Metallic nanosphere in a magnetic field: An exact solution

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We consider an electron gas moving on the surface of a sphere in a uniform magnetic field. An exact solution of the problem is found in terms of oblate spheroidal functions, depending on the parameter $p=\Phi/\Phi_0$, the number of flux quanta piercing the sphere. The regimes of weak and strong fields are discussed, and the Green's functions are found for both limiting cases in closed form. In weak fields the magnetic susceptibility reveals a set of jumps at half-integer p. The strong-field regime is characterized by the formation of Landau levels and localization of the electron states near the poles of the sphere defined by a direction of the field. The effects of coherence within the sphere are lost when its radius exceeds the mean free path. [S0163-1829(99)12905-9]

I. INTRODUCTION

The electronic properties of cylindrical and spherical carbon macromolecules[1] have attracted much theoretical interest recently. A large part of these studies is devoted to band structure calculations[2] and to the effects of topology for the transport and mechanical properties of these nanostructures. The main interest in the topological aspects is connected with the carbon nanotubes, where one can investigate the effective models, based on the band calculations and incorporating the particular geometry of the object.[3]

At the same time, the problem of the topology appears to be a general one and may be studied independently from the physics of carbon materials. Recent advances in technology[4] let one think of a wider class of the spherical nanostructures, e.g., spheres coated by metal films,[5] whose properties may differ from those of planar objects.

In this paper we study a gas of electrons moving within a thin spherical layer in an applied magnetic field. We find the exact solution of this problem in terms of oblate spheroidal functions. This physical application of the theory of spheroidal functions was not discussed previously.[6–8] We show the jumps in the susceptibility of the system in ''weak'' fields and the localization of the electronic states in ''strong'' fields. The last effect could be experimentally investigated for the hemispherical tips of nanotubes.[9] At intermediate fields the sophisticated structure of the functions makes an analytical treatment impossible and numerical methods should be used for an analysis of observable quantities.

II. SETTING UP THE PROBLEM

Let us consider electrons moving on a surface of a sphere of radius r_0. In the presence of a uniform magnetic field $\mathbf{B}$ we choose the gauge of the vector potential as a vector product $\mathbf{A}=\frac{1}{2}(\mathbf{B}\times\mathbf{r})$. The Hamiltonian of the system is given by

$$\mathcal{H}=\frac{1}{2m_e}(-i\nabla+e\mathbf{A})^2+U(r), \tag{1}$$

where m_e is the (effective) mass of an electron and $-e$ its charge; we have set $\hbar=c=1$ and omitted the trivial term connected with the spin of the particle. We assume that the total number of particles, N (with one projection of spin), is fixed and defines the value of the chemical potential μ and the areal density $\nu=N/(4\pi r_0^2)$. The confining potential $U(r)=0$ at $r_0<r<r_0+\delta r$ and $U(r)\to\infty$ otherwise. We focus our attention on the limit $\delta r\ll r_0$, when the variables of the problem are separated. The radial component $R(r)$ of the wave function is a solution of the Shrödinger equation with the quantum well potential. Henceforth we ignore the radial component and put $r=r_0$ in the remaining angular part of the Hamiltonian $\mathcal{H}_\Omega$; it can be done if μ lies below the first excited level of $R(r)$, which in turn means $\delta r\lesssim\nu^{-1/2}$. We choose the direction of the field $\mathbf{B}$ along the $\hat{\mathbf{z}}$ axis and as a north pole of the sphere ($\theta=0$) and look for the eigenfunctions to the Schrödinger equation $\mathcal{H}_\Omega\Psi=E\Psi$ in the form $\Psi(\theta,\phi)=S(\theta)e^{im\phi}$. Defining the dimensionless energy $\varepsilon=2m_e r_0^2 E$ and setting $\eta=\cos\theta$, we write

$$\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial S}{\partial\eta}+\left[\varepsilon-2mp-\frac{m^2}{1-\eta^2}-p^2(1-\eta^2)\right]S=0. \tag{2}$$

We introduced here the important dimensionless parameter

$$p=eBr_0^2/2=\pi Br_0^2/\Phi_0=r_0^2/(2l_*^2)=m_e r_0^2\omega_c/2, \tag{3}$$

with the magnetic flux quantum $\Phi_0=2\pi/e=2\times10^{-15}$ T m^2, magnetic length $l_*=(eB)^{-1/2}$, and cyclotron frequency $\omega_c=eB/m_e$. Note that for a sphere of radius $r_0=10$ nm one has $p=1$ at the field $B\simeq6$ T.

