

Weak localization of disordered quasiparticles in the mixed superconducting state

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Starting from a random-matrix model, we construct the low-energy effective-field theory for the noninteracting gas of quasiparticles of a disordered superconductor in the mixed state. The theory is a nonlinear σ model, with the order-parameter field being a supermatrix whose form is determined solely on symmetry grounds. The weak-localization correction to the field-axis thermal conductivity is computed for a dilute array of s -wave vortices near the lower critical field H_{c1} . We propose that weak-localization effects, cut off at low temperatures by the Zeeman splitting, are responsible for the field dependence of the thermal conductivity seen in recent high- T_c experiments by Aubin *et al.* [S0163-1829(99)08205-3]

I. INTRODUCTION

The long-wavelength physics of phases of matter with spontaneously broken symmetries is commonly described by an effective-field theory for the relevant order parameter. For the problem of localization and transport in disordered metallic systems at low temperature, the appropriate “order parameter” is known¹ to be a supermatrix (or a matrix of dimension zero if the replica trick is used), conventionally denoted by Q . Three universality classes, differing by their behavior under time reversal and spin rotations, are widely known.² They are labeled by an index $\beta=1,2,4$, and are traditionally referred to as the classes with orthogonal, unitary, and symplectic symmetry. We denote them by AI, A, and AII for short.³ In each case, the field theory for Q belongs to the general family of nonlinear σ models. The field Q contains the Goldstone modes of a hidden symmetry⁴ connecting retarded and advanced single-electron Green's functions, which is broken by a nonzero density of states. At tree level one recovers the classical diffusion approximation, which neglects quantum interference corrections due to electron paths with loops. Anharmonic terms in the field theory represent “interactions” between the diffusion modes, giving rise to so-called weak-localization corrections to diffusion. For the classes AI and A in dimension $d \leq 2$, these interactions become strong at large distance scales and thus cause localization of all states, regardless of the strength of the disorder.

In recent years it was found that the $\beta=1,2,4$ classification is not exhaustive: for systems with symmetries of the particle-hole (p - h) type, the invariance group of the order-parameter field Q becomes *enlarged* in the vicinity of the p - h -symmetric point. One instance of such symmetry enhancement are Dirac fermions in a random gauge field,^{5,6}

another are disordered quasiparticles that exchange charge (but no energy) with a superconducting condensate.³ Such systems exhibit novel spectral statistics and transport properties. Seven symmetry-enhanced universality classes have been identified, and the corresponding order parameters Q were constructed from their random-matrix limit in Ref. 7. The main message is that Q lives on a symmetric space—in precise technical language, on a Riemannian symmetric superspace—in all cases. The nonlinear σ models defined over such spaces are known⁸ to be *attractive under the flow of the renormalization group*. Therefore the order parameter of a given disordered single-particle system, and in most cases its low-energy effective-field theory as well, can be inferred quite simply by investigating the ergodic (or random matrix) limit.

The present report focuses on class C, which emerges for noninteracting low-energy quasiparticles in a magnetic field and in contact with a spin-singlet superconductor. The defining condition³ is that the quasiparticle Hamiltonian be invariant under SU(2) rotations of the electron spin, whereas time-reversal invariance has to be broken. Since a superconductor screens magnetic fields, this universality class can only be realized in an *inhomogeneous* superconducting state — unless time-reversal invariance is broken spontaneously. In the very recent literature the following realizations have appeared:⁹ (i) a metallic quantum dot in the form of a chaotic billiard, subject to a magnetic flux and bordering on a superconductor;^{10,11} (ii) quasiparticles in the core of an isolated vortex in a disordered s -wave superconductor;¹² and (iii) a (quantum) disordered version of a $d_{x^2-y^2}$ superconductor with orbital coupling to a magnetic field.¹³ The hallmark of class C is that, in contrast with the metallic class A, the weak localization correction does *not* vanish,¹⁴ in spite of the presence of a magnetic field. The persistence of weak

localization in a field is caused by nonstandard modes of quantum interference that appear when impurity and Andreev scattering are simultaneously present. In a semiclassical picture, the effect can be understood as being due³ to quasiparticle *paths in which a loop is circled twice*, with the charge states during the first and second looping being exactly opposite to each other.

To identify the order-parameter field Q and its low-energy effective theory for class C , one may proceed in several ways. The direct method, due to Ref. 15 and worked out in detail for isolated vortices of an s -wave superconductor in Ref. 12, is to start from the BCS mean-field Hamiltonian for the quasiparticles, set up a supersymmetric generating functional for the Gorkov Green's function, introduce a composite field Q to decouple the 4-vertices produced by averaging over the disorder, integrate out the quasiparticle fields, solve two saddle-point equations for Q in sequence (the second of which turns out to coincide with the Usadel equation¹⁶), and finally expand in gradients of Q to obtain the low-energy effective theory. The field theory so obtained is a nonlinear σ model, with Q taking values in a Riemannian symmetric superspace of type $DIII|CI$, in agreement with the random-matrix analysis of Ref. 7. Its coupling constant has the universal meaning of a conductivity for the conserved probability (or energy) current transported by the quasiparticles. Because quasiparticles also carry spin, the coupling constant may be reinterpreted¹³ as a spin conductivity in the present context. (The latter interpretation fails for systems with spin-orbit scattering or magnetic impurities, where spin is not conserved.)

