Soliton dynamics in two coupled-ring Josephson junctions

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(Received 21 November 1997; revised manuscript received 26 August 1998)

The interaction of nonrelativistic (anti-) fluxons in two coupled-ring Josephson junctions under the action of external currents is considered by using adiabatic perturbation theory. The asymptotic static and dynamic states and conditions for their coexistence are found from qualitative analysis of the system in the phase space. These states correspond to separate stable branches in the current-voltage characteristics, so that hysteresis occurs in a system. Comparison with numerical calculations shows a qualitative agreement with analytical predictions. [S0163-1829(99)01001-2]

I. INTRODUCTION

The dynamics of magnetic flux (fluxons) in a system of coupled long Josephson junctions (LJJ's) is the object of intensive analytical and experimental investigations (see Ref. 1 and referencs therein). Various interesting phenomena, such as phase locking between fluxon modes, flux-flow oscillators, and hyperradiance[2–8] were investigated for the system of coupled LJJ's.

The problem of the dynamics of fluxons in coupled LJJ's was posed in the work of Mineev *et al.*[9] In Refs. 2 and 3, it was shown that, for positive coupling coefficient [$\epsilon>0$, see Eq. (2.1)], slow fluxons with different polarity in different junctions interact attractively and can create a bound state (fluxon-antifluxon pair), while unipolar fluxons interact repulsively. Later it was shown numerically,[5] experimentally,[10] and analytically,[4] that unipolar fluxons in different junctions with velocity larger than some threshold (relativistic effect) can be synchronized and can create a bound (phase-locking, bunched) state.

In the present work the nonrelativistic dynamics of fluxons in two coupled ring LJJ's is considered. We should note that the dynamics in the ring geometry differs essentially from the dynamics in infinite LJJ's (Refs. 2–4) and in finite LJJ's in the ''flux-flow'' regime.[1,7] For example, a new bound state of unipolar fluxons, which corresponds to fluxons in diametrically opposite positions, exists in ring LJJ's (in the absence of external currents). In Sec. II the general statement of the problem, described by the set of coupled Sine-Gordon equations with periodic boundary conditions, is presented. The main idea for qualitative analysis consists of a transformation of the phase space for the infinite system to the phase space of a periodic system. By using this approach, the dynamics of fluxons under the action of a constant ''field'' (see below) is studied. We show that for relatively small velocities (or for small values of direct current) the system has two stationary states. The coexistence of these states leads to hysteretic behavior in the current-voltage (I-V) characteristics. The dependence of the lower threshold current for hysteresis on the ring length and the system parameters is found. Comparison of numerical calculations for the ordinary differential equations (ODE's) and for the full Sine-Gordon equations shows qualitative agreement with analysis on the phase space. In Sec. III we summarize the results obtained.

II. THE MODEL AND RESULTS

The system of coupled long Josephson junctions is described by the set of coupled sine-Gordon equations:[9,2,3,5]

$$\varphi_{n,tt}-\varphi_{n,xx}+\sin(\varphi_n)=-\alpha\varphi_{n,t}-\gamma_n+\epsilon\varphi_{m,xx}, \quad (2.1)$$

where φ_n are the phase differences of wave functions across the junctions, α is the dissipation coefficient, γ_n is the bias current, ϵ is the inductive coupling parameter, $n,m=1,2$; $n \neq m$. The variable x, normalized to the Josephson penetration depth λ_J, is the distance around the mean circumference of the ring. The time t is normalized to the inverse plasma frequency ω_p^{-1}. We consider the parameters α, γ_n, ϵ to be small ($\ll 1$), which corresponds to real LJJ samples.[11] Quasiperiodic boundary conditions should be considered:

$$\varphi_n(x+L,t)=\varphi_n(x,t)+2\pi k_n, \quad (2.2)$$

where $x\in[-L/2,L/2]$, L is the ring perimeter, and k_n is integer. We deal with only $k_n=\pm 1$ case.

