Image-soliton method applied to finite multiple tunnel junctions

M. B. A. Jalil
Microelectronics Research Centre, Cavendish Laboratory, Cambridge CB3 0HE, United Kingdom

M. Wagner
Hitachi Cambridge Laboratory, Cavendish Laboratory, Cambridge CB3 0HE, United Kingdom

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We show that the simple and physically intuitive method of soliton images can be used not only for a semi-infinite multiple tunnel junction (MTJ) array of identical junction capacitances C and stray capacitances C_0, but is also applicable to both homogenous and inhomogenous MTJ's of a *finite* number of junctions. In the latter, a junction capacitance $C' \neq C$ is replaced by a homogenous chain of equivalent length, thereby extending the method to important circuits relying on inhomogenous MTJ's, such as the multijunction trap and the multijunction turnstile. [S0163-1829(99)01908-6]

In the field of charge transport in Coulomb blockade devices, much interest has recently arisen in the theoretical treatment of a single-charge soliton moving along a one-dimensional chain of metallic islands, weakly coupled to each other by tunnel junctions. So far, the problem of a soliton in an infinite multiple tunnel junction (MTJ) array has been analytically solved by recursion,[1,2] with a later extension to the case of a semi-infinite MTJ by the method of images, where a soliton near the edge of the MTJ is assumed to induce an antisoliton image.[2] For the case of an MTJ of *finite* length, however, the solution to the potential distribution has been obtained by algebraically inverting a symmetric tridiagonal matrix obtained by applying Kirchhoff's laws and the conservation of charge on each of the MTJ islands.[3–5] The drawback of this approach is that for different MTJ circuits, such as the multijunction trap[4] or the multijunction turnstile,[5] the resulting tridiagonal matrices have different structures, each requiring a different method of inversion. In this article, we generalize the method of image charges to obtain the analytical solution for the potential in both homogenous and inhomogenous MTJ's of finite length. Our method not only involves much less algebraic manipulation and is applicable to both the multijunction trap and turnstile circuits, but more importantly it is conceptually simple and physically intuitive. The crucial quantity to calculate is the total electrostatic potential, which includes the localized (hence solitonlike) potential distribution due to an excess charge within the MTJ array. From the total potential, we subsequently derive quantities such as capacitances, critical charges,[6] and the Gibb's and tunneling energies—the latter determining the rate of charge transport through the MTJ.[2]

In an infinite MTJ of uniform junction capacitance C and stray capacitance C_0, the potential is analytically solved by reducing the electrostatics equations for each island to a recursive relation. The potential $\phi_{jk}^{(\infty)}$ on the jth island due to a charge e on the kth island is then found to exponentially decay with the separation distance $|j-k|$ and is given by[1,2]

$$\phi_{jk}^{(\infty)}=\frac{e}{C_{\text{eff}}}\exp(-\lambda|j-k|), \tag{1.1}$$

where

$$\lambda=\ln\left(\frac{C_{\text{eff}}+C_0}{C_{\text{eff}}-C_0}\right), \quad C_{\text{eff}}=\sqrt{C_0^2+4CC_0}. \tag{1.2}$$

In a semi-infinite MTJ, the potential $\phi_{jk}^{(\infty/2)}$ can be obtained by employing the method of images, where one assumes that the soliton centered on the kth island from the edge induces an antisoliton of the same strength at the same distance k on the opposite side of the edge, giving[2]

$$\phi_{jk}^{(\infty/2)}=\frac{e}{C_{\text{eff}}}\{\exp(-\lambda|j-k|)-\exp[-\lambda(j+k)]\}. \tag{2}$$

The introduction of the image soliton ensures that the boundary condition at the start of the semi-infinite chain, $\phi_{0k}^{(\infty/2)}=0$, is satisfied for all k.

