

## Wilson-renormalization-group approach of the principal chiral model around two dimensions

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We study the principal chiral Ginzburg-Landau-Wilson model around two dimensions within the local potential approximation of an exact renormalization-group equation. This model, relevant for the long-distance physics of classical frustrated spin systems, exhibits a fixed point of the same universality class as does the corresponding nonlinear sigma model. This allows to shed light on the long-standing discrepancy between the different perturbative approaches of frustrated spin systems. [S0163-1829(99)00606-2]

There is now a general agreement about the field theoretical treatment of the  $SO(N)$  spin system. The perturbative approaches performed around four dimensions on the Ginzburg-Landau-Wilson (GLW) model, around two dimensions on the nonlinear sigma ( $NL\sigma$ ) model and in a  $1/N$  expansion give a consistent picture of the critical physics of this system everywhere between  $D=2$  and  $D=4$ .<sup>1</sup> This picture has also been confirmed by nonperturbative methods based on truncations of Wilson exact renormalization-group (RG) equations.<sup>2-8</sup> Amazingly, there is no such agreement for many systems whose symmetry-breaking pattern is not given by  $SO(N) \rightarrow SO(N-1)$  among which are superfluid  $^3\text{He}$ ,<sup>9,10</sup> frustrated antiferromagnets,<sup>11-14</sup> superconductors,<sup>15,16</sup> electroweak phase transition,<sup>17,18</sup> etc. Generically perturbation theories predict that these systems undergo a first-order phase transition near  $D=4$  and a second-order one around  $D=2$ .<sup>19,17,13,14,18</sup> The origin of this discrepancy is not yet understood and calls for a nonperturbative approach.

In this paper we study, by means of the Wilson renormalization-group approach, the principal chiral (PC) model, corresponding to the symmetry-breaking scheme  $SO(N) \otimes SO(N) \rightarrow SO(N)$ , which is the simplest one exhibiting the nontrivial features previously quoted. The PC model is the low-energy effective field theory of a whole class of systems among which are frustrated antiferromagnets. A particularly important example is the Heisenberg antiferromagnet on the triangular lattice (AFT). Due to the frustration, the order parameter is a triad of orthonormal vectors, i.e., a  $SO(3)$  rotation matrix  $R = (e_1, e_2, e_3)$ .<sup>20,14</sup> We consider, in the following, the generalization to  $N$  orthonormal vectors  $e_\alpha$ 's with  $N$  components, i.e.,  $SO(N)$  matrices. The long-distance physics of this generalized AFT model is thus equivalent to that of orthonormal frames interacting ferromagnetically:

$$H = -J \sum_{(i,j)} \sum_{\alpha=1}^N e_\alpha^i \cdot e_\alpha^j = -J \sum_{(i,j)} \text{Tr} R^i R^j. \quad (1)$$

The Hamiltonian (1) is invariant under the  $SO(N) \otimes SO(N)$  group of left  $U \in SO(N)$  and right  $V \in SO(N)$  global trans-

formations:  $R^i \rightarrow UR^iV$ . Since, in the low-temperature phase, the residual symmetry group consists in a (diagonal)  $SO(N)$ , Eq. (1) is indeed a lattice version of the PC model. Whereas the microscopic derivation for frustrated spin systems leads in general to anisotropic interactions between the  $e_\alpha$ 's, i.e.,  $J$  is  $\alpha$  dependent, we consider here the isotropic case where all the  $J_\alpha$ 's are equal. It has been shown for a large class of frustrated spin systems, among which is the AFT model, that the anisotropy is anyway irrelevant, at least near two dimensions, for the critical properties we are interested in.<sup>13,14</sup>