Because the further analysis is based on dynamics for the *infinite* system we present some results obtained from perturbation theory. The unperturbed ($\alpha=\gamma_n=\epsilon=0$) (anti-) kink solution for Eq. (2.1) are known to be

$$\varphi_n=4\tan^{-1}\{\exp[\sigma_n(x-X_n)/\sqrt{1-v_n^2}]\}. \quad (2.3)$$

Here $\sigma_n=\pm 1$ for fluxons and antifluxons, respectively; X_n and v_n are the fluxon center and velocity correspondingly.

When two such fluxons are affected by the perturbation and coupling terms, equations for the adiabatic dynamics in the nonrelativistic case ($v_n^2\ll 1$) have the form[3]

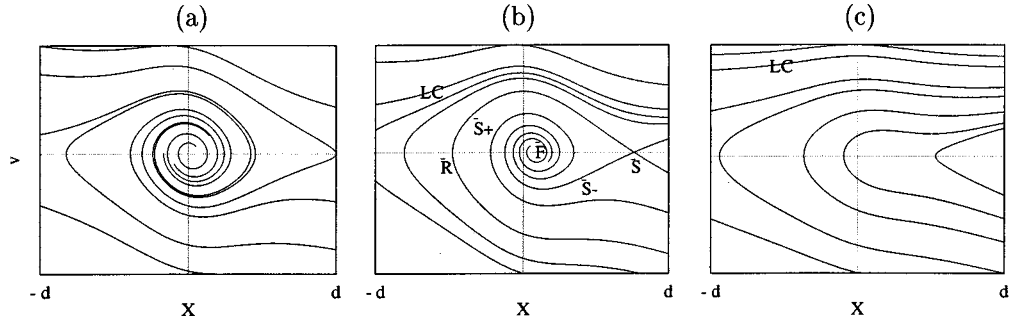

FIG. 1. Three types of cylindrical phase space, K-A case. (a) Stable point $\tilde{F}$ only; (b) stable point $\tilde{F}$ and limit cycle LC; (c) LC only. The phase space for the K-K case is similar, with a shift along X axis.

$$\frac{dv_n}{dt}=-\alpha v_n+\frac{\sigma_n\pi\gamma_n}{4}+\frac{2\epsilon\sigma_1\sigma_2}{\sinh(Z_n)}\left(1-\frac{Z_n}{\tanh(Z_n)}\right),$$
$$\frac{dX_n}{dt}=v_n,\quad Z_n=X_m-X_n. \tag{2.4}$$

Then as follows from Eq. (2.4) with γ_n constant (dc current), the velocity of the pair center tends to the asymptotic value:

$$(v_1+v_2)/2\to\pi\frac{(\sigma_1\gamma_1+\sigma_2\gamma_2)}{8\alpha}. \tag{2.5}$$

Introducing the new variable $X=X_2-X_1$, the distance between fluxons, one can obtain from the set (2.4) the equation[3]

$$\frac{d^2X}{dt^2}+\frac{a}{\sinh(X)}\left(1-\frac{X}{\tanh(X)}\right)=-\alpha\frac{dX}{dt}+\Gamma, \tag{2.6}$$

where $a=2\epsilon\sigma_1\sigma_2$ and $\Gamma=(\pi/4)(\sigma_2\gamma_2-\sigma_1\gamma_1)$. This is Newton's equation for motion of classical particle under an external constant ''field.'' The total energy $E(X,v)$ and the potential energy $U(X)$ of ''the particle'' are

