Integrable models of strongly correlated particles with correlated hopping

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The exact solution is obtained for the eigenvalues and eigenvectors for two models of strongly correlated particles with single-particle correlated and uncorrelated pair hoppings. The asymptotic behavior of correlation functions are analyzed in different regions, where the models exhibit different physical behavior. [S0163-1829(99)03805-9]

Integrable strongly correlated electron systems (see, e.g., Ref. 1) have been an important subject of research activity in recent years since they are believed to play a promising role in unraveling the mystery of high-T_c superconductivity. Several integrable correlated fermion systems that manifest superconducting properties so far have appeared in the literature. Most famous are the supersymmetric t-J (Refs. 2–4) model and the Hubbard model.[5] Other integrable correlated electron systems of interest include the correlated hopping model[6] and its generalization extensively investigated in Refs. 7 and 8. The models considered in these last papers describe the dynamics of two type of particles (spin up and spin down) with kinetic terms given by correlated single-particle hopping and uncorrelated hopping in the case of pair motion.

In this paper, we introduce two new integrable quantum chains with correlated single-particle and uncorrelated pair hopping but that have only one type of particle.

We suppose that particles on the chain may be isolated (both nearest-neighbor sites are empty) or part of a two-atom molecule (one of its nearest-neighbor sites is occupied and the other one is empty). This supposition is achieved by imposing the restriction of no simultaneous occupancies of three nearest-neighbor sites. It means that two particles on the neighbor sites create the two-atom molecule that can hop as a whole or disintegrate. The Hamiltonian of such system in the most general form can be presented as follows:

$$\begin{aligned} H=-\sum_j \mathcal{P}\{&(\sigma_{j+1}^+\sigma_j^- + \sigma_j^+\sigma_{j+1}^-)[(1-n_{j-1})(1-n_{j+2}) \\ &+ t_1 n_{j-1}(1-n_{j+2}) + t_2(1-n_{j-1})n_{j+2} + t_{12}n_{j-1}n_{j+2}] \\ &+ t_p(\sigma_{j+1}^+\sigma_{j-1}^- + \sigma_{j-1}^+\sigma_{j+1}^-)n_j + u n_j n_{j+1} \\ &+ n_{j-1}n_{j+1}[V_{11} + V_{12}n_{j+2} + V_{21}n_{j-2} \\ &+ V_{22}n_{j-2}n_{j+2}]\}\mathcal{P}, \end{aligned} \tag{1}$$

where $\sigma_j^+(\sigma_j^-)$ creates (annihilates) a particle at site j and

$$n_j=\sigma_j^+\sigma_j^-$$

is the corresponding occupation number. The model contains correlated single-particle hopping, which is described by the parameters t_1, t_2, t_{12}, and uncorrelated pair hopping described by the parameter t_p. We have also two-, three-, and four-particle static interactions between nearest neighbors. The operator $\mathcal{P}$ in Eq. (1) projects out any configuration with simultaneous occupancies of three nearest-neighbor sites and we assume periodic boundary conditions.

Certainly the Hamiltonian (1) is not integrable for an arbitrary choice of parameters. Therefore our problem is to find under what conditions the Hamiltonian (1) can be treated by the coordinate-space Bethe-ansatz technique.

The obvious way to identify a state of the Hamiltonian (1) with n particles is to specify their positions $x_1, \ldots, x_n$ ordered so that

$$1 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant N. \tag{2}$$

We assume the following ansatz for the wave function. If we have only particles ($x_{i+1} \neq x_i+1$, $i=1,2,\ldots,n-1$) we write

$$\Psi(x_1,\ldots,x_n)=\sum_P A^{1\ldots1}_{P_1\ldots P_n} \exp\left(i\sum_{j=1}^n k_{P_j}x_j\right), \tag{3}$$