Given the proper identification of the order-parameter field Q , a few qualitative conclusions are *immediate*. According to the renormalization theory of nonlinear σ models,⁸ the sign of the one-loop renormalization-group beta function in two space dimensions is completely determined by the sign of the (Ricci) curvature tensor relative to the metric tensor. Since the curvature of the Riemannian symmetric superspace of type $DIII|CI$ is positive,¹⁷ the weakly coupled two-dimensional theory renormalizes by logarithmic corrections towards strong coupling (i.e., strong disorder), which ultimately leads to localization of all quasiparticle states at $T=0$. This localized phase was called a ‘‘spin insulator’’ in Ref. 13. In dimension $d=1$ the same corrections are present, but with a linear dependence on the cutoff length. In $d=3$ the theory supports a delocalization transition to a phase of extended states, the ‘‘spin metal.’’¹³ The addition of random classical Heisenberg impurity spins (at subcritical concentration, so as to maintain superconductivity) causes crossover from class C to class D ,³ with the nonlinear σ model changing to type $CI|DIII$.⁷ In the process, the sign of the symmetric space curvature gets reversed, whence weak localization turns into weak *antilocalization*, making it possible for extended states to exist *already in two dimensions*.

Our goal here is to extend the treatment of Ref. 12 and illustrate some of the above general facts at the thermal transport of the class- C quasiparticles of a disordered s - or d -wave superconductor in the mixed state.¹⁸ We assume the presence of (nondescript) nonmagnetic impurities, which disorder the vortex array and cause elastic scattering of the quasiparticles. To tackle this problem we will use a coarse-

grained or random-matrix type of approach, placing the emphasis on *symmetry considerations*.

II. EFFECTIVE-FIELD THEORY FROM AN N -ORBITAL MODEL

We begin our treatment by partitioning the superconductor into cells of equal size, with each cell containing one vortex segment with a length of the order of the elastic mean free path ℓ . Within each cell we introduce (in the spirit of the real-space renormalization group) a basis of N quasiparticle wave functions that comprise the relevant low-energy configurations. The matrix of the Hamiltonian in such a basis assumes a sparse block structure, with one block on the diagonal for each cell, and with off-diagonal blocks that couple neighboring cells. If i labels the cells and $a=1, \dots, N$ the orbitals inside a cell, the ‘‘coarse-grained’’ Hamiltonian is of the form

$$H = \sum_{\langle i,j \rangle} \sum_{ab} (h_{ia,jb} (c_{ia\uparrow}^\dagger c_{jb\uparrow} + c_{ia\downarrow}^\dagger c_{jb\downarrow}) + \Delta_{ia,jb} (c_{ia\uparrow}^\dagger c_{jb\downarrow} - c_{ia\downarrow}^\dagger c_{jb\uparrow})/2 + \text{H.c.}),$$

where the sum over i,j is restricted to $i=j$ and pairs of neighboring cells. The spin-singlet nature ($\uparrow\downarrow\text{-}\downarrow\uparrow$) of the coupling to the pairing field is dictated by conservation of spin. Fermi statistics then requires the complex matrix Δ to be symmetric: $\Delta_{ia,jb} = \Delta_{jb,ia}$.¹⁹ If we temporarily suppress the cell and orbital indices, H can be written in the schematic form $H = \text{Tr}(\mathcal{H}\tilde{\mathbf{c}}\tilde{\mathbf{c}}) + \text{const}$, where

$$\tilde{\mathbf{c}} = \begin{pmatrix} c_{\uparrow}^\dagger \\ c_{\downarrow} \end{pmatrix}, \quad \mathbf{c} = (c_{\uparrow} \ c_{\downarrow}^\dagger), \quad \mathcal{H} = \begin{pmatrix} h^T & \Delta^\dagger \\ \Delta & -h \end{pmatrix}.$$

The symmetries of the Hamiltonian matrix \mathcal{H} are summarized by the equation $\mathcal{H} = -\mathcal{C}\mathcal{H}^T\mathcal{C}^{-1}$, with \mathcal{C} being the symplectic unit $\mathcal{C} = i\sigma_2 \otimes \mathbf{1}$. Note that when the Zeeman energy $H_Z = \mu B \sum_{ia} (c_{ia\uparrow}^\dagger c_{ia\uparrow} - c_{ia\downarrow}^\dagger c_{ia\downarrow})/2$ is taken into account, the $SU(2)$ spin rotation invariance of H is broken down to a $U(1)$ symmetry.

Disorder in the microscopic Hamiltonian gives rise to randomness in \mathcal{H} . Because the universal properties at long wavelengths are insensitive to the microscopic details, we have considerable freedom in choosing the random Hamiltonian \mathcal{H} . The simplest choice is an N -orbital model with locally gauge-invariant disorder of the type invented by Wegner²⁰ for the purpose of describing the universal physics of the Anderson localization transition for $\beta=1,2,4$. The crucial new feature in the present case is the relation $\mathcal{H} = -\mathcal{C}\mathcal{H}^T\mathcal{C}^{-1}$, which is invariant under symplectic transformations $\mathcal{H} \mapsto S\mathcal{H}S^{-1}, S^T\mathcal{C}S = \mathcal{C}$. We therefore adopt a model with local $\text{Sp}(2N)$ gauge invariance: the elements of the matrix \mathcal{H} are taken to be Gaussian-distributed uncorrelated random variables with zero mean, $\langle \mathcal{H} \rangle = 0$, and second moments specified by

$$\langle \text{Tr} A \mathcal{H}_{ij} \text{Tr} B \mathcal{H}_{kl} \rangle = \frac{w_{ij}}{2N} (\delta_{ij}^{kl} \text{Tr} AB - \delta_{ij}^{kl} \text{Tr} ACB^T\mathcal{C}^{-1}),$$

where $\delta_{ij}^{kl} = \delta_{ik}\delta_{jl}$, and w_{ij} is a rapidly decreasing function of the distance between the cells i and j . Aside from respect-

ing the symmetries and locality of the Hamiltonian, this choice has the virtue of maximizing the information entropy. The main benefit from using such a maximum entropy model is that the introduction of the supermatrix Q , usually a tricky step that requires some expertise, becomes straightforward as we now proceed to show.