Let us first sketch out the experimental and numerical situation for frustrated spin systems which, in  $D=3$ , is far from being clear. Indeed, the behavior of systems that are supposed to be described by the PC model like AFT ( $\text{CsVCl}_3, \text{RbNiCl}_3$ ) and helimagnets ( $\text{Ho}, \text{Dy}, \text{Tb}$ ) are affected by the presence of disorder localized near the sample surface and, possibly, by topological defects. As a consequence, the critical exponents strongly vary from one compound to another.<sup>21,22</sup> Numerically, the situation is also confused since simulations performed on the PC model and directly on the AFT model lead respectively to first order and second order behavior with exponents of an unknown universality class.<sup>23</sup>

Beyond this apparent lack of universality at the experimental and numerical level, the theoretical situation already exhibits the puzzling features previously mentioned. Around  $D=2$ , the critical physics is obtained by means of a low-temperature expansion performed on the PC  $NL\sigma$  model. A fixed point is found for any  $N > 2$  in  $D = 2 + \epsilon$  dimensions so that a second-order phase transition is expected.<sup>24,25</sup> On the other hand, the weak-coupling expansion performed in  $D = 4 - \epsilon$  on the PC GLW model suggests a first-order phase transition since no fixed point is found for any  $N > 2$ .<sup>11</sup> As such, the situation is not paradoxical since perturbation theories are only trustworthy in the immediate vicinity of their respective critical dimension. However if, as usual, we extrapolate the perturbative results to  $D=3$ , the two results come into conflict. It is thus of first importance to clarify this theoretical situation before hoping to describe real materials.

From the theoretical point of view, the crucial fact is that the calculation of the  $\beta$  functions in the two different perturbative approaches relies on qualitatively different grounds. Indeed, the  $\beta$  function of a  $NL\sigma$  model built on a manifold  $G/H$  only depends on the symmetry-breaking scheme  $G \rightarrow H$  (Ref. 25)—i.e., on Goldstone modes—whereas that of the GLW near  $D=4$  is sensitive to the representation of  $G$  spanned by the order parameter chosen to realize the symmetry-breaking scheme. This feature can be fully appreciated in the  $N=3$  PC model. Indeed, since  $SO(3) \otimes SO(3)$  is isomorphic to  $SO(4)$  the symmetry-breaking pattern is that of the usual four-component spin system:  $SO(4) \rightarrow SO(3)$ . The perturbative  $\beta$  function of the  $N=3$  PC  $NL\sigma$  model in  $D=2+\epsilon$  is thus identical to that of the  $N=4$  vector model, although the symmetry-breaking scheme is realized with a rotation matrix which is a  $SO(4)$  tensor and *not* with a four-component vector.<sup>13,14</sup> If this perturbative result remained true beyond  $D=2+\epsilon$ , as it is believed in the  $SO(N)$  vectorial case, we could expect the critical behavior of the PC model to be determined by the same fixed point as the  $N=4$  vector model everywhere between two and four dimensions. This is, however, not the case, at least perturbatively in the vicinity of  $D=4$ , since there is no fixed point in the GLW approach.

The origin of the discrepancy between the two approaches can be ultimately traced back to the (nonperturbative) spectrum of both models. Whereas it is very likely that in the  $SO(N)$  case with a vectorial order parameter the  $NL\sigma$  and GLW models share the same low-energy degrees of freedom everywhere between two and four dimensions, it is no longer the case for models with more general order parameters and symmetries. For example, for the  $N=3$  PC model, the spectrum of the  $D=2$   $NL\sigma$  model consists in four massive particles<sup>26</sup> whereas the spectrum of the  $D=4$  GLW in the high-temperature phase involves nine massive particles. Ideally, we should understand at a nonperturbative level how these two field contents are linked together in  $D=3$  and how they are related to the degrees of freedom of the underlying microscopical system. This is a formidable task that will not be tackled here.

