perp}+\frac{e^2}{2c^2}\mathcal{K}_{\parallel}(\nabla\times\mathbf{a})^2_{\parallel}\right]+\cdots,\tag{B12}$$

where

$$\mathcal{K}_{\perp,\parallel}=K_{\perp,\parallel}-\frac{K^2_{\perp,\parallel}}{T}\lim_{q\to 0^+}\int d^3(\mathbf{r}-\mathbf{r}')e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\times\langle(\nabla\times\mathbf{S}(\mathbf{r}))_{\perp,\parallel}(\nabla'\times\mathbf{S}(\mathbf{r}'))_{\perp,\parallel}\rangle.\tag{B13}$$

The anisotropic charges $\tilde{e}_{\perp,\parallel}(T,H)$ and coupling constants $K_{\perp,\parallel}(T,H)$ are defined in Eqs. (15) and (4), respectively. The thermal average $\langle\cdots\rangle$ is over the gauge theory defined by the free energy functional $\mathcal{F}_{\rm eff}$, Eq. (4). Combining Eqs. (B1) and (B13), we get, as before

$$\frac{c^2}{e^2}\Upsilon_{\mu\nu}(\mathbf{q})=(\mathcal{K}_{\parallel}-\mathcal{K}_{\perp})\epsilon_{z\alpha\mu}\epsilon_{z\beta\nu}q_{\alpha}q_{\beta}+\mathcal{K}_{\perp}\epsilon_{\rho\alpha\mu}\epsilon_{\rho\beta\nu}q_{\alpha}q_{\beta}.\tag{B14}$$

Comparing this to the general expression for the helicity modulus of the GL theory at finite field, given by Eq. (B7), we conclude that $\tilde{\chi}_{\perp,\parallel}=\mathcal{K}_{\perp,\parallel}$.

A *very* important point is that *below* $T_{\Phi}(H)$ we have $\mathcal{K}_{\perp,\parallel}=K_{\perp,\parallel}$. This is a mathematical consequence of the following physical picture (Sec. V). In the Φ-ordered state, *only* the field-induced *(s)* vortex lines can ''screen'' the test vector potential $\mathbf{a}(\mathbf{r})$. All thermally generated vortex loops are of *finite* size and cannot contribute to the ''screening'' in the long-wavelength limit, described by $\mathcal{K}_{\perp,\parallel}$ in Eq. (B12). This implies that the new order parameter Φ is *finite* and therefore $\langle(\nabla\times\mathbf{S})^2_{\perp,\parallel}\rangle$, Eq. (B13), *vanishes* in the long-wavelength limit, as $\sim q^2/|\langle\Phi\rangle|^2$. Therefore, we can *uniquely fix* the coupling constants $K_{\perp,\parallel}(T,H)$ [and corresponding anisotropic charges $\tilde{e}_{\perp,\parallel}(T,H)$] that enter the gauge theory description (4), by connecting them directly to the components of the helicity modulus tensor right below $T_{\Phi}(H)$ [or, more precisely, for $T_{\Phi}(H)-T\to 0^+$]. In particular, the functions $c_{\perp,\parallel}(T,H)$, introduced below Eq. (4) and referred to at various points in the main text, follow from the general expression (B9):

$$c_{\perp}(T,H)=\mathcal{C}\mathcal{R}^{\Lambda}_{\parallel}(q_0^2),\quad c_{\parallel}(T,H)=\mathcal{C}\frac{[\mathcal{R}^{\Lambda}_{\perp}(q_0^2)]^2}{\mathcal{R}^{\Lambda}_{\parallel}(q_0^2)},\tag{B15}$$

where $\mathcal{R}^{\Lambda}_{\perp,\parallel}(q_0^2)$ are the scaling functions introduced in Eq. (B10) and are to be evaluated below $T_{\Phi}(H)$. The gauge theory scenario predicts that the *fundamental anisotropy* ratio $c_{\parallel}/c_{\perp}$ takes on a universal value along $T_{\Phi}(H)$.