where P is the permutation of $1,2,\ldots,n$ and k_P are unknown quasiparticle momenta. The n superscript 1 in the amplitude indicates we have only isolated particles. In the case where we have a pair at the position $x_l, x_{l+1}(x_{l+1}=x_l+1)$ the ansatz is

$$\begin{aligned} &\Psi(x_1,\ldots,x_l,x_{l+1},\ldots,x_n) \\ &\quad=\sum_P A^{1\ldots\overline{11}\ldots1}_{P_1\ldots P_l P_{l+1}\ldots P_n} \exp\left(i\sum_{j=1}^n k_{P_j}x_j\right), \end{aligned} \tag{4}$$

where the bar at the lth and $(l+1)$th position of the superscript indicates the pair's position. The general case with many isolated particles and pairs follows from Eqs. (3) and (4).

It is not difficult to consider the eigenvalue equations if $n=1,2$ or for general n in the case where we have only

isolated particles. These equations give the connections between the coefficients $A^{1\cdots 1}_{P_1\cdots P_n}$.

$$\begin{aligned} &s_{P_jP_{j+1}}A^{\cdots 11\cdots}_{\cdots P_jP_{j+1}\cdots}+[P_j\leftrightarrow P_{j+1}]=0,\\ &N^{(1)}_{P_jP_{j+1}}A^{\cdots 11\cdots}_{\cdots P_jP_{j+1}\cdots}-c^{(1)}_{P_jP_{j+1}}A^{\cdots 11\cdots}_{\cdots P_jP_{j+1}\cdots}\\ &\quad+[P_j\leftrightarrow P_{j+1}]=0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} s_{P_1P_2}&=N^{(1)}_{P_1P_2}c^{(2)}_{P_1P_2}-N^{(2)}_{P_1P_2}c^{(1)}_{P_1P_2},\\ N^{(1)}_{P_1P_2}&=1-e^{ik_{P_2}}V_{11}+e^{i(k_{P_1}+k_{P_2})},\\ N^{(2)}_{P_1P_2}&=t_2e^{ik_{P_2}}+t_1e^{i(k_{P_1}+2k_{P_2})},\\ c^{(1)}_{P_1P_2}&=t_1+t_2e^{i(k_{P_1}+k_{P_2})},\\ c^{(2)}_{P_1P_2}&=-t_p+e^{ik_{P_1}}+e^{ik_{P_2}}-Ue^{i(k_{P_1}+k_{P_2})}\\ &\quad+e^{i(2k_{P_1}+k_{P_2})}+e^{i(k_{P_1}+2k_{P_2})}-t_pe^{2i(k_{P_1}+k_{P_2})}. \end{aligned} \tag{6}$$

In contrast to the XXZ model,[2,9] this is not sufficient to prove that the Bethe ansatz works. In order to do that we must consider the eigenvalue equations at the boundary of the inequalities (2) for the case $n=3$ and 4. This gives a complicated system of equations for the coupling constants of the Hamiltonian (1). We have treated this system on a computer and found the following integrable cases:

$$\begin{aligned} &t_1=t_2=2\cosh\eta,\quad t_{12}=1,\quad t_p=-\varepsilon,\\ &V_{11}=V_{22}=0,\ \ U=2t_p,\quad V_{12}=V_{21}=\varepsilon(t_1^2-1), \end{aligned} \tag{7}$$

and

$$\begin{aligned} &t_1=\frac{\sinh 2\eta}{\cosh 3\eta}e^{2\eta},\quad t_2=-\frac{\sinh 2\eta}{\cosh 3\eta}e^{-2\eta},\\ &t_{12}=1,\quad t_p=\frac{\varepsilon\cosh\eta}{\cosh 3\eta};\\ &V_{11}=2t_p\cosh 2\eta,\quad U=2t_p(1+4\sinh^2\eta\cosh 2\eta),\\ &V_{12}=(e^{-4\eta}-2\cosh 2\eta)t_p,\\ &V_{21}=(e^{4\eta}-2\cosh 2\eta)t_p,\quad V_{22}=0, \end{aligned} \tag{8}$$

where $\varepsilon=\pm 1$.