Equation (2) is known as the spheroidal differential equation and was extensively studied previously.[6–8] The solutions to it are given by oblate (angular) spheroidal functions $S_{lm}(p,\eta)$ with the corresponding eigenvalues $\varepsilon_{lm}(p)$.

It is known that spheroidal functions belong to the simplest class of special functions which are not essentially hypergeometric ones. For spheroidal functions there are no recurrence relations, generating function representations, etc., which are characteristic for classical special functions. The spectrum $\varepsilon_{lm}(p)$ is found as the eigenvalues of infinite matrices. For different sets of functions orthogonal on the interval $(-1,1)$, these matrices are reduced to tridiagonal form and could be further analyzed within the chain fraction formalism or by explicit (numerical) diagonalization.[6–8] Perhaps, the most known quantum-mechanical application of the spheroidal functions is the problem of an electron in a two-center Coulomb potential (H_2^+ molecule).

III. WEAK FIELD, SUSCEPTIBILITY

Let us first concentrate on the case of the weak field. In the absence of the field, the spectrum is that of a free rotator model and the solutions to (2) are the associated Legendre polynomials[14]

$$\varepsilon_{lm}(0)=l(l+1),$$

$$S_{lm}(0,\eta)=\sqrt{\frac{2l+1}{4\pi r_0^2}\frac{(l-|m|)!}{(l+|m|)!}}P_l^m(\eta). \tag{4}$$

According to Eqs. (4) the wave functions are the spherical harmonics $\Psi_{lm}(\theta,\phi)=r_0^{-1}Y_{lm}(\theta,\phi)$ and are normalized on the surface of the sphere, $r_0^2\int|\Psi|^2\sin\theta d\theta d\phi=1$. This normalization facilitates the comparison of our results with the case of planar geometry.

For $p\neq 0$ one develops perturbation theory around the initial wave functions (4) as long as $p^2\leqslant 4l$.[6–8] The energies and wave functions in this case are given by a series in p^2:

$$\varepsilon_{lm}(p)=l(l+1)+2pm+\frac{p^2}{2}\left[1+\frac{m^2}{l^2}\right]+O\left[\frac{p^2}{l}\right], \tag{5}$$

$$S_{lm}(p,\eta)\propto P_l^m(\eta)+P_{l\pm 2}^m(\eta)O[p^2/l]. \tag{6}$$

To clarify the criterion $p^2/l\lesssim 1$ we consider the case when the field ceases to be small for electrons near the Fermi level, $l\simeq r_0\sqrt{4\pi\nu}$, mostly contributing to the physical properties. This case corresponds to the relation

$$Br_0^3\sim[\hbar c/e^2]^2e^2\sqrt{\nu}\sim 137^2\omega_{pl}.$$

Hence the field cannot be treated as a perturbation if the energy of the magnetic field in the volume of the sphere is 10^4 times larger than the characteristic plasma frequency. For densities $\nu\sim 10^{14}$ cm^{-2} and $r_0=10$ nm it corresponds to fields greater than 40 T.

For the weak-field regime it is interesting to observe the following property of the spectrum (5). For simplicity we consider the situation when the Lth unperturbed level is completely filled and the $(L+1)$th level is empty. Linearizing the spectrum we have $\varepsilon_{lm}\simeq 2L(l-L-1/2+pm/L)$. It is clear that at $p>1/2$ the state $(l=L+1, m=-L-1)$ is energetically more favorable than the state $(l=L, m=L)$, with the resulting change in the occupation of the levels.