Above $T_{\Phi}(H)$, in the true normal state, infinite vortex loops proliferate across the system and *can* contribute to screening. $\langle(\nabla\times\mathbf{S})^2_{\perp,\parallel}\rangle$ in Eq. (B13) becomes finite and $\sim 1/\xi_\Phi$. This causes the nonanalytic drop in the q^2 term of the helicity modulus and the corresponding screening lengths (B9), just as discussed in Sec. V [see Eq. (21)].

The reader should note that the set of results presented in this appendix, connecting the properties of the gauge theory description (4) around $T_\Phi(H)$ to the general long-distance form of the helicity modulus and screening lengths of the GL theory at finite field (1), is not only physically transparent and appealing but also exact, provided our assumptions (i)–(iv) (Sec. II) are satisfied.

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[25] $\mathbf{U}(\mathbf{r})$ is the vector potential associated with a given configuration of *s* vortices, with unit fluxes attached. In the notation of Ref. 9, $\mathbf{U}$ would correspond to $\nabla\Theta$, where Θ is the phase of the gauge transformation used in that paper. An explicit form of Θ is, however, not uniquely determined by the positions of *s* vortices: one must also specify the positions of branch cuts across which the phase changes from 0 to 2π. The present formulation eliminates this unnecessary complication since $\mathbf{U}$ itself *is* uniquely determined for any given configuration of *s* vortices.

[26] See, for example, V. N. Popov, *Functional Integrals and Collective Excitations* (Cambridge University Press, Cambridge, England, 1987), Chaps. 7 and 8.

[27] There is, however, no simple correspondence at the level of correlation functions. The equivalence between the two representations, Eqs. (1) and (2), works only for the full partition functions and not at the level of individual configurations. This is clear since, for a given configuration of Z (1) with $N_\Phi+N_a$ vortex lines ''percolating'' across the system along $\mathbf{H}$, we cannot tell which ones are field induced and which are thermally generated. All we know is that each configuration must have at least N_Φ of such vortex lines. For example, this precludes a direct connection between Φ and the original Ψ.

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[33] Extreme anisotropy and potential finite ''mass terms'' have been considered in Ref. 9. For example, footnote 20 of that paper considers the possibility of a finite helicity modulus $\|\mathbf{H}$ immediately below $T_\Phi(H)$.

[34] As will be discussed later in the text, the Φ transition in the gauge theory (4) is governed by an *isotropic* critical point. However, with $K_\perp/K_\|\rightarrow\infty$ and, potentially, finite $M_\|^2\mathbf{S}_\|^2$ and/or Chern-Simons terms, the same transition could be governed by an *anisotropic* critical point, with a different set of critical exponents in $\|$ and $\perp$ directions and correspondingly different details of critical thermodynamics and transport.

[35] ''Perturbed'' comes with quotation marks since, even though the magnetic field enters Eq. (4) through a small parameter $\tilde{e}$, this perturbation is strongly relevant and it immediately destabilizes the zero-field (''neutral'') 3D *XY* critical point.

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[53] Unlike the mean-field-based approach including only the field-induced London vortex lines, where the strength of this interaction is set by the mean-field *amplitude* squared of $\Psi(\mathbf{r})$.

[54] Equation (11) for $t_\Phi(H)$ differs from the estimate reported in Ref. 9 in two ways: first, the previous estimate was made for the weakly coupled layered case and, second, it was based on the mean-field approximation to Eq. (2). In contrast, the present expression (11) is exact, provided d_Φ is specified.

