

## Higher-order effects on Shapiro steps in Josephson junctions

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(Received 22 July 1998)

We demonstrate that the well known phase-locking mechanism leading to Shapiro steps in ac-driven Josephson junctions is always accompanied by a higher-order phase-locking mechanism similar to that of the parametrically driven pendulum. This effect, resulting in a  $\pi$ -periodic effective potential for the phase, manifests itself clearly in the parameter regions where the usual Shapiro steps are expected to vanish.

[S0163-1829(99)02502-3]

The harmonically driven pendulum has been studied extensively over the past decades in many different contexts. The driven pendulum has been one of the key systems in nonlinear dynamics due to its simplicity and richness in nonlinear phenomena such as phase locking<sup>1-4</sup> and chaos.<sup>5-7</sup> Another reason for studying the driven pendulum is that the pendulum equation is the most widely used model for superconducting Josephson junctions.<sup>1,8</sup> Particularly phase-locking of pendulum motion to an ac perturbation has been extensively studied in the literature due to the general interests in synchronization of oscillators and specifically due to the related Josephson junction, where phase-locking of the voltage response to an ac current can occur in certain regions of the parameter space.<sup>9</sup> There are two distinctively different ways of driving a pendulum; direct and parametric drive. The direct (torsional) ac drive, which we will study in this paper, is the most relevant for technologically interesting systems such as the Josephson junction, and it has been found to produce efficient harmonic<sup>1,9</sup> and subharmonic<sup>10</sup> phase locking of great interest for, e.g., the voltage standard.<sup>11</sup> The parametric drive (e.g., pivot oscillations) has mainly been studied from the point of view of nonlinear dynamics. The usual theoretical technique for predicting and analyzing the ac-driven pendulum is to assume that the driving frequency is much larger than the natural oscillations of the unperturbed pendulum. Within this framework, one can develop time separation analyses separating a fast linear response to the ac drive from the overall slow nonlinear behavior of the system. The main difference between the effects of the two driving mechanisms is that the parametric drive usually leads to a  $\pi$ -periodic effective potential for the slow nonlinear behavior,<sup>12-14</sup> while the direct drive leads to the usual  $2\pi$  periodicity.<sup>1,8</sup> We will demonstrate in this paper that the direct ac drive may lead to a  $\pi$ -periodic effective potential in certain regions of the parameter space and that this result has consequences for the well known Shapiro steps in the current-voltage (*IV*) characteristics of ac-driven Josephson junctions.

We study the pendulum equation in the form,

$$\ddot{\phi} + \alpha \dot{\phi} + \sin \phi = \eta + \varepsilon \sin \Omega t, \quad (1)$$

where  $\phi$  is the pendulum angle relative to vertical (down) and  $\alpha$  is a normalized friction coefficient. Time is normalized to  $\tau = \sqrt{g/l}$ , where  $g$  is the gravitational constant and  $l$  is the length from the pivot to the pendulum bob,  $\eta$  is a dc-torque, normalized to  $mgl$ ,  $m$  being the mass of the bob, and the normalized frequency,  $\Omega$ , and amplitude,  $\varepsilon$ , define the direct ac drive of the pendulum. In the context of Josephson junctions,  $\phi$  is the phase difference between the quantum mechanical wave functions of the superconductors defining the junction,  $\alpha$  is the normalized dissipation coefficient due to transport of quasiparticles,  $\eta$  and  $\varepsilon$  are currents normalized to the critical current of the junction, and time is normalized to the inverse Josephson plasma frequency,  $\tau = \sqrt{\hbar C/2eI_c}$ , where  $\hbar$  is Planck's constant,  $C$  is the device capacitance, and  $I_c$  is the critical current. Voltage across the Josephson device is given by  $V = \dot{\phi} \hbar/2e$ , uniquely relating Josephson voltage to pendulum speed.<sup>8</sup>