This phenomenon is accompanied by jumps in the static susceptibility. Consider the free energy $F=-T\Sigma_{lm}\ln(e^{(\mu-E_{lm})/T}+1)$. Then, apart from the Pauli spin contribution, the magnetic (differential) susceptibility $\chi=\partial M/\partial B=-\partial^2F/\partial B^2$ is given by

$$\chi=-\frac{\mu_B^2 m_e r_0^2}{2}\sum_{lm}\left[\frac{\partial^2\varepsilon_{lm}}{\partial p^2}n_F(\varepsilon_{lm})+\left(\frac{\partial\varepsilon_{lm}}{\partial p}\right)^2 n_F'(\varepsilon_{lm})\right],$$

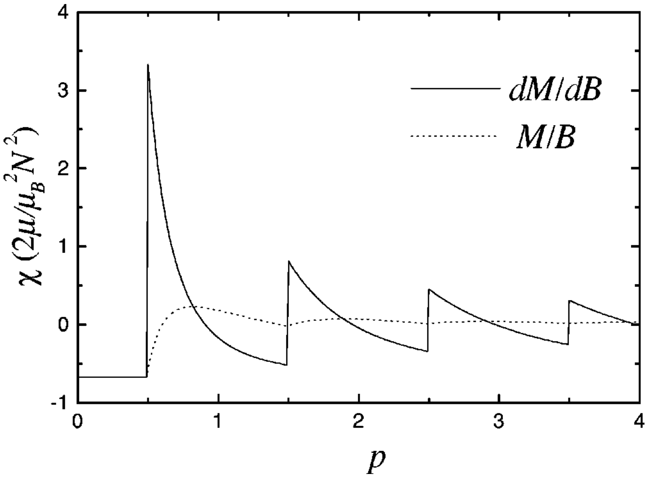

FIG. 1. The dependence of the static susceptibility χ on the number p of magnetic flux quanta piercing the sphere.

with the Bohr magneton $\mu_B=e/2m_e$, the Fermi function $n_F(\varepsilon)$, and $n_F'(\varepsilon)$ its derivative. The first term here gives the diamagnetic contribution $\chi^d\simeq -2/3\mu_B^2 m_e r_0^2 N$ and the second term is the paramagnetic one. At low temperatures $\omega_c\lesssim T\ll\mu$ we let $n_F'(E_{lm})\simeq -\delta(\varepsilon_{lm}-\mu)$ and change $\Sigma_m\simeq\int_{-l}^{l}dm$. Performing the integration we get

$$\chi=N^2(\mu_B^2/2\mu)\left[-2/3+p^{-3}\sum_l (l-L-1/2)^2\right], \tag{7}$$

where the summation is restricted by $|l-L-1/2|<p$. From Eq. (7) we see that χ exhibits jumps at $p=1/2,3/2,5/2\ldots$, while $\chi\to 0$ in the formal limit $p\to\infty$. This behavior is shown in the Fig. 1, together with the quantity M/B. At the other fillings, i.e., at the other values of μ, the jumps of χ take place at other values of p and the qualitative picture remains true. The amplitude of the jumps is N times larger than the Pauli spin susceptibility $\sim N\mu_B^2/\mu$; this coherent effect vanishes if the coherence on the sphere is lost due to the finite quasiparticle lifetime (see below).

IV. WEAK FIELD, GREEN'S FUNCTION

Let us discuss the properties of the electron Green's function. For the two points on the sphere, $\mathbf{r}\leftrightarrow(\theta,0)$, and $\mathbf{r}'\leftrightarrow(\theta',\phi)$, we define

$$G(\mathbf{r},\mathbf{r}',\omega)=\sum_{lm}\frac{\Psi_{lm}^*(\theta',\phi)\Psi_{lm}(\theta,0)}{\omega+\mu-E_{lm}}, \tag{8}$$

with $E_{lm}=(2m_e r_0^2)^{-1}\varepsilon_{lm}$.

In the absence of a magnetic field $\Psi_{lm}(\theta,\phi)=r_0^{-1}Y_{lm}(\theta,\phi)$ and one can find (see below) an exact representation of G through the Legendre function

$$G(\omega)_{B=0}\equiv G^0(\omega)=-\frac{m_e}{2\cos\pi a}P_{-1/2+a}(-\cos\Omega), \tag{9}$$

where we introduced $a=\sqrt{2m_e r_0^2(\mu+\omega)+1/4}$ and the distance Ω on the sphere:

$$\cos\Omega=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi.$$

We immediately see from Eq. (9) the logarithmic singularity in the one-point correlator ($\Omega=0$) as it should be. It is instructive to find out how the usual expressions for the planar geometry are recovered for the large radius of the sphere, $r_0\to\infty$; in this case $a\propto r_0$, while $\Omega\to|\mathbf{r}-\mathbf{r}'|/r_0\equiv r/r_0$. In the limit $a\gg 1$ and for $a\sin\Omega\gtrsim 1$ one has, in the main order of a^{-1},[10]

$$G^0(\omega)\simeq-\frac{m_e}{\sqrt{2\pi a\sin\Omega}}\frac{\cos(a\pi-a\Omega-\pi/4)}{\cos\pi a}. \quad (10)$$