$$E(X,v)=\frac{v^2}{2}+U(X),\quad U(X)=\frac{aX}{\sinh(X)}-\Gamma X. \tag{2.7}$$

Since Eq. (2.6) is symmetric with respect to the transformation $\Gamma\to-\Gamma$, $X\to-X$, one can assume $\Gamma>0$ without loss of generality. We consider both the case of identical polarity of the fluxons with one fluxon (kink) in each junction (K-K case) and the case of opposite polarity (K-A case). Two stationary points, the saddle S and the focus F, can exist on the phase space of Eq. (2.6) with $\Gamma=0$. The coordinates X_S and X_F of these points, found from equation $dU(X)/dX=0$, are, for $\Gamma>0$, both negative (positive) for the K-K (K-A) case. The points S and F exist, only if

$$|\Gamma|\leqslant\Gamma_0\simeq\frac{16}{3\pi^2\sqrt{3}}|a|\simeq0.31|a|. \tag{2.8}$$

Thus, for small current ($|\Gamma|<\Gamma_0$) fluxons in infinite LJJ's stand or move either with constant separation (bound state) or with increasing separation (unbound state), depending on initial conditions. For large current ($|\Gamma|>\Gamma_0$) the distance between fluxons will always increase.

Now let us consider the finite ring geometry. Strictly one should consider the interaction of two periodic waves, each described by Jacobi elliptic functions. However, a good but distinctly simple treatment uses results based on Eq. (2.6), the only difference being that the variable X is finite and restricted to $[-d,d]$, where $d=\frac{1}{2}L$. Additionally, it is necessary to change the phase space (PS) of the system as follows: *cut the plane phase space for the infinite system along the lines* $X=\pm d$ *and ''glue'' it along these lines in order to obtain a cylindrical phase space*. We use tilde to denote points and curves in the cylindrical PS. If $d<\max(|X_S|,|X_F|)$, equilibrium points on $X=d$ should be considered, so that the coordinate of the new saddle point $\tilde{S}$ is defined by $|X_{\tilde{S}}|=\min(|X_S|,d)$. The separatrix branches $\tilde{S}^+,\tilde{S}^-$ coincide with the branches S^+,S^- in the plane PS, if $X_{\tilde{S}}<d$, and differ from them otherwise.

For $\Gamma=0$, there are stable and unstable static states both for K-K and for K-A pairs. The stable state for the K-K case is the state, when fluxons are in diametrically opposite positions. The existence of such a state is special to ring LJJ's. For small Γ ($|\Gamma|<\Gamma_{c1}$) phase space contains only one stable state—the fixed point $\tilde{F}$ [Fig. 1(a)], so fluxons will move with constant separation, $v\equiv v_2-v_1=0$. We call this stationary state *static*. For $\Gamma_{c1}<|\Gamma|<\Gamma_{c2}$ the phase space has two stable states—the point $\tilde{F}$ and the limit cycle (LC) [Fig. 1(b)]. For this region of Γ the fluxon dynamics depends on the way in which the ''field'' Γ changes. For increasing Γ the corresponding state will be the point $\tilde{F}$, while for decreasing Γ the asymptotic state is the *dynamic* stationary state LC. The last case means that fluxons are unbound, but they move with the time-averaged relative velocity having the constant value

$$\langle v\rangle=\langle v_2-v_1\rangle=\Gamma/\alpha. \tag{2.9}$$

Since $\langle v_n\rangle$ is proportional to voltage across the nth junction,[11] two *stable* branches, which correspond to bound and unbound fluxons, in the I-V characteristics may exist, so giving hysteresis phenomena. We should note that hysteresis occurs due to the coexistence of two stable asymptotic states in the fluxon dynamics. Such I-V dependence was also observed in Ref. 7, numerically modeling the ''flux-flow'' regime in finite linear LJJ's. To simplify the analysis of the linear LJJ they used periodic boundary conditions (2.2) and single-mode approximation, which is valid for small junctions, while our consideration is valid for sufficiently long junctions. For $|\Gamma|>\Gamma_{c2}$ the only stable state is a LC [Fig.