We now apply the method of images to a *finite* MTJ of n identical junction and stray capacitances C and C_0, respectively, and a charge e on the kth island. The potential $\phi_{jk}^{(n)}$ in the MTJ can be solved by extending the MTJ infinitely on both sides and introducing two antisolitons on either side at a distance k and $(n-k)$ from the edges, with strengths of φ_1 and φ_2, respectively, as shown in Fig. 1(a). $\phi_{jk}^{(n)}$ is then given by the contributions from all three solitons, i.e.,

$$\phi_{jk}^{(n)}=\frac{e}{C_{\text{eff}}}\exp(-\lambda|j-k|)-\varphi_1\exp[-\lambda(j+k)]-\varphi_2 \times\exp[-\lambda(2n-j-k)], \tag{3.1}$$

where, by solving the simultaneous equations obtained from the boundary conditions $\phi_{jk}^{(n)}=0$, for $j=0$ and $j=n$, we obtain the antisoliton strengths

$$\varphi_1=\frac{e}{C_{\text{eff}}}[\exp(2\lambda n)-\exp(2\lambda k)]/[\exp(2\lambda n)-1],$$

$$\varphi_2=\frac{e}{C_{\text{eff}}}[\exp(2\lambda n)-\exp(2\lambda n-2\lambda k)]/[\exp(2\lambda n)-1]. \tag{3.2}$$

Combining Eqs. (3.1) and (3.2) yields

$$\phi_{jk}^{(n)}=\frac{e}{C_{\text{eff}}}\left[\frac{\text{ch}(n-|j-k|)\lambda-\text{ch}(n-j-k)\lambda}{\text{sh}\, n\lambda}\right], \tag{4}$$

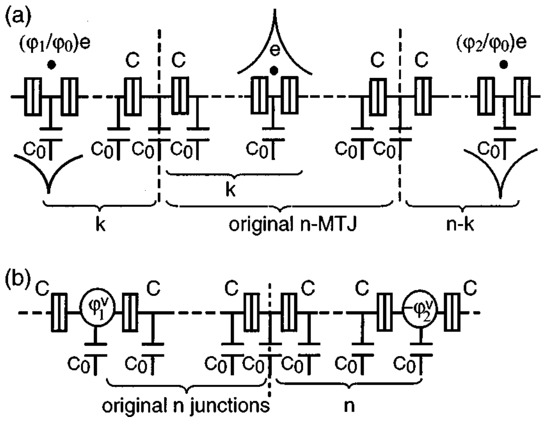

FIG. 1. (a) A charge e on the kth island of a (finite) n MTJ can be thought of as inducing two image charges, with the planes of reflection being the edges of the MTJ. $\varphi_0 = e/C_{\text{eff}}$ is the peak potential on the kth island, and the relative strengths of the images $\varphi_{1,2}/\varphi_0$ are such that the potentials from all three charges cancel out at both edges of the MTJ. (b) The effect of an external bias on an n MTJ is equivalent to two biases of opposite signs in an infinite MTJ, separated by $2n$ junctions, with the plane of reflection at the grounded edge of the original finite MTJ.

which has been derived earlier in Ref. 3. In the next step, we need to evaluate the potential $\phi_{jV}^{(n)}$ due to an external bias V, applied at one end (the zeroth island) of the n MTJ. As before, we extend the MTJ infinitely on both sides, replace the external bias V by a bias ϕ_1^v, and introduce an image bias $-\varphi_2^v$, a distance n from the right-hand edge, as shown in Fig. 1(b). $\phi_{jV}^{(n)}$ is then given by

$$\phi_{jV}^{(n)} = \varphi_1^v e^{-\lambda j} - \varphi_2^v e^{-(2n-j)\lambda}, \tag{5.1}$$

where the respective bias strengths $\varphi_{1,2}^v$ are again evaluated from the boundary conditions $\phi_{0,V}^{(n)} = V$ and $\phi_{n,V}^{(n)} = 0$, yielding

$$\varphi_1^v = \varphi_2^v = V/(1 - e^{-2\lambda n}). \tag{5.2}$$

Substituting Eq. (5.2) into Eq. (5.1), we obtain

$$\phi_{jV}^{(n)} = V \frac{\text{sh}(n-j)\lambda}{\text{sh}\, n\lambda}, \tag{6}$$

which in the limit of $n \to \infty$ gives the known result $\phi_{jV}^{(\infty/2)} = Ve^{-\lambda j}$ for a semi-infinite array.[2] From Eq. (6), one can then evaluate the capacitance $C^{(n)}$ between one end of the n MTJ and ground as

$$C^{(n)} = \frac{C}{V}(\phi_{0,V}^{(n)} - \phi_{1,V}^{(n)}) = C\left[1 - \frac{\text{sh}(n-1)\lambda}{\text{sh}\, n\lambda}\right]. \tag{7.1}$$