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[57] While the rescaling procedure described here is a straightforward generalization of the familiar $H=0$ rescaling of anisotropy, it does contain one salient point: the rescaled gauge theory looks just like the original Eq. (4) but with the Φ-dependent part isotropic, $K_{\perp,\|} \rightarrow K'_{\perp,\|}$, $\mathbf{r} \rightarrow \mathbf{r}' = (\Gamma^{1/3} r_\perp, \Gamma^{-2/3} r_\|)$, and $\mathbf{S} \rightarrow \mathbf{S}' = (\Gamma^{-1/3} S_\perp, \Gamma^{2/3} S_\|)$. However, the gauge is *different*: instead of the original Coulomb gauge $\nabla \cdot \mathbf{S} = 0$ we now have $\Gamma^2 (\nabla' \cdot \mathbf{S}')_\perp + (\nabla' \cdot \mathbf{S}')_\| = 0$. This is dealt with by introducing a gauge transformation: $\mathbf{S}' \rightarrow \mathbf{S}'' = \mathbf{S}' - \nabla' \Pi$ and $\Phi \rightarrow \Phi'' = \Phi \exp i\Pi$, where $\Pi(\mathbf{r}')$ is a regular phase defined so that the new gauge potential $\mathbf{S}''$ satisfies the Coulomb gauge condition, i.e., $\nabla' \cdot \mathbf{S}'' = \nabla' \cdot \mathbf{S}' - \nabla' \cdot \nabla' \Pi = 0$. Such gauge transformation is both perfectly justified and desirable. It is justified because, even though our fictitious electrodynamics (4) is always defined in a *specific* gauge and does not possess the local gauge *freedom* of its real namesake, its free energy is still invariant under gauge transformations. It is desirable since the Coulomb gauge ensures that Φ (or Φ'' in the anisotropic case) can be used as a true physical order parameter of our Φ transition: see T. Kennedy and C. King, Phys. Rev. Lett. **55**, 776 (1985). Related issues involving gauge transformations are discussed in Ref. 9. In the isotropic case [$\gamma_\perp = \gamma_\|$ in Eq. (1)] none of these niceties are necessary and we can stay in the Coulomb gauge throughout. The fundamental anisotropy of the gauge theory, Eq. (4), expressed as $K_\perp \neq K_\|$ $(c_\perp \neq c_\|)$, always remains.

[58] One naively expects the range of fields over which d_m is universal to be smaller than the corresponding range for d_Φ: the melting line is first order and involves a nonuniform low-temperature state.

[59] The exponent p is *assumed* to have the same value, $1 > p \cong 1/2$, throughout the Φ-ordered phase. We can show that this is the case in the regime where the average size of vortex loops and s vortex overhangs is $\ll l$ and we assume that it remains true everywhere below $T_\Phi(H)$. This is consistent with our physical expectation that the long-wavelength limit of the Φ-ordered phase can be related to a suitably defined ''line liquid,'' in the sense of Nelson (Ref. 8). An alternative possibility, that of p being a continuously varying exponent, changing from $p \cong 1/2$

far below $T_{\Phi}(H)$ to $p \to 1$ as we approach $T_{\Phi}(H)$, appears less likely in view of the overall picture painted by the gauge theory scenario. For example, the Φ-ordered state of Eq. (4) has a true long-range order and is not a critical phase with continuously varying exponents.

[60] G. W. Crabtree and D. R. Nelson, Phys. Today **50**(4), 38 (1997).

[61] ''Vortex tachyons'' denote infinite vortex paths that traverse the system along x or y direction without winding along the z axis. Thus, in the boson analogy, they travel longer intervals across ''space'' (xy plane) than along ''time'' (z axis $\|\mathbf{H}$).

[62] In principle, however, such a phase could result in a new transition line between $T_{\Phi}(H)$ and $T_m(H)$, at least in a certain window of material parameters.

[63] J. Lidmar, M. Wallin, C. Wengel, S. M. Girvin, and A. P. Young, Phys. Rev. B **58**, 2827 (1998), and references therein.

[64] For a useful summary of various anisotropy effects, see E. Sardella, Phys. Rev. B **53**, 14 506 (1996), and references therein.

[65] The same would be true for the LL-based theories, if uncritically extended to the $H \to 0$ limit.

[66] Eventually, for extremely low fields, the effect of electromagnetic screening (i.e., finite κ) becomes important and $T_m(B) \to 0$ as $B \to 0$ (Ref. 8).

[67] A. K. Nguyen and A. Sudbø (unpublished and private communication). Also, see S.-K. Chin, A. K. Nguyen, and A. Sudbø, cond-mat/9809115 (unpublished). The author is grateful to Prof. A. Sudbø and Dr. A. K. Nguyen for numerous discussions and the opportunity to see their unpublished numerical results.

[68] L. P. Pitayevski, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].

[69] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 1995), Chap. 9.

[70] G. Parisi, *Statistical Field Theory* (Addison-Wesley, New York, 1988), Chap. 16.