The question we address here is the possibility of a matching between the  $NL\sigma$  and GLW approaches when varying the dimension. This allows us, at the same time, to test the validity of the  $NL\sigma$  model for frustrated systems, at least around  $D=2$ . Indeed, due to the discrepancy between the two perturbative approaches and the absence of experimental and numerical evidence of an  $O(4)$  critical behavior, the reliability of the  $NL\sigma$  model approach has been questioned.<sup>27</sup> Clearly, the answer to these questions escapes a perturbative treatment. In general, the  $1/N$  expansion provides a powerful tool to link up different perturbative methods. In the case of matrix models such an analysis is however plagued by technical difficulties. Some progress has been recently obtained but is confined to the leading order.<sup>28,29</sup> The Wilson's scheme, which has been successfully used in many topics,<sup>30–34,39,8</sup> turns out to be the most efficient approach. In this paper, we study the PC GLW model near  $D=2$  by means of the Wilson-Polchinski exact renormalization-group within the local potential approximation (LPA). We show that the GLW and  $NL\sigma$  approaches can be reconciled in the vicinity of two dimensions. More

precisely we show by a RG analysis that the two models belong to the same universality class near two dimensions since the GLW model exhibits a nontrivial fixed point identical to that of the  $NL\sigma$  model.

The partition function of the PC GLW model is obtained by writing the most general  $SO(N) \otimes SO(N)$  invariant potential, at most quartic in  $N \times N$  real matrices  $M$ , that favors orthogonal matrices for the field  $M$ :

$$Z = \int DM \exp - \left[ \int d^D x \frac{1}{2} \text{Tr}(\nabla^t M \cdot \nabla M) + \frac{r}{2} \text{Tr}^t M M + \mu \text{Tr}^t M M)^2 + \lambda (\text{Tr}^t M M)^2 \right]. \quad (2)$$

The domain of parameters of interest for us is given by  $\lambda > 0$  since, in this case, the minimum of the potential in the broken phase is given by  $M(x) = R_0$  where  $R_0 \in SO(N)$ . In this phase, the model displays a  $SO(N)$  symmetry, so that the symmetry-breaking scheme is  $SO(N) \otimes SO(N) \rightarrow SO(N)$  and thus corresponds to the GLW version of the PC model.

Our aim being to make contact with the  $NL\sigma$  model, let us show how the orthogonality of the lattice order parameter of Eq. (1) can be recovered from Eq. (2). Let  $r$  and  $\mu$  go to infinity, the ratio  $r/4\mu$  being fixed. In this limit, one recovers the partition function of the PC  $NL\sigma$  model where a delta function enforces the orthogonality constraint on  $M$  at each point:

$$Z = \int DM \exp - \frac{1}{2} \int d^D x \left[ \text{Tr}(\nabla^t M \cdot \nabla M) + 2\mu \text{Tr} \left( {}^t M M + \frac{r}{4\mu} \right)^2 + 2\lambda (\text{Tr}^t M M)^2 \right] \quad (3)$$

$$\rightarrow \int DM \delta \left( {}^t M M - \frac{1}{g_0^2} \right) \exp - \frac{1}{2} \int d^D x \text{Tr}(\nabla^t M \cdot \nabla M) \quad (4)$$

up to an overall constant. The quantity  $1/g_0^2 = -r/4\mu$  which corresponds to the minimum of Eq. (3) (when  $\lambda \ll \mu$ ) identifies with the inverse temperature of the  $NL\sigma$  model. Of course, since the preceding limit is made on the bare action, it does not allow to conclude how both models are related under RG transformations. We shall show that, around two dimensions, the GLW and  $NL\sigma$  models actually converge to the *same* renormalized trajectory in the continuum limit.

To realize this program we now study the evolution of the PC GLW model under RG transformations within the LPA. This approximation consists in truncating the effective Wilsonian action to its potential part  $V(M) = \int d^D x v[M(x)]$ . Note that the LPA thus misses the field renormalization. The Wilson-Polchinski equation for the potential density  $v(M)$  is given by:<sup>2,4</sup>

$$\frac{\partial v}{\partial t} = Dv - \frac{D-2}{2} M_{ij} v'_{ij} + \frac{1}{4\pi} v''_{ij,ij} - v'_{ij} v'_{ij}, \quad (5)$$

where  $v'_{ij} = \delta v / \delta M_{ij}$  and  $t = \ln \Lambda$ ,  $\Lambda$  being the dimensionless running scale.