The existence of two oscillating exponents $\exp\pm i(a\pi-a\Omega-\pi/4)$ corresponds to quantum coherence of two waves. One propagates along the shortest path between two points and another wave goes along the longest path, turning around the sphere. In the theory of metals one considers $|\omega|\ll\mu=k_F^2/(2m_e)$, while the finite quasiparticle lifetime can be modeled by ascribing the imaginary part to ω. We write in this sense $\sqrt{2m_e\omega}=k_F+i/l_{mfp}$ with the mean free path l_{mfp}. When $r_0\gtrsim l_{mfp}$, we see that the coherence breaks and the only surviving exponent in Eq. (10) has the form

$$G^0_{damped}(\omega)\simeq-\frac{m_e}{\sqrt{2\pi k_F r}}\exp\left[ik_F r+i\frac{\pi}{4}-\frac{r}{l_{mfp}}\right], \quad (11)$$

which expression coincides with the usual findings.[11]

It is possible to obtain the closed form of the Green's function in the weak-field regime. We sketch the corresponding derivation below.

First we neglect the p^2/l terms in the wave function (6) and write $Y^*_{lm}(\theta',\phi)Y_{lm}(\theta,0)=(4\pi)^{-1}(2l+1)e^{-im\phi}P_l^m(\cos\theta)P_l^{-m}(\cos\theta')$. Using Eq. (5) and dropping the terms containing $p^2/a\sim p^2/l$, we have

$$\frac{2l+1}{\omega-\varepsilon_{lm}}\simeq\left[a+\frac{pm}{a}-l-\frac{1}{2}\right]^{-1}-\left[a+\frac{pm}{a}+l+\frac{1}{2}\right]^{-1}.$$

Now we represent the appearing fraction as the Taylor series

$$(a+pm/a-z)^{-1}=\exp\left(\frac{pm}{a}\frac{\partial}{\partial a}\right)(a-z)^{-1}$$

and substitute m by $i\partial/\partial\phi$ in the last exponent. After that we can sum over m in Eq. (8) (Ref. 10) and represent the remaining sum over l as the contour integral, to obtain

$$G(\omega)=\exp\left[i\frac{p}{a}\frac{\partial^2}{\partial a\partial\phi}\right]\oint\frac{d\nu}{2\pi i}\frac{m_e}{2\sin\pi\nu}\frac{P_\nu(-\cos\Omega)}{a-\nu-1/2}$$
$$=\exp\left[i\frac{p}{a}\frac{\partial^2}{\partial a\partial\phi}\right]G^0(\omega), \quad (12)$$

with $G^0(\omega)$ given by Eq. (9). When $a\gg 1$ the expression (12) could be simplified at the above condition $a\sin\Omega\gtrsim 1$. In this case we use Eq. (10) and observe that, upon differentiating over ϕ, the main contribution of order of a stems from the numerator $\exp(\pm ia\Omega)$ of Eq. (10). Then one uses the identity

$$\exp\left[\frac{z}{a}\frac{\partial}{\partial a}a\right]=\frac{1}{a}\exp\left[z\frac{\partial}{\partial a}\right]a$$

and finds

$$G(\omega)\simeq-\frac{m_e}{2}\frac{\exp[ip\beta(\pi-\Omega)]}{\sqrt{2\pi a\sin\Omega}}\left[\frac{e^{ia(\pi-\Omega)-i\pi/4}}{\cos\pi(a+p\beta)}+\frac{e^{-ia(\pi-\Omega)+i\pi/4}}{\cos\pi(a-p\beta)}\right], \quad (13)$$

with $\beta=\sin\theta\sin\theta'\sin\phi/\sin\Omega$. As before, in the presence of damping $r_0>l_{mfp}$, the coherence on the sphere breaks and the only surviving term in Eq. (13) has the form

$$G_{damped}(\omega)\simeq\exp(i\hat{\mathbf{z}}(\mathbf{r}'\times\mathbf{r})/2l_*^2)G^0_{damped}(\omega), \quad (14)$$

in accordance with previous results (see, e.g., Ref. 12).