We replace the operators $\tilde{\mathbf{c}}, \mathbf{c}$ by classical fields $\tilde{\psi}, \psi$: $\tilde{\mathbf{c}}_{i\alpha\alpha'} \mathbf{c}_{j\beta\beta'} \rightarrow \sum_{\sigma} \tilde{\psi}_{i\alpha\alpha, \sigma} \psi_{\sigma, j\beta\beta'}$ and integrate bilinears in $\tilde{\psi}, \psi$ against $\exp i \text{Tr}(\mathcal{H} - E) \tilde{\psi} \psi$ in the usual way to generate the Gorkov Green's function at energy E . The introduction of a bosonic partner ($\sigma = B$) for each fermionic field ($\sigma = F$) serves to cancel vacuum graphs by the mechanism of supersymmetry. There is one complication, however: the matrix $\tilde{\psi} \psi$ does not share the symplectic symmetry of the Hamiltonian. To remedy this mismatch, we introduce an extra quantum number ("pseudo charge") $c = \pm 1$, so that the quasiparticle fields expand to tensors $\tilde{\psi}_{i\alpha\alpha, \sigma c}$ and $\psi_{\sigma c, i\alpha\alpha}$.⁷ On imposing the conditions $\tilde{\psi} = C \psi^T \gamma^{-1}$ and $\psi = -\gamma \tilde{\psi}^T C^{-1}$, where γ is a real orthogonal matrix that will be specified shortly, we have the symmetry $\tilde{\psi} \psi = -C(\tilde{\psi} \psi)^T C^{-1}$ as desired.

Since the order parameter Q is a local field, its nature can be uncovered by looking at the Hamiltonian truncated to a single cell. With this truncation temporarily in force, we introduce Q as follows:

$$\begin{aligned} & \int d\mathcal{H} \exp(-N \text{Tr} \mathcal{H}^2 / 2w_0 + i \text{Tr} \mathcal{H} \tilde{\psi} \psi) \\ &= \int dQ \exp(-N \text{STr} Q^2 / 2w_0 + i \text{STr} Q \psi \tilde{\psi}). \end{aligned}$$

The equality is verified by using the cyclic invariance of the (super)trace: $\text{Tr}(\tilde{\psi} \psi)^2 = \text{STr}(\psi \tilde{\psi})^2$. The Hubbard-Stratonovitch field Q is a 4×4 supermatrix which, by its coupling to $\psi \tilde{\psi}$, inherits the symmetry

$$Q = -\gamma Q^T \gamma^{-1}. \quad (1)$$

The constraints relating $\tilde{\psi}$ and ψ to one another are compatible only if γ^2 equals the superparity matrix (+1 on bosons, -1 on fermions). To meet this condition we put

$$\gamma = E_{BB} \otimes \sigma_1 + E_{FF} \otimes i\sigma_2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & i\sigma_2 \end{pmatrix}.$$

Relation (1) is the defining equation of an orthosymplectic Lie algebra and is invariant under $Q \mapsto TQT^{-1}$ with $\gamma(T^{-1})^T \gamma^{-1} = T \in \text{OSp}(2|2)$. In the general case, where Green's functions at n different energies are to be averaged, Q acquires matrix dimension $4n \times 4n$, and the symmetry group gets enlarged to $\text{OSp}(2n|2n) \equiv G$.

Returning now to the full lattice problem, introducing Q_i for every cell i , and integrating over the quasiparticle fields $\psi, \tilde{\psi}$ we arrive at the following action functional:

$$S/N = \sum_{ij} w^{-1}_{ij} \text{STr} Q_i Q_j / 2 + \sum_i \text{STr} \ln(Q_i - \omega \otimes \Sigma_3),$$

where $(\Sigma_3)_{\sigma c, \sigma' c'} = \delta_{\sigma \sigma'} (\sigma_3)_{cc'}$ and ω is a diagonal matrix containing the energies at which the quasiparticle Green's functions are to be evaluated. Variation of S yields the saddle-point equation $\sum_j w^{-1}_{ij} Q_j = (\omega \Sigma_3 - Q_i)^{-1}$, whose physical solution (dictated by causality of the Green's function) at $\omega = 0$ and homogeneous in space is $Q^0 = i\nu \Sigma_3$ with $\nu^{-2} = \sum_j w^{-1}_{ij}$. Low-energy fluctuations result from setting $Q_i = T_i Q^0 T_i^{-1}$ and taking $T_i \in G$ to vary slowly with the position of the cell i . By expanding in gradients, the low-energy effective action for such configurations at $\omega = 0$ is easily seen to be a nonlinear σ model,

$$S_0 = -\frac{\pi\nu}{8} \int d^3x \text{STr}(D_{\perp}(\nabla_{\perp} Q)^2 + D_{\parallel}(\nabla_{\parallel} Q)^2), \quad (2)$$

where we have switched to continuous coordinates $x_{\perp} = (x, y)$ and $x_{\parallel} = z$. The parameter ν is the density of states of the superconductor, and D_{\parallel}, D_{\perp} are the field-axis and transverse effective diffusion constants of the quasiparticle gas. We are using units $\hbar = 1$. At finite ω , the field-theory action is perturbed by a term

$$S_{\omega} = \frac{i\pi\nu}{2} \int d^3x \text{STr} \omega \Sigma_3 Q, \quad S = S_0 + S_{\omega}. \quad (3)$$