The Bethe-ansatz equations are derived following the standard procedure.[1,2] Each state of the Hamiltonian is specified by a set of particle rapidities $\lambda_j(j=1,\ldots,n)$ related to the momenta of particles k_j. The rapidities have to satisfy the Bethe-ansatz equations. For the model (7) these equations have the following form:

$$\left[\frac{\varepsilon\sin(\lambda_j+i\eta)}{\sin(\lambda_j-i\eta)}\right]^{N-n}=e^{-iP}\prod_{l=1}^{n}\frac{\sin(\lambda_j-\lambda_l+i\eta)}{\sin(\lambda_j-\lambda_l-i\eta)}, \tag{9}$$

$$e^{ik_j}=\varepsilon\frac{\sin(\lambda_j+i\eta)}{\sin(\lambda_j-i\eta)}, \tag{10}$$

and for the model (8) these equations are

$$\begin{aligned} &\left[\frac{\varepsilon\sin(\lambda_j-i\eta)}{\sin(\lambda_j+i\eta)}\right]^{N-n}\\ &\quad=e^{-iP}\prod_{l=1}^{n}\frac{\cos(\lambda_j-\lambda_l+i\eta)}{\cos(\lambda_j-\lambda_l-i\eta)}\frac{\sin(\lambda_j-\lambda_l-2i\eta)}{\sin(\lambda_j-\lambda_l+2i\eta)}, \end{aligned} \tag{11}$$

$$e^{ik_j}=\varepsilon\frac{\sin(\lambda_j-i\eta)}{\sin(\lambda_j+i\eta)}. \tag{12}$$

In both cases

$$P=\sum_{l=1}^{n}k_l \tag{13}$$

is the momentum and the energy of the system is given by

$$E=-2\varepsilon\sum_{j=1}^{n}\left[\cosh 2\eta-\frac{\sinh^2 2\eta}{\cosh 2\eta-\cos 2\lambda_j}\right]. \tag{14}$$

It is interesting to observe the similarity of the Bethe-ansatz equations (9) and (11), and those of the spin-1 Zamolodchikov-Fateev model[10] and Izergin-Korepin model,[11] respectively. Although both models are exactly integrable, let us restrict ourselves to the more physically interesting model (7) with density $\rho<1/2$. There are different regions, with distinct physical properties depending on the parameter $\Delta=-\varepsilon\cosh\eta$.

(1) $\Delta=<-1,\varepsilon=+1,\eta$ is real. We have a similar situation as in the ferromagnetic noncritical XXZ chain with fixed magnetization.[2,12] The ground state must contain exactly one string of maximum length. It means that there is a gap for the arbitrary concentration of particles and the system is in a phase-separated region, where all the particles prefer to stay together.

(2) $\Delta>1,\varepsilon=-1,\eta$ is real. We may analyze this case by considering the limiting case $\eta\to+\infty$. From this analysis it is clear that the ground state contains $n/2$ bound pairs characterized by a pair of complex particle rapidities,

$$\lambda^{\pm}_{\alpha}=\frac{1}{2}(\lambda^{(2)}_{\alpha}\pm i\eta). \tag{15}$$

Inserting Eq. (15) into Eqs. (10) and introducing the density function $\rho(\lambda)$ for the distribution of $\lambda^{(2)}_{\alpha}$ in the thermodynamic limit, we obtain the linear integral equation

$$\begin{aligned} &2\pi\rho(\lambda)+\int_{-\lambda_0}^{\lambda_0}[2\Phi^{(1)}(\lambda-\lambda';2\eta)\\ &\quad+\Phi^{(1)}(\lambda-\lambda';4\eta)]\rho(\lambda')d\lambda'\\ &\quad=\left(1-\frac{n}{N}\right)[\Phi^{(1)}(\lambda,\eta)+\Phi^{(1)}(\lambda,3\eta)], \end{aligned} \tag{16}$$