To determine the energy of tunneling in the MTJ, we first need to calculate the external capacitance C_j^{ext} across the jth junction. This quantity is defined as the equivalent capacitance across the junction due to the rest of the circuit (with all the biases in the latter being shorted), and arises from the Norton-Thevenin simplification of the circuit.[7] Using Eq. (7.1) and $C_0 = 2C(\cosh\lambda - 1)$, derived from Eq. (1.2), C_j^{ext} is found to be

$$C_j^{\text{ext}} = \frac{(C_0 + C^{(j-1)})(C_0 + C^{(n-j)})}{2C_0 + C^{(j-1)} + C^{(n-j)}} = C\left(\frac{f_{j-1}f_{n-j} - 1}{f_{j-1} + f_{n-j} - 2} - 1\right), \tag{7.2}$$

where $f_j = \sinh\lambda(j+1)/\sinh\lambda j$. With the above result and the total potential of the jth island given by $(\phi_{jk}^{(n)} + \phi_{jV}^{(n)})$, we are now in a position to evaluate the tunneling energies that determine the rate of charge transport through the MTJ. We consider a tunnel sequence $\{1,2,3,\ldots,n\}$, in which tunneling occurs sequentially along the MTJ. In Ref. 3, the jth tunneling energy δE_j is obtained by calculating the resulting change in the Gibb's energy. An equivalent and slightly less complicated way of evaluating δE_j is through the following equation, which involves only linear terms in the potential

$$\begin{aligned} \delta E_j &= \frac{e}{C}(q_j^c - q_j) \\ &= \frac{e^2}{2(C + C_j^{\text{ext}})} - e(\phi_{j-1,j-1}^{(n)} - \phi_{j,j-1}^{(n)}) - e(\phi_{j-1,V}^{(n)} - \phi_{jV}^{(n)}) \\ &= \delta E_j^{(1)} + \delta E_j^{(2)} + \delta E_j^{(3)}, \end{aligned} \tag{8}$$

where q_j is the jth junction charge, and $q_j^c = eC/2(C + C_j^{\text{ext}})$, the corresponding so-called critical charge, which must be exceeded by q_j for the tunneling event to be energetically favorable.[6] Note that the first two energy terms in Eq. (8) give the ''internal'' change in energy, which is dependent on the position of the charge in the MTJ, while the third is the contribution due to the external bias. Using Eqs. (7.2), (4), and (6), we obtain for $\delta E_j^{(1)}$, $\delta E_j^{(2)}$, and $\delta E_j^{(3)}$

$$\delta E_j^{(1)} = \frac{e^2}{C_{\text{eff}}}\left[\frac{\text{sh}\,\lambda j \text{sh}\,\lambda(n-j) + \text{sh}\,\lambda(j-1)\text{sh}\,\lambda(n-j+1) - 2\text{sh}\,\lambda(j-1)\text{sh}\,\lambda(n-j)}{\text{sh}\,\lambda n}\right],$$

$$\delta E_j^{(2)} = \frac{2e^2}{C_{\text{eff}}}\left[\frac{\text{sh}\,\lambda(j-1)\text{sh}\,\lambda(n-j) - \text{sh}\,\lambda(j-1)\text{sh}\,\lambda(n-j+1)}{\text{sh}\,\lambda n}\right],$$

$$\delta E_j^{(3)} = \text{eV}\left[\frac{\text{sh}\,\lambda(n-j) - \text{sh}\,\lambda(n-j+1)}{\text{sh}\,\lambda n}\right], \tag{9.1}$$