We have rescaled the field to $Q = T \Sigma_3 T^{-1}$. Since this expression for Q is invariant under $T \rightarrow Tk$ for $k = \Sigma_3 k \Sigma_3 \in \text{GL}(n|n) \equiv K$, the supermatrix Q lives on a coset space G/K . If we parametrize Q by

$$Q = \exp \begin{pmatrix} 0 & X \\ \tilde{X} & 0 \end{pmatrix} \Sigma_3,$$

positivity of S_0 or equivalently, stability of the functional integral, requires $\tilde{X}_{BB} = +X_{BB}^{\dagger}$ and $\tilde{X}_{FF} = -X_{FF}^{\dagger}$. In invariant mathematical language, this means $Q_{BB} \in \text{SO}^*(2n)/\text{U}(n)$ (Ref. 21) and $Q_{FF} \in \text{Sp}(2n)/\text{U}(n)$, which are symmetric spaces of type $DIII$ and CI — hence the name $DIII|CI$ for the present nonlinear σ model. The same effective theory (restricted to the FF sector due to the use of fermionic replicas) was obtained in Ref. 13, based on a quasiparticle Hamiltonian for a dirty $d_{x^2-y^2}$ superconductor with orbital coupling to a magnetic field. This is no surprise, as that system belongs to symmetry class C and the order-parameter field Q and its low-energy effective theory are determined solely by symmetry. (Incidentally, the classification scheme of Ref. 3 assigns the quasiparticles of the $d_{x^2-y^2}$ superconductor in zero field to class CI . According to Ref. 7, the corresponding symmetric superspace is $D|C$, also in agreement with the findings of Ref. 13.)

In order to break parity and account for the Hall angle, one would need to add to the Lagrangian a topological density proportional to $\epsilon^{kl} \text{STr} Q \partial_k Q \partial_l Q$, which is closely related to Pruisken's θ term²² well known from the theory of the integer quantum Hall effect. In two dimensions this term integrates to a winding number and is nontrivial, since the fundamental group of $\text{U}(n)$ is $\Pi_1(\text{U}(n)) = \mathbf{Z}$ and there exists the topological identity $\Pi_1(\text{U}(n)) = \Pi_2(\text{Sp}(2n)/\text{U}(n))$. However, such a topological term does not affect the results

for the longitudinal spin and thermal conductivities presented below and will therefore be omitted.

The maximum entropy derivation presented here does not supply microscopic expressions for the couplings νD_{\parallel} and νD_{\perp} . (We can express them in terms of the random-matrix parameters w_{ij} , but this is neither illuminating nor useful.) These parameters can either be calculated from (quasi)classical transport theory²³ or, better yet, taken from experiment. In the latter case we extract the (bare) coupling constants from experiments conducted at temperatures high enough so that the transport is classical, and then use the field theory (2) and (3) to predict the quantum corrections that emerge at lower temperatures.

III. WEAK LOCALIZATION IN CLASS C

The field theory (2) does not apply to charge transport, as the condensate carries charge and quasiparticle charge is not a constant of motion. The energy, however, and for class C also the spin of a quasiparticle, are conserved, which allows one to probe for quasiparticle transport and localization by measuring the *thermal* and *spin* transport. To obtain the relevant transport coefficients we start from the bilocal conductivity tensor

$$\tau_{ll}(\mathbf{x}, \mathbf{x}'; E) = \sum_{\alpha\alpha'} v_{l\alpha}^{(\mathbf{x})} G_{\alpha\alpha'}^R(\mathbf{x}, \mathbf{x}'; E) v_{l\alpha'}^{(\mathbf{x}')} G_{\alpha'\alpha}^A(\mathbf{x}', \mathbf{x}; E),$$

which describes the nonlocal linear response of the spin current to a perturbation due to the Zeeman coupling with an applied field. The quantities G^R and G^A are the retarded and advanced Gorkov Green's functions and $v_{l\alpha} = [i(\sigma_3)_{\alpha\alpha}(\tilde{\partial}_l - \tilde{\partial}_l) - 2eA_l]/2m$ is the l component of the velocity operator. We use the relation $\mathcal{H} = -\mathcal{C}\mathcal{H}^T\mathcal{C}^{-1}$ to express G^A at energy E by G^R at energy $-E$.¹² Disorder averaging and the mapping on the nonlinear σ model with $n=2$ then turn the tensor τ_{ll} into a correlation function of the conserved $\text{OSp}(4|4)$ Noether current $\mathcal{J} = (Q\nabla Q)^{\text{B11, B22}}$ of the field theory:

$$\langle \tau_{ll}(\mathbf{x}, \mathbf{x}') \rangle = (\pi\nu D_l)^2 \langle \mathcal{J}_l(\mathbf{x}) \bar{\mathcal{J}}_l(\mathbf{x}') \rangle. \quad (4)$$

The second superscript in the expression for the Noether current refers to the pseudocharge, while the third distinguishes between the two Green's functions. The symmetry-breaking perturbation due to the quasiparticle energy E is incorporated into the formalism by setting $\omega = \text{diag}(E^+, -E^-)$, where $E^{\pm} = E \pm i0$.