where

$$\Phi^{(1)}(\lambda,\eta)=\frac{\sinh\eta}{\cosh\eta-\cos\lambda}. \tag{17}$$

The parameter λ_0 is determined by the subsidiary condition for the total density $\rho=n/N$ of particles

$$\int_{-\lambda_0}^{\lambda_0}\rho(\lambda)d\lambda=\frac{1}{2}\rho. \tag{18}$$

Similar calculations gives us the ground-state energy

$$\frac{1}{N}E=2\rho\cosh 2\eta-2\sinh 2\eta\int_{-\lambda_0}^{\lambda_0}[\Phi^{(1)}(\lambda,\eta)+\Phi^{(1)}(\lambda,3\eta)]\rho(\lambda)d\lambda. \tag{19}$$

The other two regions are obtained from $-1<\Delta=-\varepsilon\cosh\eta<1$. Denoting $\eta=i\gamma,\lambda=i\mu$, instead of Eq. (9) we have

$$\left[\frac{\varepsilon\sinh(\mu_j+i\gamma)}{\sinh(\mu_j-i\gamma)}\right]^{N-n}=e^{-iP}\prod_{l=1}^{n}\frac{\sinh(\mu_j-\mu_l+i\gamma)}{\sinh(\mu_j-\mu_l-i\gamma)}. \tag{20}$$

(3) In the case $-1<\Delta<0,0<\varepsilon=+1,0,\gamma<\pi/2$ we solved numerically Eq. (20) for N up to 100 and checked that the state now contains only strings of minimal length 1,[13] i.e., all particle rapidities $\{\mu_j\}$ have imaginary part $i\pi/2$ (antiparticles). In the thermodynamic limit we then have the following integral equations for the distribution function:

$$2\pi\sigma(\mu)-\int_{-\mu_0}^{\mu_0}\Phi^{(2)}(\mu-\mu',2\gamma)\sigma(\mu')d\mu'=-\left(1-\frac{n}{N}\right)\Phi^{(3)}(\mu,2\gamma), \tag{21}$$

$$\int_{-\mu_0}^{\mu_0}\sigma(\mu')d\mu'=\rho, \tag{22}$$

where

$$\Phi^{(2)}(\mu,\gamma)=\frac{\sin\gamma}{\cosh\mu-\cos\gamma},$$

$$\Phi^{(3)}(\mu,\gamma)=\frac{-\sin\gamma}{\cosh\mu+\cos\gamma}, \tag{23}$$

and the ground-state energy is given by

$$\frac{1}{N}E=-2\rho\cos 2\gamma+2\sin 2\gamma\int_{-\mu_0}^{\mu_0}\Phi^{(3)}(\mu,2\gamma)\sigma(\mu)d\mu. \tag{24}$$

(4) In the case $0<\Delta<1,\varepsilon=-1,0<\gamma<\pi/2$ our numerical results of Eq. (20) for lattice sizes N up to 100 indicate that the ground state contains only bound pairs, characterized by a pair of complex-particle rapidities[14,15] as in region (15),