There are two different ways to exploit Eq. (5). The first one is to search for an exact solution in any dimension, having recourse to numerical integration. This provides a powerful way to obtain precise values for critical quantities.<sup>8</sup> The second one is to solve Eq. (5) in a low-temperature expansion. We follow this route since we are interested in qualitative features of the RG flow and we want to make contact with the standard perturbative analysis of the NL $\sigma$  model. Mitter and Ramadas used the same techniques in the  $SO(N)$  case for a proof of perturbative renormalizability of the NL $\sigma$  model.<sup>35</sup>

Let us parametrize the potential density  $v$  by  $v(M) = u(\chi)/g_t^2$  with  $\chi = g_t^2 t M M$ . In a perturbative approach the running potential has always a minimum as it is the case for the initial potential in Eq. (3) for  $t M M = 1/g_0^2$ . The running temperature  $g_t$  is thus defined *via*:

$$\left. \frac{\partial u}{\partial \chi} \right|_{\chi=1} = 0. \quad (6)$$

We now write the Wilson-Polchinski equation for the potential density  $u$  within the LPA:

$$\begin{aligned} \frac{\partial u}{\partial t} = & D u - (D-2) \chi_{ij} u'_{ji} \\ & - (u'_{ji} u'_{jk} + u'_{jl} u'_{kj} + u'_{ij} u'_{jk} + u'_{lj} u'_{kj}) \chi_{lk} \\ & + \frac{g_t^2}{4\pi} [(u''_{js,jp} + u''_{sj,jp} + u''_{js,pj} + u''_{sj,pj}) \chi_{sp} + 2N u'_{ii}] \\ & + g_t^2 \frac{d}{dt} \left( \frac{1}{g_t^2} \right) (\chi_{ij} u'_{ji} - u). \end{aligned} \quad (7)$$

Under the iterations of the RG, all  $SO(N) \otimes SO(N)$  invariant terms are generated so that the evolved potential writes:

$$\begin{aligned} u(\chi, \{\lambda_{p_1, q_1; \dots; p_n, q_n}(t)\}) = & \sum_i \sum_{\{p_k, q_k\}} \lambda_{p_1, q_1; \dots; p_i, q_i}(t) \\ & [\text{Tr}(\chi - 1)^{p_1}]^{q_1} \dots [\text{Tr}(\chi - 1)^{p_i}]^{q_i}. \end{aligned} \quad (8)$$

The Wilson-Polchinski Eq. (7) generates the flow of all the  $\lambda_{p_1, q_1; \dots; p_n, q_n}(t)$ 's. When combined with Eq. (6) we also get the evolution of  $g_t$ :

$$\frac{d g_t^2}{d t} = -(D-2) g_t^2 + \frac{1}{4\pi} \frac{g_t^4}{2\lambda_{2,1}(t) + 2N\lambda_{1,2}(t)}$$

$$\begin{aligned} & [(12N+12)\lambda_{3,1}(t) + 24N\lambda_{1,3}(t) + 4(N^2+N+4)\lambda_{2,1;1,1}(t) \\ & + 4(2N+1)\lambda_{2,1}(t) + 4(N^2+2)\lambda_{1,2}(t)]. \end{aligned} \quad (9)$$

The flow analysis shows that all the  $\lambda_{p_1, q_1; \dots; p_n, q_n}(t)$ 's are irrelevant coupling constants: after an exponentially rapid transient regime, their scale dependence is entirely controlled by that of  $g_t$ :

$$\begin{aligned} \lambda_{p_1, q_1; \dots; p_n, q_n}(t) \rightarrow & \bar{\lambda}_{p_1, q_1; \dots; p_n, q_n}^{(0)} + \bar{\lambda}_{p_1, q_1; \dots; p_n, q_n}^{(1)} g_t^2 \\ & + O(g_t^4). \end{aligned} \quad (10)$$