Next, let $\sigma_{ll} = (2\pi)^{-1} \int \langle \tau_{ll}(\mathbf{x}, \mathbf{x}') \rangle d^3x'$ be the local ‘spin’ conductivity for quasiparticles with fixed spin up or down. To compute this quantity from the correlator (4), we adopt a rational parametrization for Q ,

$$Q = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1}.$$

Inserting this parametrization into the field-theory action (2), and doing the functional integral in Gaussian approximation (tree level), we obtain $\sigma_{ll} = \sigma_{ll}^0$ with

$$\sigma_{ll}^0 = \nu D_l,$$

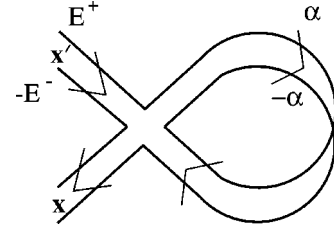


FIG. 1. One-loop diagram contributing to the correlator $\langle \tau_{ll}(\mathbf{x}, \mathbf{x}') \rangle$. The electric charge of the quasiparticle during the second looping ($-\alpha$) is opposite to the charge during the first looping (α).

which is the result expected from quasiclassical transport theory. The weak-localization correction to $\langle \tau_{ll}(\mathbf{x}, \mathbf{x}') \rangle$ arises from one-loop graphs of the kind shown in Fig. 1; see Ref. 12. The basic element of this graph is a 4-vertex representing the fourth-order term in the Taylor expansion of S with respect to Z, \tilde{Z} . A double line oriented by an arrow stands for the bare propagator $\langle Z\tilde{Z} \rangle_0$. All one-loop graphs are composed of three propagators and one 4-vertex. Although these graphs appear as a calculational device for organizing the field-theoretic perturbation expansion, they do have a direct physical meaning, as follows. Each of the two single lines in Fig. 1 stands for a Feynman path contributing to the Gorkov Green's function $G^R(\mathbf{x}, \mathbf{x}'; \pm E)$. Double lines represent sums of impurity ladders with an arbitrary number of Andreev scattering events inserted. It is seen from Fig. 1 that one of the two Green's function lines proceeds directly from the point \mathbf{x}' to the point \mathbf{x} , whereas the other one makes an excursion in the form of a double loop. The propagator associated with the double loop is called the D -type cooperon.³ What is essential here is that the charge of the quasiparticle during the second looping is exactly opposite to the charge during the first looping. This feature makes the D -type cooperon stable with respect to disorder averaging irrespective of the orbital coupling to a magnetic field by canceling the Aharonov-Bohm phase $\oint \mathbf{A} \cdot d\mathbf{l}$ accumulated in the loop. Figure 1 also indicates the fact¹⁰ that the present variant of the weak-localization phenomenon already affects a single Green's function and thus the density of states.

By evaluating the one-loop graphs in a similar manner as in Ref. 12, we obtain

$$\delta\sigma_{ll} = -\frac{D_l}{\pi} \text{Re} \int \frac{d^3k}{(2\pi)^3} (D_{\parallel}k_{\parallel}^2 + D_{\perp}k_{\perp}^2 + 2iE)^{-1}. \quad (5)$$

The full spin conductivity is $\sigma_{ll} = \sigma_{ll}^0 + \delta\sigma_{ll} + \dots$. Note that the correction is formally similar²⁴ to that for class AI, except that it explicitly depends on energy. In fact, it *disappears with increasing excitation energy*, or temperature, in agreement with the fact¹² that moving up in energy causes crossover from class C to class A, where weak localization is absent. In dimension $d \leq 2$ the integral over wave numbers is cut off in the infrared by the inverse of the dephasing length L_{ϕ} due to inelastic (or quasielastic²⁵) scattering, while for dimension $d \geq 2$ it is UV regularized by the inverse elastic mean free path.

Next recall that the quasiparticle spin is assumed to be conserved, which allows one to consider the sectors with spin up ($s = +1/2$) and spin down ($s = -1/2$) separately.

Turning on the Zeeman coupling is equivalent to shifting the excitation energy $E \rightarrow E - s\mu B$. As a result, the energy dependence of the weak-localization correction translates into a *field dependence*. Note that this effect differs from weak localization in disordered metals,²⁴ where the orbital coupling to a magnetic field causes class AI to cross over to class A. In that case, the field scale is set by $B_0 = (eD\tau_\varphi)^{-1}$ with τ_φ being the dephasing time. In the present case the relevant field scale is $B_Z = (\mu\tau_\varphi)^{-1}$.

To compute the thermal conductivity κ at temperature T , we use the relation

$$\kappa_{ll} = \sum_{s=\pm 1/2} \int_0^\infty \sigma_{ll}(E - s\mu B) \frac{\partial f_T}{\partial T}(E) E dE,$$

where $f_T(E) = (1 + e^{E/T})^{-1}$ is the Fermi-Dirac distribution, and our unit of temperature is such that $k_B = 1$. If the energy dependence of σ^0 can be neglected in the range $0 < E \lesssim T$, and if $T \lesssim \text{Max}(\mu B, \Gamma_\varphi)$, i.e., $\delta\sigma$ is cut off by the Zeeman energy or the dephasing rate $\Gamma_\varphi = \tau_\varphi^{-1}$, rather than by the temperature, we may pull out $\sigma_{ll}(E)$ from under the integral sign, thus obtaining an analog of the Wiedemann-Franz law:

$$\frac{\kappa_{ll}(B)}{T} = \frac{\pi^2}{3} \sigma_{ll}(\mu B). \quad (6)$$

Here we have combined the spin-up and spin-down contributions by assuming the quasiclassical term σ_{ll}^0 to be unaffected by the Zeeman splitting.