Let us write the phase in the following form,

$$\phi = \theta + \xi t + \Xi(t), \quad (2)$$

where  $\Xi(t)$  is a function that oscillates with frequency  $\Omega$ ,  $\xi$  is a constant to be determined, and  $\theta$  is a phase,  $\langle \dot{\theta} \rangle = 0$ . Inserting Eq. (2) into Eq. (1) we obtain

$$\ddot{\theta} + \ddot{\Xi} + \alpha \dot{\theta} + \alpha \dot{\Xi} + \alpha \xi + \alpha \ddot{\Xi} + \sin(\theta + \xi t + \Xi) = \eta + \varepsilon \sin \Omega t. \quad (3)$$

We will choose  $\Xi$  so that,  $\ddot{\Xi} + \alpha \dot{\Xi} = \varepsilon \sin \Omega t$ , and we then obtain

$$\Xi(t) = -\frac{\varepsilon}{\Omega \sqrt{\Omega^2 + \alpha^2}} \sin(\Omega t + \gamma), \quad (4)$$

where  $\gamma = \tan^{-1}(\alpha/\Omega)$  is a constant phase. Equation (3) then takes the form

$$\ddot{\theta} + \alpha \dot{\theta} + \sin(\theta + \xi t + \Xi) = \eta - \alpha \xi. \quad (5)$$

Rewriting Eq. (1) in the form of Eq. (5) demonstrates how the direct ac drive can lead to parametric effects similar to those of the parametrically driven pendulum equation.

We can now apply three consecutive transformations that will lead to an equation in variables where time-dependent terms are of order  $\Omega^{-3}$ . We describe the general procedure as a generalization of the analysis presented in Ref. 17 (which goes back to Poincare, see, e.g., Ref. 18) in the Appendix.

To average the equation we rewrite it as a system of two ordinary differential equations<sup>19</sup> (ODEs)

$$\dot{\theta} = p, \quad (6)$$

$$\dot{p} = \eta - \alpha \xi - \alpha p - A \sin \theta - B \cos \theta,$$

where  $A$  and  $B$  are given by

$$A = \cos(\xi t + \Xi),$$

$$B = \sin(\xi t + \Xi).$$

Carrying out the procedure described in the Appendix three times and neglecting terms  $\mathcal{O}(\Omega^{-3})$  or higher we obtain

$$\dot{\Theta} = P \quad (7)$$

$$\dot{P} = \eta - \alpha \xi - \alpha P - G_1 \sin(\Theta + \delta_{G_1}) - G_2 \sin(2\Theta + \delta_{G_2}),$$

where

$$G_1 = \begin{cases} -J_k(\Gamma), & \text{if } \xi/\Omega = k \text{ integer} \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

$$G_2 = \begin{cases} -\sum_{n \neq k} \frac{J_n(\Gamma) J_{2k-n}(\Gamma)}{\Omega^2 (n-k)^2}, & \text{if } \xi/\Omega = k \text{ integer} \\ -\sum_n \frac{J_n(\Gamma) J_{1-n}(\Gamma)}{\Omega^2 \left(n - \frac{1}{2}\right)^2}, & \text{if } \xi/\Omega = k = \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

$$\Gamma = -\frac{\varepsilon}{\Omega \sqrt{\Omega^2 + \alpha^2}},$$

and

$$\delta_{G_1} = \tan^{-1} \frac{\langle B \rangle}{\langle A \rangle},$$

$$\delta_{G_2} = \tan^{-1} \frac{2\langle \{B\}_{-1} \{A\}_{-1} \rangle}{\langle \{B\}_{-1}^2 \rangle - \langle \{A\}_{-1}^2 \rangle}.$$

Brackets,  $\langle \dots \rangle$ , denote time average and  $\{f\}$  is given by  $\{f\} = f - \langle f \rangle$ , where  $f$  is a periodic function. The mean-zero antiderivative,  $\{f\}_{-1}$ , is defined as