Therefore, for any initial conditions, the flow is driven towards a one-dimensional renormalized trajectory parametrized by  $g_t$  whose evolution, obtained from Eqs. (9) and (10), is given at leading orders by:

$$\frac{d g_t^2}{d t} = -(D-2) g_t^2 + \frac{N-1}{4\pi} g_t^4 + O(g_t^6). \quad (11)$$

This  $\beta$  function identifies with that of the temperature of the PC NL $\sigma$  model calculated perturbatively.<sup>24</sup> It however differs from the standard expression where the coefficient  $N-1$  is replaced by  $N-2$ . The origin of this difference is that, within the LPA, the field renormalization is set equal to one. If the field renormalization had been taken into account, which is the case in the next orders in the derivative expansion, we would have obtained the correct coefficient. This difference is irrelevant for our purpose.

Let us indicate how, in two dimensions, our previous results allow us to recover, in the continuum limit, the hard constraint of the NL $\sigma$  model (4). After the transient regime—i.e., Eq. (10)— $u(\chi, \{\lambda_{p_1, q_1; \dots; p_n, q_n}(t)\})$  has converged towards  $\bar{u}(\chi, g_t)$  which can be expanded in powers of  $g_t^2$ :

$$\bar{u}(\chi, g_t) = \sum_{k \geq 0} (g_t^2)^k \bar{u}^{(k)}(\chi). \quad (12)$$

We have found the *exact* form of  $\bar{u}^{(0)}(\chi)$  so that the dominant part of the potential density at low temperature writes:

$$v(M) \sim \frac{1}{g_t^2} \bar{u}^{(0)}(\chi) = \frac{1}{2g_t^2} \text{Tr}[\sqrt{g_t^2 t M M} - 1]^2. \quad (13)$$

Suppose now that, after blocking, the model having converged to the one-dimensional renormalized trajectory, the effective running temperature has reached the value  $g_\mu$ , at scale  $\mu$ . Reversing the flow on this trajectory, towards the ultraviolet, Eq. (11) gives the bare temperature  $g_0$  at scale of the overall cutoff  $\Lambda_0$  (typically, the inverse lattice spacing). Due to asymptotic freedom  $g_0$  goes to zero when taking the continuum limit  $\Lambda_0 \rightarrow \infty$ . It is easy to see from Eq. (13) that, in this limit, the configurations contributing to the partition function correspond to  $SO(N)$  matrices (up to a normalization): the delta constraint of Eq. (4) is recovered from RG transformations. Thus, in the continuum limit, the GLW and NL $\sigma$  models coincide. The statistical interpretation of this is that the soft field GLW model appears as the block-spin iterated NL $\sigma$  model.

These results show that the PC GLW and NL $\sigma$  models belong to the same universality class near two dimensions. This is a strong evidence of the validity of the NL $\sigma$  model approach and of the existence of a second-order phase transition near two dimensions. Thus the critical behavior of the PC GLW model must change as  $D$  varies between  $D=2$  and  $D=4$ . This, of course, relies on the assumptions that our results persist beyond the low-temperature expansion and the LPA, and that the  $\epsilon=4-D$  expansion of the GLW model is meaningful. The change of critical behavior could be a general feature of models that are afflicted by the same troubles

even if their origins—presence of topological excitations, role of irrelevant operators—certainly depend on the precise model under study. In any case, analyzing this requires us to vary the dimension and to use the next orders of approximation in the derivative expansion.<sup>36–38</sup> A somewhat similar study has been performed for superconductors<sup>39</sup> and for the

Kosterlitz-Thouless phase transition.<sup>40</sup> In the context of the PC model, it will be addressed in a future publication.

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