the sum of which gives $\delta E_j = g_j - g_{j-1}$, where

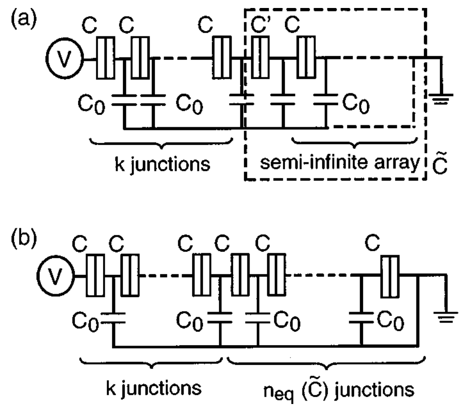

FIG. 2. (a) An inhomogenous semi-infinite MTJ with its $(k+1)$th junction capacitance replaced by $C' \neq C$. The portion of the circuit beyond the kth junction can be replaced by a single capacitance $\tilde{C}$, (b) which in turn is equivalent to $n_{\mathrm{eq}}(\tilde{C})$ links of the homogenous array, as given by Eq. (11).

$$g_j = \frac{e^2}{C_{\mathrm{eff}}} \frac{\mathrm{sh}\, j\lambda\, \mathrm{sh}(n-j)\lambda}{\mathrm{sh}\, n\lambda} + \mathrm{eV} \frac{\mathrm{sh}(n-j)\lambda}{\mathrm{sh}\, n\lambda}, \tag{9.2}$$

and the cumulative energy change after m events becomes

$$\Delta E_m = \sum_{j=1}^{m} \delta E_j = g_m - g_0. \tag{9.3}$$

The Gibb's energy F_m of the circuit is then given by $\Delta E_m + E_0$, the offset $E_0 = C^{(n)} V^2/2$ being the initial capacitive energy (before the first tunnel event). We thus obtain the Gibb's energy expression of Ref. 3,

$$F_m = \frac{e^2}{C_{\mathrm{eff}}} \frac{\mathrm{sh}\, m\lambda\, \mathrm{sh}(n-m)\lambda}{\mathrm{sh}\, n\lambda} - \mathrm{eV}\left(1 - \frac{\mathrm{sh}(n-m)\lambda}{\mathrm{sh}\, n\lambda}\right) + \frac{CV^2}{2}\left(1 - \frac{\mathrm{sh}(n-1)\lambda}{\mathrm{sh}\, n\lambda}\right). \tag{10}$$

To demonstrate the generality of our method we now apply it to an *inhomogenous* MTJ, where for the sake of simplicity all but one of the junction capacitances are identical. Let us consider the semi-infinite array of Fig. 2, where the $(k+1)$th junction has a capacitance C' compared with C for the rest. The semi-infinite chain beyond C' has a capacitance $C_h = (C_{\mathrm{eff}} - C_0)/2$,[1] and so the circuit can be reduced to a finite chain terminated by $\tilde{C} = (C_h + C_0)C'/(C_h + C_0 + C')$. The next step is to ''homogenize'' the chain by replacing $\tilde{C}$ with an equivalent number n_{eq} of (C, C_0) links, where by inverting Eq. (7.1), n_{eq} is found to be

$$n_{\mathrm{eq}}(\tilde{C}) = \frac{1}{\lambda} \tanh^{-1}\left(\frac{\mathrm{sh}\, \lambda}{\mathrm{ch}\, \lambda - 1 + \tilde{C}/C}\right), \tag{11}$$

and the total capacitance of the array is then given by $C^{(k+n_{\mathrm{eq}})}$. Note that the equivalent number of junctions n_{eq} is now a continuous variable. The above generalization enables us to analyze a wider range of circuits as before, including the multijunction trap shown in Fig. 3. We will limit ourselves to obtaining the analytical expression for the potential profile in the trap, since the corresponding expression for the tunneling energies follows directly from the potential profile,

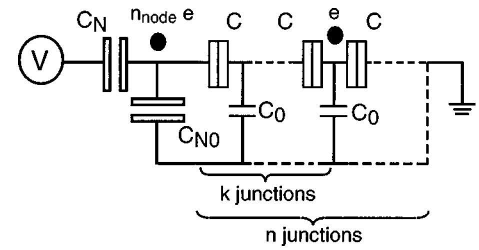

FIG. 3. Multijunction trap circuit, with C_N and C_{N0} as the gate and stray capacitance of the node, respectively. The node is attached to a finite MTJ of n junctions. An escaping charge e is at the kth island, while $n_{\mathrm{node}}e$ charges are stored on the node.

as was demonstrated above for the case of an MTJ with stray capacitances.