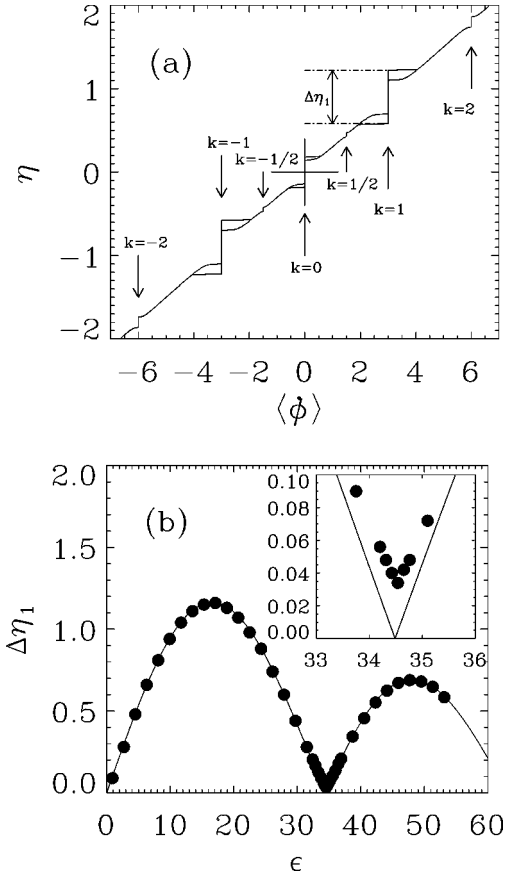


FIG. 1. (a) Simulated IV ( $\eta, \langle \phi \rangle$ ) characteristics of the ac-driven system described by Eq. (1). Parameters are  $\Omega = 3, \alpha = 0.3$ , and  $\Gamma = 5$ . Relevant resonant  $k$  steps are indicated with arrows, and the range of phase locking,  $\Delta \eta_k$ , is indicated for the  $k=1$  step. (b) Magnitude,  $\Delta \eta_1$ , of the phase-locked step as a function of  $\varepsilon$  for parameters:  $\Omega = 3$  and  $\alpha = 0.1$ . Solid line represents the usual prediction of the Shapiro step,  $\Delta \eta_1 = 2|G_1|$  for  $k=1$  and markers are results of numerical simulations. Inset shows details near the point  $J_1(\Gamma) = 0$ .

$$\{f\}_{-1} = \int \{f\} dt, \quad \langle \{f\}_{-1} \rangle = 0.$$

It is now clear that phase locking of the pendulum motion can exist for values of  $\eta$  given by

$$|\eta - \alpha \xi| < |G_1 \sin(\Theta + \delta_{G_1}) + G_2 \sin(2\Theta + \delta_{G_2})|. \quad (10)$$

For  $G_2 = 0$  this leads directly to the well-known Bessel function expression for the Shapiro steps in ac-driven Josephson junctions<sup>1</sup> and this result is correct up to order  $\mathcal{O}(\Omega^{-2})$ . However, we find that, for  $G_1 = 0$ , i.e., at every node of the Bessel function  $J_k$ , we have phase-locking originating from the coefficient  $G_2$  to the  $\pi$ -periodic term in Eq. (7). As a consequence, we may predict that the Shapiro steps in the IV characteristics of ac-driven Josephson junctions do *not* vanish for parameter values given by  $J_k(\Gamma) = 0$ .

In order to demonstrate this we have performed numerical simulations of (1) and measured the ranges of phase-locking in  $\eta$  as a function of  $\Omega, \varepsilon, k$ , and  $\alpha$ . Figure 1(a) shows a typical normalized IV ( $\eta, \langle \phi \rangle$ ) characteristic of an ac-driven system and we use this to define the magnitude of the locking range,  $\Delta \eta_k$ . The system parameters have here been cho-