$$\mu_\alpha^{\pm}=\frac{1}{2}(\mu_\alpha^2\pm i\gamma). \tag{25}$$

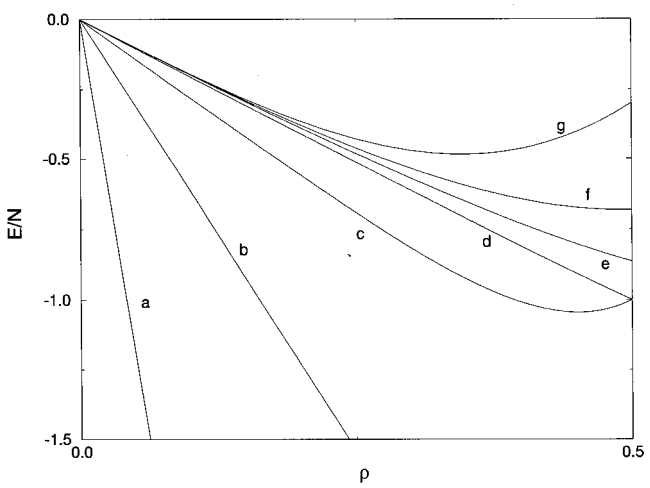

FIG. 1. The ground-state energy as a function of the density ρ for some values of $\Delta=-\varepsilon\cosh\eta$, in regions (2), (3), and (4). *a*, $\Delta=11.5920$; *b*, $\Delta=2,5092$; *c*, $\Delta=0.7071$; *d*, $\Delta=0.1423$; *e*, $\Delta=-0.9239$; *f*, $\Delta=-0.8090$; *g*, $\Delta=-0.5$.

For the density function $\sigma^{(2)}(\mu)$ we have the following integral equations:

$$2\pi\sigma^{(2)}(\mu)+\int_{-\mu_0^{(2)}}^{\mu_0^{(2)}}[2\Phi^{(2)}(\mu-\mu';2\gamma)+\Phi^{(2)}(\mu-\mu';4\gamma)]\sigma^{(2)}(\mu')d\mu'=(1-\rho)[\Phi^{(2)}(\mu,\gamma)+\Phi^{(2)}(\mu,3\gamma)] \tag{26}$$

$$\int_{-\mu_0^{(2)}}^{\mu_0^{(2)}}\sigma^{(2)}(\mu')d\mu'=\frac{1}{2}\rho, \tag{27}$$

and the ground-state energy is given by

$$\frac{1}{N}E=2\rho\cos 2\gamma-2\sin 2\gamma\int_{-\mu_0^{(2)}}^{\mu_0^{(2)}}[\Phi^{(2)}(\mu,\gamma)+\Phi^{(2)}(\mu,3\gamma)]\sigma^{(2)}(\mu)d\mu. \tag{28}$$

Solving numerically the corresponding integral equations in regions 2, 3, and 4 we show in Fig. 1 the ground-state energy as a function of density for some values of Δ. Except for region I, in all regions we expect gapless excitation for $\rho<1/2$. In order to understand the physical properties of the model under consideration we shall investigate the long-distance behavior of the correlation functions. For this purpose we shall use two-dimensional conformal field theory[16,17] and analytic methods[18] to extract finite-size corrections from the Bethe-ansatz equations. The results of these calculations indicate that the critical fluctuations are described by a conformal field theory with the central charge $c=1$. The long-distance power-law behavior of the density-density correlation functions is given by the general form

$$\langle\rho(r)\rho(0)\rangle\simeq\rho^2+A_1r^{-2}+A_2r^{-\alpha}\cos(2k_Fr);$$