IV. ISOLATED VORTICES

We now specialize to an extreme type-II s -wave superconductor in a weak magnetic field (but well into the mixed state so that the field is approximately homogeneous), where quasiparticles are bound to a dilute array of vortex cores and the amplitude to hop from one vortex to another is negligibly small. In this case the problem reduces to a set of decoupled one-dimensional theories, one for each vortex, and we formally set $D_\perp = 0$. The parameters of the one-dimensional nonlinear σ model were calculated by solving the Usadel equation for a single vortex in Ref. 12, where we found $D_\parallel = C_2 v_F / 3 C_1$, and $\nu \int d^2 x_\perp = 2 \nu_N \pi \xi^2 C_1$ if the integral extends over the area occupied by one vortex. The parameter ξ is the dirty coherence length, ν_N is the density of states of the normal metal, and $C_1 = 3.16$ and $C_2 = 1.20$ are numerical constants dependent on the vortex profile. Using the fact that the total number of vortices equals the transverse area of the sample divided by half the square of the magnetic length $l_B = \sqrt{2\pi/eB}$, we obtain $\sigma_{ll}^0 = 4\pi C_2 \nu_N (\xi/l_B)^2 v_F / 3$ for the quasiclassical limit of the spin conductivity. The weak-localization correction is given by

$$\delta\sigma_{ll} = -\frac{2D_\parallel}{\pi l_B^2} \text{Re} \int \frac{dk}{2\pi} [D_\parallel k^2 + \Gamma_\varphi + 2i(E - s\mu B)]^{-1},$$

where inelastic events were incorporated by shifting the denominator by Γ_φ . This result applies when the dephasing length $L_\varphi = \sqrt{D_\parallel/\Gamma_\varphi}$ is shorter than the vortex length L_\parallel . In the opposite, mesoscopic regime ($L_\parallel \ll L_\varphi$) the weak localization effect was worked out in Ref. 12.

The low-temperature behavior of the thermal conductivity depends on how Γ_φ varies with T . If we assume a power law $\Gamma_\varphi \sim T^p$ with exponent $p < 1$,²⁵ then σ_{ll} becomes constant in the energy range where $df_T(E)/dT$ is essentially different from zero, and we get the Wiedemann-Franz law (6) with

$$\sigma_{ll}(\mu B) = \sigma_{ll}^0 - (\pi l_B^2)^{-1} \text{Re} \sqrt{D_\parallel / (\Gamma_\varphi + i\mu B)}.$$

In the high-field regime $\mu B \gg \Gamma_\varphi$ the weak-localization correction to the thermal conductivity is cut off by the Zeeman splitting, giving a characteristic dependence $\delta\kappa_{ll}/T \sim -1/\sqrt{B}$. On the other hand, if $p > 1$ then the relevant low- T regime is $T \gg \Gamma_\varphi$, and the weak-localization effect is cut off by T for low fields. In that case, one finds

$$\kappa_{ll}/T = \frac{\pi^2}{3} \sigma_{ll}^0 - \frac{3}{4} \sqrt{\frac{\pi}{2}} (\sqrt{2} - 1) \zeta(3/2) L_T / \pi l_B^2,$$

i.e., the quantum correction is determined by the thermal length $L_T = \sqrt{D_\parallel/T}$.

The above considerations apply to a vortex array in the dilute limit near H_{c1} . As the field is increased, the quasiparticle hopping rate between vortices in an s -wave superconductor grows strongly. When the field is tuned close to H_{c2} , where the system of vortex cores becomes dense, the diffusion constant D_\perp gets large and the anisotropic field theory (2) three dimensional. Since the quasiparticle states of the weakly disordered three-dimensional system are extended, a delocalization transition must take place with increasing field. Note that this transition is not in a new universality class, as the breaking of spin-rotation invariance by the Zeeman coupling reduces class C to class A.¹³ Nevertheless, the occurrence of such a delocalization transition may be of experimental interest, for it can be observed by *varying the magnetic field* (instead of the disorder strength or the chemical potential).

V. WEAK LOCALIZATION IN THE CUPRATES

We now adapt our results to the very interesting case of quasi two-dimensional d -wave superconductors such as the cuprates. As was stated before, the low-energy quasiparticles of a dirty d -wave superconductor in zero magnetic field belong to symmetry class CI. Weak-localization effects in that class arise from two distinct modes of quantum interference:³ the cooperon of type A, and the cooperon of type D. The former is the natural analog of the cooperon mode well known from the theory of disordered metals.²⁴ When time-reversal symmetry is broken by a magnetic field penetrating the superconductor, the A-type cooperon becomes massive and disappears over a scale given by $B_0 = (eD\tau_\varphi)^{-1}$. This crossover takes class CI into class C, while leaving weak localization due to the D-type cooperon intact. As we have seen, the latter mode is cut off only by the Zeeman energy, which becomes effective over the characteristic field scale $B_Z = (\mu\tau_\varphi)^{-1}$. Using $\mu = 2\mu_B = e/m$ and $D \sim k_F \ell / m$ we see that the two scales are separated by a large factor: $B_Z/B_0 \sim k_F \ell$, i.e., the elimination of the A-type cooperon by the orbital coupling to the magnetic field takes place at much smaller fields than does the removal of the D-type cooperon by the Zeeman energy. This justifies our explicitly retaining the Zeeman coupling, while burying the orbital coupling via

the introduction of a maximum entropy model. In the following, we take the magnetic field to be applied along the c axis, and assume the system to be well into the mixed state so that the field is approximately homogeneous.