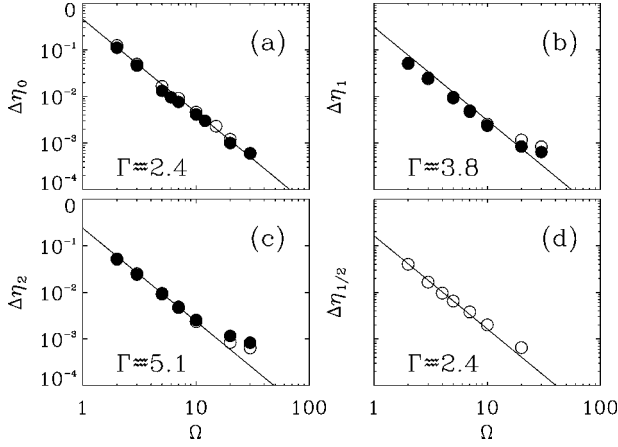


FIG. 2. Minimum of the locking range in  $\eta$  as a function of the driving frequency  $\Omega$  near the first node of  $G_1$  for nonzero  $\Gamma$ . Solid lines show the prediction,  $\Delta\eta_k = 2|G_2|$  and markers are results of numerical simulations of Eq. (1). Open markers are for  $\alpha=0.1$  and closed are for  $\alpha=0.05$ . (a)  $\Gamma \approx 2.4$  and  $k=0$ . (b)  $\Gamma \approx 3.8$  and  $k=1$ . (c)  $\Gamma \approx 5.1$  and  $k=2$ . (d)  $\Gamma \approx 2.4$  and  $k=\frac{1}{2}$ .

sen to  $\alpha=0.3$ ,  $\Omega=3$ , and  $\Gamma=5$ . In Fig. 1(b) we show the magnitude of the locking range,  $\Delta\eta_k$ , against  $\varepsilon$  for  $\Omega=3, \alpha=0.1$ , and  $k=1$ . The solid line represents the usual Bessel function prediction,  $\Delta\eta_1 = 2|G_1|$ , and the markers represent the numerical simulations. It is clear that as long as  $G_1$  (the relevant Bessel function) is not close to one of the nodes the comparison between the solid line and the markers is good. However, from the inset we see that close to the node of  $G_1$ , we observe a relatively large discrepancy, which is obviously due to the correction from the  $\pi$ -periodic effective potential given by  $G_2$ . The best parameter range to study the effect of the  $\pi$ -periodic effective potential is therefore to choose parameters such that  $G_1 \approx 0$ ; i.e., for  $J_k(\Gamma) = 0$  when  $k$  is an integer.

Figure 2 shows direct comparisons between numerical simulations (markers:  $\alpha=0.05$  closed,  $\alpha=0.1$  open) and our predictions (solid line),  $\Delta\eta_k = 2|G_2|$ , for parameter values leading to  $G_1=0$ . The comparisons are performed at the smallest nonzero value of  $\Gamma$  for which  $J_k(\Gamma)=0$ , and we show comparisons for  $k=0$  [Fig. 2(a)],  $k=1$  [Fig. 2(b)],  $k=2$  [Fig. 2(c)], and  $k=\frac{1}{2}$  [Fig. 2(d)] keeping  $\Gamma$  constant for each figure. Note that in the latter case,  $k=\frac{1}{2}$ ,  $G_1$  is always zero. We have here, arbitrarily, chosen  $J_0(\Gamma=2.4) \approx 0.0$ . It is obvious that our comparisons demonstrate an excellent agreement between simulations and prediction of the magnitude of the phase-locked region in  $\eta$  for all the different parameter values. The comparisons are performed in the frequency range between  $\Omega=1$  and  $\Omega=40$  since driving frequencies smaller than  $\Omega=2$  typically lead to low stability of the phase locked states (the analysis is developed for high  $\Omega$ ) and since phase-locking becomes impractical to identify for frequencies larger than  $\Omega=30$ . Our data shows a slight trend of overestimating the locking range for large frequencies. We have identified this to be an artifact of numerically solving the pendulum equation with discrete time. Particularly, we have chosen the time step for the simulations to be a fraction of the period of the driving frequency, and thus, when looking for extremely small phase-locked steps in the

normalized  $IV$  characteristics, we are finding an artificial phase locking of the dynamics to the temporal discretization.