V. STRONG-FIELD REGIME

Let us now turn to the case of strong fields, $p\to\infty$. We define the integer number $n\geq 0$ as follows: $2n=l-|m|$ for even $l-|m|$ and $2n+1=l-|m|$ for odd $l-|m|$. The value of n has a simple meaning; it corresponds to the number of zeros of the wave function $S_{lm}(p,\cos\theta)$ within the interval $\theta\in(0,\pi/2)$, by analogy with the known property of $P_l^m(\cos\theta)$. The spectrum of Eq. (2) is then given by a series[6–8]

$$\varepsilon_{lm}(p)=4p[n+(m+|m|+1)/2]-(s^2-m^2+1)/2$$
$$+O[s^3/p], \quad (15)$$

with $s=2n+|m|+1$; one has $s=l+1$ ($s=l$) for even (odd) values of $l-m$. The eigenfunctions are given by

$$S^{\pm}_{lm}(p,\eta)=\tilde{S}_{lm}(p,\eta)\pm\tilde{S}_{lm}(p,-\eta), \quad (16)$$

where the plus (minus) sign corresponds to even (odd) values of $l-m$.[8] The functions $\tilde{S}_{lm}(p,\eta)$ are found as a series in the Laguerre polynomials $L_n^m(x)$; in the main order of p^{-1} they can be written as

$$\tilde{S}_{lm}(p,\eta)\simeq c(1-\eta)^{|m|/2}e^{-p(1-\eta)}L_n^{|m|}(2p(1-\eta)),$$
$$c=\left[\frac{2^{|m|}p^{|m|+1}n!}{2\pi r_0^2(n+|m|)!}\right]^{1/2}. \quad (17)$$

We see that in the main order of p the Landau quantization takes place; i.e., the spectrum (15) is that of a quantum oscillator with the cyclotron frequency being the energy quantum.[13] The convergence and hence the applicability of the series (15) is given by the condition $p\gtrsim s\sim l$. In its turn, it means[10] that all n zeros of the approximate eigenfunction $L_n^{|m|}(2p(1-\eta))$ in Eq. (17) lie in the northern hemisphere, $\eta>0$, as they should.

We notice the following important property of the spectrum (15). For given nonpositive m the values of ε_{lm} corresponding to $l=2n+|m|$ and $l=2n+|m|+1$ coincide. This property and the form of the wave function (16) can be understood as follows. At $p\to\infty$ the field-induced potential $p^2\sin^2\theta$ in (2) localizes the particles near the poles of the sphere. This form of two-well potential leads to the discussed degeneracy of energy levels, while the total wave function is given by a symmetrization (16) of the wave functions (17) related to each of the wells. The possibility of quantum tunneling between the wells lifts the degeneracy and produces an exponentially small energy splitting ($\sim e^{-2p}$) between the states $S^{+}_{lm}(p,\eta)$ and $S^{-}_{lm}(p,\eta)$, Eq. (16).[8]

It would be tempting to obtain the Green's function at $p\to\infty$ in a closed form. It can be done with the following simplifications. First, we neglect the exponentially small splitting and notice that $S^{+}(\eta)S^{+}(\eta')+S^{-}(\eta)S^{-}(\eta')=2[\tilde{S}(\eta)\tilde{S}(\eta')+\tilde{S}(-\eta)\tilde{S}(-\eta')]$; i.e., the correlations within one hemisphere only survive. Now we leave only the leading terms (17) of the wave functions and $O(p)$ terms in energies (15). The necessity to restrict the summation in Eq. (8) by terms with $s\lesssim p$ could be modeled by inclusion of the cutoff factor $e^{-s\delta}$ with $\delta\sim p^{-1}$ into Eq. (8). We put $\mu=0$ for simplicity of writing and raise the denominator into the exponent, $(\omega-E_{lm})^{-1}=i\int_0^\infty dt\, e^{it(E_{lm}-\omega)}$. Next we perform the summation over n with the use of bilinear generating function for the Laguerre polynomials:[10]