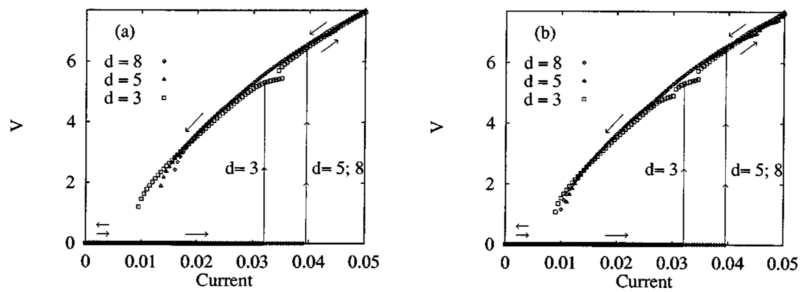

FIG. 2. I-V characteristics $V(|\gamma_n|)$ for different sizes of system. (a) K-K case; (b) K-A case.

1(c)], so that for any initial conditions unbound fluxons will move asymptotically with constant $\langle v \rangle$.

Let us find the threshold values Γ_{c1} and Γ_{c2}. First note that the condition $|\Gamma| > \Gamma_{c1}$ corresponds to the existence condition for the LC. By using the existence theorem and Bendixson criterion for cylindrical PS,[12] one can show that the only possible stationary states for the system are $\tilde{F}, \tilde{S}$, and LC. The limit cycle exists, if

$$d > \tilde{d}_0, \quad \text{otherwise}, \tag{2.10}$$

$$v(d, \tilde{S}^-) \geqslant v(-d, \tilde{S}^+). \tag{2.11}$$

The value $\tilde{d}_0$ is determined from the relation $U(-\tilde{d}_0) - E_{diss}(-\tilde{d}_0, X_{\tilde{S}}, \tilde{S}^+) = U(X_{\tilde{S}})$, where $E_{diss}(a,b,C)$ is the loss of energy due to dissipation as the particle moves along the trajectory C from $x=a$ to $x=b$, so that $E_{diss}(a,b,C) = \alpha \int_a^b v(x,C)dx$, where $v(x,C)$ is the velocity on the trajectory C.

The relation $d < \tilde{d}_0$ means that the separatrix $\tilde{S}^+$ does not intersect the X axis, so that no point $\tilde{R}$ [see Fig. 1(b)] exists. In this case, velocities on the separatrix branches must be compared and Eq. (2.11) can be written by using Eq. (2.7) as

$$|\Gamma| \geqslant \Gamma_{c1} = \left| \frac{E_{diss}(-d, X_{\tilde{S}}, \tilde{S}^+) + E_{diss}(X_{\tilde{S}}, d, \tilde{S}^-)}{2d} \right|. \tag{2.12}$$

Estimating E_{diss} using the dependence $v(X, S^{\pm})$ for $G = \alpha = 0$, Eq. (2.12) can be approximated as

$$|\Gamma| \geqslant \Gamma_{c1} = \frac{\alpha J(d)}{2d}, \tag{2.13}$$

where $J(d) = \int_{-d}^{d} \{2a[1 - X/\sinh(X)]\}^{1/2} dX$ for the K-K case and $J(d) = \int_{-d}^{d} \{2a[d/\sinh(d) - X/\sinh(X)]\}^{1/2} dX$ for the K-A case.

Therefore, an unbound state of fluxons may coexist with a bound state either if the size of the ring exceeds the characteristic length $\tilde{d}_0$, or if the "field" is sufficiently large [Eqs. (2.11) or (2.13)], yet $|\Gamma| < \Gamma_{c2}$. For estimation of the threshold Γ_{c2}, above which only the unbound state is possible, one can use the value Γ_0 from Eq. (2.8).