The cuprates are highly anisotropic materials, consisting of weakly coupled Cu O_2 planes, for which $D_{\parallel} \equiv D_c \ll D_{ab} \equiv D_{\perp}$. At weak interlayer coupling, the continuum approximation leading to Eq. (2) is not justified in the c direction, and we need to restore the discrete layer structure. This is done by making the replacement $D_{\parallel} k_{\parallel}^2 \rightarrow 2t_c(1 - \cos k_{\parallel}a)$, where a is the distance between layers and t_c is the interlayer hopping rate. Then, by performing the integral in Eq. (5) over the domain $L_{\varphi}^{-1} < |k_{\perp}| < \ell^{-1}$ and $-\pi/a < k_{\parallel} < \pi/a$, we obtain

$$\delta\sigma_{\perp} = -(2\pi^2 a)^{-1} \text{Re} \ln[F_s(\Gamma)/F_s(\Gamma_{\varphi})],$$

$$F_s(\varepsilon) = \sqrt{\varepsilon + 4t_c + 2i(E - s\mu B)} + \sqrt{\varepsilon + 2i(E - s\mu B)}, \quad (7)$$

where $\Gamma = D_{\perp}/\ell^2$ and $\Gamma_{\varphi} = D_{\perp}/L_{\varphi}^2$ are the elastic and inelastic scattering rates.

To evaluate the consequences of this general formula, one needs to distinguish cases. For brevity, we concentrate on the limit defined by the condition that elastic scattering sets the largest energy scale: $\Gamma \gg \max(4t_c, 2E, \mu B)$. Consider first the case $4t_c \lesssim \max(2E, \mu B)$, which physically means that the coherence of the quantum interference modes is destroyed before quasiparticles have a chance to hop between layers. The layers then effectively decouple, yielding a two-dimensional system, and the formula for $\delta\sigma_{\perp}$ becomes

$$\delta\sigma_{\perp} = -(4\pi^2 a)^{-1} \text{Re} \ln[\Gamma/(\Gamma_{\varphi} + 2iE - 2is\mu B)].$$

The appearance of a logarithm is characteristic of weak localization in two dimensions. For the in-plane thermal conductivity we get

$$\delta\kappa_{\perp}(B)/T = -(12a)^{-1} \text{Re} \ln[\Gamma/(\Gamma_{\varphi} + i\mu B)],$$

provided that the conditions of validity of the Wiedemann-Franz law (6) are satisfied. Note that in contrast with three-dimensional metals, where weak localization is a rather minute effect, the correction here can easily exceed 10% under experimental conditions. This is because the relative size $\delta\kappa/\kappa$ is roughly given by the inverse of the dimensionless intralayer coupling constant $2\pi\nu_{2d}D_{\perp}$, whose value in zero field has been estimated^{26,13} to be not much in excess of unity.

In the opposite limit, where $4t_c$ is much larger than μB and Γ_{φ} , but still smaller than Γ , the field dependence of the weak localization correction to the thermal conductivity becomes three-dimensional:

$$\frac{\delta\kappa_{\perp}(B)}{T} = -\frac{1}{6a} [\ln\sqrt{\Gamma/t_c} - \text{Re}\sqrt{(\Gamma_{\varphi} + i\mu B)/4t_c}],$$

where again the law (6) was assumed. Note that the above expressions for $\delta\kappa_{\perp}(B)$ increase with B .

To summarize, weak localization in class C , cut off by the Zeeman splitting, causes the thermal conductivity to increase with the magnetic field at sufficiently low temperatures. To make this more quantitative, we need to specify the field/temperature range where the effect becomes observable. The

answer is provided by the value of the spin magnetic moment of the electron (with a g factor of 2), which is 1.35 K/T in suitable units. As a result, if the field strength is of the order of 1 T, the weak-localization-induced field dependence sets in at temperatures below one 1 K (unless for some unexpected reason the dephasing rate Γ_{φ} is anomalously large).

We now wish to elucidate whether such an effect might already be visible in recent experiments. The discussion is somewhat complicated by an ongoing debate concerning the leading, quasiclassical term κ^0 . Let us summarize the current situation as we see it.

Krishana *et al.*²⁷ measured the magnetic field dependence of the thermal conductivity in a $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$ system for temperatures $T \geq 6$ K. After an initial decrease at weak fields, they observed a sharp kink at $B^* \sim \sqrt{T}$, followed by a wide plateau for $B > B^*$. The nonanalyticity at B^* has been interpreted²⁸ as a phase transition to a new ground state with a secondary id_{xy} order parameter. We will not be particularly concerned with that issue here. [The addition of an id_{xy} component to the order parameter is fully compatible with the symmetries of class C and, if disorder is present, the field theory (2) for the quasiparticle excitations remains qualitatively unchanged.] From the observation of field independence over a sizable range of temperatures, one deduces²⁷ that both the electronic and the phonon contribution to the thermal conductivity must be individually constant. The constancy of the electronic part was initially attributed to the $d_{x^2-y^2} + id_{xy}$ state being fully gapped, i.e., to the complete absence of low-energy quasiparticle excitations. This explanation has been challenged by experimental data of Aubin *et al.* While confirming the results of Ref. 27 for $T > 5$ K, these data reveal the emergence of a positive thermal magnetoconductance at lower temperatures $T \lesssim 1$ K. (The data also show pronounced hysteresis effects whose interpretation remains controversial.) Taking the constancy of the phonon contribution for granted, the observation of such dependence strongly indicates a residual density of quasiparticle states at zero energy. The existence of such states is no surprise. Indeed, in the mixed state of a superconductor with $d_{x^2-y^2}$ wave symmetry a residual density of states is expected even in the absence of disorder, because some fraction of the low-energy quasiparticles (close in momentum to the d -wave nodes) are Doppler shifted to zero energy by the supercurrent circulating around the vortices (the Volovik effect³⁰), which leads to $\nu(E=0, B) = \nu_N \sqrt{B/B_{c2}}$. (For a recent discussion of the same effect for a ground state with $d_{x^2-y^2} + id_{xy}$ symmetry, see Ref. 31.) The residual density of states created by this mechanism is approximately constant in energy below the average Doppler shift scale E_B , roughly estimated by $E_B/T_c \approx \sqrt{B/B_{c2}}$. Disorder can only broaden the range of energy independence of ν . Hence, assuming $B_{c2} \sim 100$ T and a superconducting transition temperature $T_c \sim 100$ K, the energy scale E_B is of order 10 K for fields of a magnitude of about 1 T.