We have demonstrated that the  $\pi$ -periodic effective potentials can exist in the directly ac-driven pendulum and we have further given a quantitatively correct estimate of the significance of this effect. The magnitude of the  $\pi$ -periodic effective potential suggests that the phase-locking signature of the potential should be directly observable not only in the driven pendulum,<sup>15,16</sup> but also in ac-driven Josephson junctions. For relatively low driving frequencies,  $\Omega \approx 2$ , we observe locking ranges in Fig. 2 of the order of  $\Delta\eta \approx 0.1$ , indicating that a standard dc current-voltage characteristic of a current ac-driven junction will indeed show a significant locking range where the usual Bessel function amplitude would suggest that locking is not possible. A particularly convenient choice of parameters is to operate the system at the subharmonic resonance,  $k=\frac{1}{2}$ , where  $G_1$  is always zero.

This work was performed under the auspices of the U.S. Department of Energy. V.Z. acknowledges partial support from NSF Grant No. DMS-9627721.

## APPENDIX

Let us consider a system of equations (6) written in vector form

$$\dot{x} = f(x, \tau),$$

where  $\tau = \Omega t, x = (\theta, p)$ , and  $f = (f_1, f_2)$ . We will apply an averaging procedure as follows.

Let  $x = x_1 + \Omega^{-1}h_1(x_1, \tau)$  be the first transformation with, yet undefined  $h_1$ . This function is restricted to be periodic in  $\tau$  so that the new variables are close to the old ones uniformly in time. In the new variables the equation takes the form

$$(I + \Omega^{-1}D_{x_1}h_1)\dot{x}_1 + h_{1\tau} = f(x_1 + \Omega^{-1}h_1(x_1, \tau), \tau),$$

where  $I$  is the unit matrix and  $D_x$  denotes differentiation with respect to the elements in  $x$ . We obtain by Taylor expanding  $f$ ,

$$(I + \Omega^{-1}D_{x_1}h_1)\dot{x}_1 + h_{1\tau} = f(x_1, \tau) + \Omega^{-1}D_{x_1}f(x_1, \tau)h_1 + \dots$$

In order to eliminate the oscillatory part of  $f(x_1, \Omega t)$  we choose  $h_1 = \{f(x_1, \tau)\}_{-1}$  ( $\{g\} = g - \langle g \rangle$  and  $\{g\}_{-1}$  is a mean-zero antiderivative of  $\{g\}$ ) to obtain

$$\dot{x}_1 = \langle f \rangle(x_1) + \Omega^{-1}R_1(x_1, \tau, \Omega^{-1}),$$

where  $R_1$  is polynomial in  $\Omega^{-1}$ . The second transformation, given by  $x_1 = x_2 + \Omega^{-2}h_2(x_2, \tau)$  with  $h_2 = \{R_1(x_1, \tau, 0)\}_{-1}$ , moves time dependence to second order in  $\Omega^{-1}$

$$\dot{x}_2 = \langle f \rangle(x_2) + \Omega^{-1}\langle R_1 \rangle(x_2, 0) + \Omega^{-2}R_2(x_2, \tau, \Omega^{-1}).$$

Continuing this procedure one can bring the system to the form

$$\begin{aligned} \dot{x}_n = & \langle f \rangle(x_n) + \Omega^{-1}\langle R_1 \rangle(x_n, 0) \\ & + \dots + \Omega^{-n+1}\langle R_{n-1} \rangle(x_n, 0) + \Omega^{-n}R_n(x_n, \tau, \Omega^{-1}). \end{aligned}$$

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- <sup>19</sup>It is important to apply the normal form procedure, see the Appendix, to the system of two first order ODEs for  $(\theta, p)$  rather than to the original second order ODE for  $\theta$ . The reason is that the group of transformations which are used to move rapidly oscillating terms to higher order must be sufficiently large. In the first case we obtain the averaged equations by transforming  $(\theta, p)$ ; see the Appendix. In the second case it is impossible to arrive at the same result as the transformations of only one variable  $\theta$  do not give enough freedom. This is another manifestation of the well known fact that transformations in configuration space form a subgroup of canonical transformations in phase space.