In order to check the results, we have integrated both Eqs. (2.6) and (2.1) numerically with $\epsilon = 0.1$, $\alpha = 0.05$, $\sigma_1 \gamma_1 = \sigma_2 \gamma_2$. In Fig. 2 the I-V characteristics of coupled LJJ's for both the K-K and K-A cases are shown. The horizontal axis is the absolute value of the current γ_n through either junction. The signs were chosen such that the pair center was stationary [see Eq. (2.5)]. The vertical axis is the time-averaged absolute value of

$$V = \int_{-d}^{d} (\sigma_2 \varphi_{2,t} - \sigma_1 \varphi_{1,t}) dx \simeq -2\pi(v_2 - v_1). \tag{2.14}$$

It represents the *difference* of voltages at the junctions, taking into account the fluxon polarities. As can be seen from Fig. 2, if the current $|\gamma_n|$ increases from 0, then the voltage V initially vanishes, corresponding to the point $\tilde{F}$ in phase space. This situation lasts up to $|\gamma_n| \sim 0.03 - 0.04$ [see Eq. (2.8)], when switching from point $\tilde{F}$ to LC happens. For decreasing current, the voltage V is nonzero until $|\gamma_n| \sim 0.01$ [see Eqs. (2.10), and (2.11)]. The value of V in the LC state is defined by Eqs. (2.9) and (2.14) and is almost independent of the system size d. In Fig. 2 one can see also some resonances, which can be explained by the interaction of fluxon with linear modes.

The results for various d are summarized in Fig. 3. The horizontal axis corresponds to the size of the system d, while the vertical axis is the threshold amplitude: $|\gamma_n^*| = 2\Gamma_{c1}/\pi$. The points, dashed and solid lines denote the results of numerical simulations of partial differential equations (PDE's) (2.1), the ODE (2.6) and calculations using the formula (2.13), respectively. For the K-K case, the results agree well. For the K-A case, Eq. (2.13) provides at least a qualitatively correct result, showing the existence of a maximum in the dependence γ_n^* on d. As the above analysis shows, the saddle point $\tilde{S}$ and the separatrix play an important role. For the K-K case, the saddle point is within the interval $[-d,d]$ for almost all values of Γ and α, while for the K-A

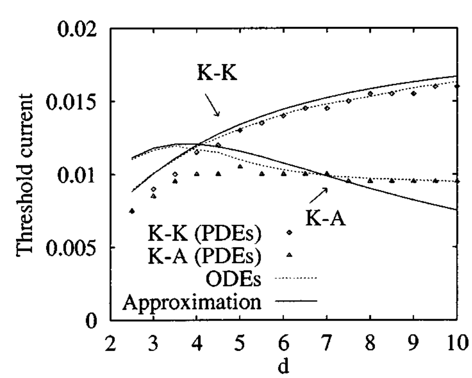

FIG. 3. The dependence of threshold dc current γ_n^*. Points correspond to numerical calculations of Eq. (2.1). Solid lines are the approximation (2.13). Dashed lines are the threshold from numerical calculations of Eq. (2.6).

case the position of the saddle point varies between $+\infty$ and ~ 1. This is a reason for some mismatch in Fig. 3. One can see that our approach is satisfactory down to $d \sim 3-4$. For smaller ring size, the excitation in the LJJ's cannot be approximated by a solitary flux, but should be treated as a periodic traveling wave.

III. CONCLUSIONS

In the present paper, the dynamics of fluxons in two coupled ring Josephson junctions under the action of direct and alternating currents was studied, using an analysis of the periodic system based on the dynamics of an infinite system. By using the glued cylindrical phase space we have shown, that, depending on the problem parameters, only three different types of fluxon behavior occur. These types include (a) motion of fluxons around the ring, with constant separation; (b) motion of fluxons with constant averaged relative velocity; (c) coexistence of the two possible motions. Comparison with numerical simulations demonstrates good qualitative agreement with analytical predictions.

ACKNOWLEDGMENTS

We are grateful to Dr. A. V. Ustinov for useful discussions. E.N.T. wishes also to acknowledge the Royal Society/NATO Postdoctoral Fellowship Program for financial support and the Department of Mathematics and Statistics, Edinburgh University, for its warm hospitality.

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