Suppose there are n_{node} charges stored on the node, with an additional escaping charge on the kth island (the node being the zeroth island). There are three contributions to the potential Φ_j on the jth island, (i) Φ_{jV} from the external bias V, (ii) Φ_{jk} induced by the escaping charge, and (iii) Φ_{j0} due to the node charges. From Eq. (6), Φ_{jV} is given by

$$\Phi_{jV} = V_{\mathrm{node}} \frac{\mathrm{sh}(n-j)\lambda}{\mathrm{sh}\, n\lambda}, \tag{12}$$

where $V_{\mathrm{node}} = C_N V/(C_N + C_{N0} + C^{(n)})$ is the node potential due to the bias V, n is the length of the MTJ, and $C_{N,N0}$ are the gate and stray capacitance of the node, respectively. To evaluate Φ_{jk}, we again homogenize the array by replacing the node with an equivalent number of (C, C_0) links. We can write the node capacitance as $(\tilde{C}_{\mathrm{node}} + C_0)$, where $\tilde{C}_{\mathrm{node}} = C_N + C_{N0} - C_0$. Thus, the extra number of links is $\Delta j = n_{\mathrm{eq}}(\tilde{C}_{\mathrm{node}})$. The potential contribution of the escaping charge is then given by $\Phi_{jk} = \phi^{(n+\Delta j)}_{j+\Delta j, k+\Delta j}$, i.e., with all island indexes being displaced by Δj compared with $\phi^{(n)}_{jk}$ in Eq. (4). Similarly, the third contribution Φ_{j0} due to the node charges is given by $n_{\mathrm{node}} \times \phi^{(n+\Delta j)}_{j+\Delta j, 0+\Delta j}$. With an equally straightforward analysis, the analytical expression for the potential in a multijunction turnstile can also be evaluated, since each branch of the turnstile may be treated as a multijunction trap.

In summary, we have applied the method of image charges, which previously had been used for the case of a semi-infinite MTJ, to obtain analytic solutions to the potential in homogenous as well as inhomogenous MTJ arrays of finite length, the latter of which cover important circuits like the multijunction trap and turnstile. The solutions are in agreement with numerical solutions obtained via Kirchhoff's laws and charge conservation, as well as previous analytic solutions evaluated via algebraic inversion of matrices. Compared to earlier analytic methods, ours has the advantage of being simple and physically more intuitive, while nevertheless having a wider scope of application.

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[1] P. Delsing, in *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992), p. 249.
[2] N. S. Bakhvalov, G. S. Kazacha, K. K. Likharev, and S. I. Serdyukova, Zh. Eksp. Teor. Fiz. **95**, 1010 (1989) [Sov. Phys. JETP **68**, 581 (1989)].
[3] G. Y. Hu and R. F. O'Connell, Phys. Rev. B **49**, 16 773 (1994).
[4] G. Y. Hu and R. F. O'Connell, Phys. Rev. Lett. **74**, 1839 (1995).
[5] Young Bong Kang, G. Y. Hu, R. F. O'Connell, and Jai Yon Ryu, J. Appl. Phys. **80**, 1526 (1996).
[6] L. J. Geerligs, V. F. Anderegg, P. Holweg, J. E. Mooij, H. Pothier, D. Esteve, C. Urbina, and M. H. Devoret, Phys. Rev. Lett. **64**, 2691 (1990).
[7] G.-L Ingold and Yu. V. Nazarov, in *Single Charge Tunneling* (Ref. 1), p. 68.