$$\sum_{n=0}^{\infty}\frac{n!}{(n+|m|)!}L_n^{|m|}(x)L_n^{|m|}(y)z^n =\frac{(xyz)^{-|m|/2}}{1-z}\exp\left(-z\frac{x+y}{1-z}\right)I_{|m|}\left(2\frac{\sqrt{xyz}}{1-z}\right),$$

with the modified Bessel function $I_m(w)$; in our case $x=2p(1-\eta), y=2p(1-\eta')$, and $z=e^{it\omega_c}$. Making use of the property $I_{|m|}(iw)=i^m J_m(w)$ and rescaling $t\omega_c/2\to t$ we arrive at an intermediate formula for $\eta,\eta'>0$:

$$G(\omega)\simeq-\frac{m_e}{4\pi}\int_0^\infty\frac{dt}{\sin(t+i\delta)}e^{-2it\omega/\omega_c-i(x+y)\cot(t+i\delta)/2}\times\sum_{m=-\infty}^{\infty}e^{i(t-\phi+\pi/2)m}J_m\left(\frac{\sqrt{xy}}{\sin(t+i\delta)}\right).$$

The summation over m is now easily done $[\Sigma_m e^{im\phi}J_m(w)=e^{iw\sin\phi}]$ and we obtain the Green's function as a sum of two terms, referring to two hemispheres. The "northern" term, corresponding to the above expression, is given by

$$G^n(\omega)\simeq-\frac{m_e}{4\pi}e^{iv}\int_{i\delta}^{i\delta+\infty}\frac{dt}{\sin t}e^{-2i(t-i\delta)\omega/\omega_c-i\rho\cot t/2}$$
$$=-\frac{m_e e^{iv-2\delta\omega/\omega_c}}{8\pi\cos\pi\frac{\omega}{\omega_c}}\times\int_{-\pi/2+i\delta}^{\pi/2+i\delta}dt\frac{\exp\left[-\frac{2i\omega}{\omega_c}t+\frac{i\rho\tan t}{2}\right]}{\cos t}, \qquad (18)$$

with the analog of the distance on the sphere, $\rho=2p[2-\eta-\eta'-2\sqrt{(1-\eta)(1-\eta')}\cos(\phi+i\delta)]$, and that of the vector product, $v=2p\sqrt{(1-\eta)(1-\eta')}\sin(\phi+i\delta)$. The form of the "southern" term is obtained by changing $\eta\to-\eta$ and $\eta'\to-\eta'$ in these expressions. The last integral is reduced for vanishing δ to the hypergeometric function $\Psi(a,b;z)$ and we obtain the "northern" term in the form

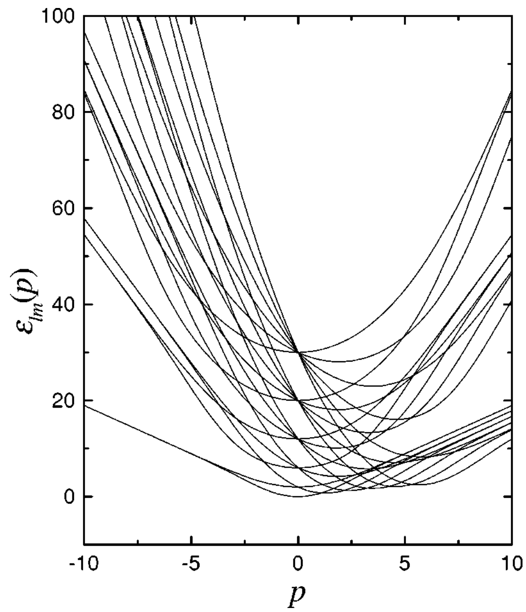

FIG. 2. The dependence of the energy levels ε_{lm} on the number p of magnetic flux quanta piercing the sphere. For convenience of presentation, we plotted the evolution of $\varepsilon_{lm}(p)$ with $m\leqslant 0$ on the right-hand side of the plot, at $p>0$. The evolution of $\varepsilon_{lm}(p)$ with $m\geqslant 0$ is depicted on the left-hand side of this plot for the formally negative p; see the text for additional explanations.

$$G^n(\omega)\simeq-\frac{m_e}{4\pi}e^{iv-\rho/2-\delta\omega/\omega_c}\Gamma\left[\frac{1}{2}-\frac{\omega}{\omega_c}\right]\Psi\left[\frac{1}{2}-\frac{\omega}{\omega_c},1;\rho\right],$$
$$=\frac{m_e}{4\pi}e^{iv-\rho/2-\delta\omega/\omega_c}\sum_{n=0}^{\infty}\frac{L_n(\rho)}{\omega/\omega_c-n-1/2}. \qquad (19)$$