$$2k_F=\pi\rho; \tag{29}$$

$$\rho(r)=\sigma_j^+\sigma_j^-, \tag{30}$$

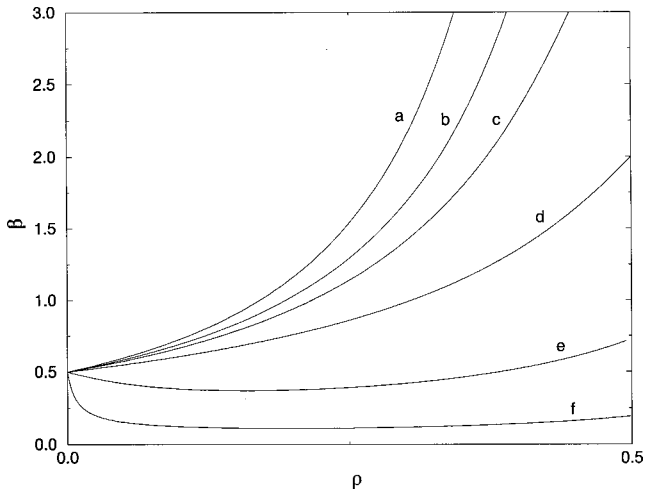

FIG. 2. The exponent β describing the pair-pair correlation function as a function of the density ρ for some values of $\Delta=-\varepsilon\cosh\eta$ in regions (2) and (4). *a*, $\Delta=11.5919$; *b*, $\Delta=1.2039$; *c*, $\Delta=0.7071$; *d*, $\Delta=0.3827$; *e*, $\Delta=0.1423$; *f*, $\Delta=0.0383$.

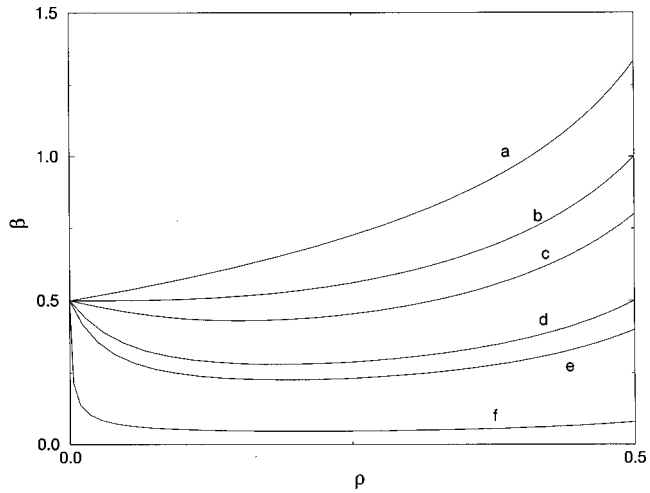

FIG. 3. The exponent β describing the spin-spin correlation function as a function of the density ρ for some values of $\Delta=-\varepsilon\cosh\eta$ in region (3). *a*, $\Delta=-0.5$; *b*, $\Delta=-0.7071$; *c*, $\Delta=-0.8090$; *d*, $\Delta=-0.9239$; *e*, $\Delta=-0.9511$; *f*, $\Delta=0.9980$.

while the pair-correlation function is given by

$$G_\rho(r)=\langle\sigma_j^+\sigma_{j+1}^+,\sigma_{j+r}^-\sigma_{j+r+1}^-\rangle\simeq Br^{-\beta}. \tag{31}$$

The exponents α and β describing the algebraic decay are calculated from the dressed charge function $\xi_0=\xi(\lambda_0)$

$$\beta=\alpha^{-1}=\frac{1}{2[\xi(\lambda_0)]^2}. \tag{32}$$

This function is obtained by the solution of the integral Eqs. (16), (21), and (26) with the right-hand side replaced by $(1-\rho)$. In Fig. 2 we show the exponent β as a function of ρ for several values of Δ in regions (2) and (4). Our numerical results indicate that as $\rho\to 1/2$ the exponent β tends toward the value $4\gamma/\pi$ and $\beta=8(\pi-2\gamma)/\pi$ in regions (2) and (4), respectively. In the region with dominant density-density correlations $\beta>1$ the particles prefer to move individually, instead by pair hopping, but in the region with dominant pair correlations $\beta<1$ they create two-atom molecules. For arbitrary values of $\Delta>1$ there exists a curve $\Delta=\Delta_0(\rho)$ separating both behaviors. An analogous behavior of correlation functions can be observed in the models[7,8] that is translated as a strong tendency to the superconductivity. In region (3) we have no pairs. In Fig. 3 we show the exponent β that describes now the spin-spin correlation function $\langle\sigma_j^+\sigma_{j+r}^-\rangle$. Our numerical results indicate that as $\rho\to 1/2$ the exponent β tends toward the value $\beta=4\gamma/\pi$.

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