Now recall the experimental observations reported in Ref. 29: an electronic thermal conductivity that is independent of the magnetic field for $T \geq 5$ K (and $H > H^*$), and begins to increase with B below $T \approx 1$ K. (According to a footnote in Ref. 29, the same effect has been seen in $\text{YBa}_2\text{Cu}_3\text{O}_{7-x}$.) The first point to address is the field independence at the higher temperatures. Franz³² has recently proposed a model

for the quasiclassical thermal conductivity κ^0 , in which the increase of ν with the field is exactly canceled by a concomitant decrease of the quasiparticle mean free path ℓ . The model assumes scattering from the superflow due to randomly positioned vortices. In contrast, another recent theory³³ argues in favor of the dominant scattering mechanism being impurities close to the unitarity limit. We will not pursue here the discussion as to which is the correct model to use. With the microscopic theory of the plateau effect being a subject of debate, our philosophy is to accept it as an *experimental fact* that the field variation of ν and ℓ is such as to cancel in $\kappa^0 \sim \nu(B)\ell(B)$. The question to address, then, is *why a field dependence sets in when the temperature is lowered*. We argue that this is at least in part due to weak localization. As we have seen, weak localization in the mixed state of dirty d -wave superconductors is a phenomenon on safe theoretical ground, is sizable in magnitude, and is expected to occur at the right temperature and field scales to match the experiment.²⁹ To preclude any confusion, we stress that the effect under consideration is distinct from weak localization in combination with Aslamasov-Larkin fluctuations, which have been invoked in Ref. 34 to explain the negative thermal magnetoconductance observed in a dirty $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ system at much higher temperatures.

Theories proposed by previous authors attribute the temperature variation of $d\kappa(B,T)/dB$ to the leading (quasiclassical) term, $\kappa^0 \sim \nu\ell$. Given the low-energy constancy of the density of states, such a variation would have to arise from an energy (or temperature) dependence of the elastic mean free path. Possible explanations are (i) low-energy transparency of d -wave vortices to quasiparticles,³² and (ii) energy dependence of the elastic scattering rate due to impurities near the unitarity limit.³³ The challenge to these scenarios is to explain why for fields $B \sim 1$ T the effect sets in at temperatures around 1 K. In the weak-localization scenario we have described, this comes about very naturally if Γ_φ is determined by thermal broadening, since $\mu = 1.35$ K/T.

A clear difference is that weak-localization effects *continue to be enhanced* with decreasing temperature—they ultimately drive the system to an insulator by localization of all quasiparticle states — whereas the energy dependence of the elastic mean free path saturates. To discriminate, it is therefore desirable to push the experimental measurements to the lowest temperatures possible. In order to achieve a quantitative description based on formula (7), it will be necessary to take the field dependence of ℓ into account. Our suggestion is to extract the density of states $\nu(B)$ from measurements of

the specific heat, and then deduce $\ell(B)$ from the quasiclassical formula $\kappa(B) \sim \nu(B)\ell(B)$, valid at high temperatures ($T \gtrsim 5$ K). As far as the temperature dependence of the dephasing rate Γ_φ is concerned, a phenomenological model needs to be used. To our knowledge, a theory for this quantity in the mixed state of dirty d -wave superconductors does not exist. In the long run, weak localization may turn out to be the appropriate tool to *measure* Γ_φ , as is established practice in disordered metals.^{24,25}

VI. CONCLUSION

Noninteracting electrons subject to disorder and a magnetic field are well known to belong to the standard universality class A (unitary symmetry, $\beta = 2$). When spin-singlet pairing correlations are added, the universality class of the low-energy quasiparticles changes to type C . It has been shown that the transport properties of these quasiparticles are unconventional. In particular, there exist modes of destructive quantum interference that survive the orbital coupling to a magnetic field. They are cut off at higher fields by the Zeeman coupling, thereby giving rise to a field-dependent quantum (or weak-localization) correction to the low-temperature thermal conductivity, with the characteristic scale given by $\mu = 1.35$ K/T. A good place to look for such corrections experimentally are disordered low-dimensional superconductors, such as the cuprates, in the mixed state.

On general symmetry grounds, the low-energy effective-field theory for quasiparticles in class C is predicted to be a nonlinear σ model of type $DIII|CI$. The Lagrangian of this field theory has a universal form, independent of the symmetry of the order parameter (s, d , etc.), as long as the superconductor conserves the quasiparticle spin and is penetrated by magnetic flux. The role of the superconducting ground state is merely to determine the values of the field-theory coupling constants, their anisotropy, and their dependence on energy and magnetic field. Quantitative predictions for the weak-localization corrections to transport can be made once the values of the couplings and their dependences have been obtained, either from quasiclassical transport theory or from experiment. We advocate the use of such predictions in understanding the low-temperature experiments of Aubin *et al.*

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