The last equation is similar to previous findings.[12]

Let us discuss the applicability of Eq. (19). Our derivation was straightforward until Eq. (18), while the last step demanded $\delta\sim1/p\to0$. The incomplete restructuring of the spectrum into the Landau level scheme is absent in the usual planar geometry, wherein one would put $\delta=0$ and the expression (19) would be exact. In the spherical case we cannot treat the higher levels with $l\gtrsim p$ in an analytical way; it is mimicked in Eq. (19) by the appearance of the exponential cutoff at $\omega\gtrsim p\omega_c$.

A subtler issue in justifying Eq. (19) is the shift of the last integration in Eq. (18) to the interval $(-\pi/2,\pi/2)$. The integration over the remaining segments $(\pm\pi/2,\pm\pi/2+i\delta)$ yields the factor $\cos(\pi\omega/\omega_c)$; thus the contribution of these segments to the Green's function has no poles in ω. This contribution could be combined with the smooth (real) part of $G(\omega)$ stemming from the consideration of the highest-energy levels.

As a result, one may conclude that the expression (19) correctly reproduces the basic properties of the Green's function for $\omega<p\omega_c$.

The finite value of the cutoff parameter δ in Eq. (19) becomes important for the one-point correlation function, i.e., at $\eta=\eta'$ and $\phi=0$. The residues of the Green's function define the local density of states (LDOS) by the relation $N(\mathbf{r})=\int d\omega\, n_F(\omega)[G(\mathbf{r},\mathbf{r},\omega-i0)-G(\mathbf{r},\mathbf{r},\omega+i0)]/(2\pi i)$. If we assume that n lowest Landau levels are filled by the electrons, then it follows from Eq. (19) that the LDOS is given by $N(\mathbf{r})=npe^{iv-\rho/2}/(2\pi r_0^2)$. In this case finite $\delta \sim 1/p$ provides a smooth variation of the LDOS of the form

$$N(\mathbf{r})\propto \exp[-O(\sin^2\theta)],$$

which variation is absent in the usual planar geometry. This result, however, may depend on the approximations made and requires further numerical investigation.

It should be stressed that at intermediate fields, at $p>1$ and $p\lesssim l\lesssim p^2$, the spectrum and wave functions are not principally reduced to closed expressions of hypergeometric type. Numerical methods are indispensable here. We calculated the evolution with p of the energy levels with $l=0,\ldots,5$ by diagonalizing the 200×200 tridiagonal matrices in the basis of $P_l^m(\eta)$. The results are shown in the Fig. 2 where we made the following convention. From the general property of Eq. (2) it follows that $\varepsilon_{l,-m}(p)=\varepsilon_{l,m}(-p)$ with formally negative p.[14] For better readability of the graph, it is possible then to show all the data, plotting only half of them with one sign of m but in the formally extended region of p. One can see in Fig. 2 that the degeneracy at $p=0$ is eventually changed by the Landau levels formation at $p=10$. In the intermediate region $p\sim3$ the absence of any structure in the levels' scheme is noted. The mesh of lines at the intermediate fields mimicks the chaotic behavior, although there is no chaos here. Both in the quantum problem considered here and in its classical counterpart one has two variables (θ,ϕ) and two integrals of motion, the energy and the projection of the angular momentum onto the field direction. To obtain chaos, it suffices to break the rotation symmetry, $\phi\rightarrow\phi+\delta\phi$. The latter problem is, however, beyond the scope of this study.

In conclusion, we demonstrated the exact solution of the electron gas on the sphere in the magnetic field. In the limits of weak and strong fields this solution is reduced to the hypergeometric functions and the observable quantities are found in closed form. In the case of intermediate fields, the solution is not essentially a hypergeometric function and the observables require further numerical analysis.

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[14] This property is seen in the low-field limit, Eq. (5), but is obscure in the expression for the strong fields, Eq. (15). The reason is that the function $\varepsilon_{lm}(p)$ has cuts in the complex plane of p; hence the analytical continuation $p\rightarrow -p$ is not straightforward for p near the complex infinity (Ref. 7).