Classification and stability of phases of the multicomponent one-dimensional electron gas

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The classification of the ground-state phases of complex one-dimensional electronic systems is considered in the context of a fixed-point strategy. Examples are multichain Hubbard models, the Kondo-Heisenberg model, and the one-dimensional electron gas in an active environment. It is shown that, in order to characterize the low-energy physics, it is necessary to analyze the perturbative stability of the possible fixed points, to identify all discrete broken symmetries, and to specify the quantum numbers and elementary wave vectors of the gapless excitations. Many previously proposed exotic phases of multichain Hubbard models are shown to be unstable because of the ''spin-gap proximity effect.'' A useful tool in this analysis is a generalization of Luttinger's theorem, which shows that there is a gapless even-charge mode in any incommensurate N-component system. [S0163-1829(99)03124-0]

I. INTRODUCTION

The basic theory of the low-energy physics of the interacting one-dimensional electron gas (1DEG), both with and without spin, has been well established for two decades. The purpose of this paper is to extend this general analysis to obtain a classification of the stable fixed points of *multicomponent* one-dimensional electronic systems. Examples of such problems include one-dimensional metals with several bands crossing the Fermi surface, such as multichain Hubbard ladders,[1–4] and the ''1DEG in an active environment,''[5] of which the most studied example is the Kondo-Heisenberg model,[6,7] i.e., a 1DEG interacting with a periodic array of localized spins. While these models are still one dimensional, and are amenable to the same methods of solution as the 1DEG, their added richness brings in significant new physics. In particular, in the context of the theory of high-temperature superconductivity, this class of models includes some in which a spin gap and a strongly divergent superconducting susceptibility derive from purely repulsive interactions. Moreover, in these cases, the driving force for the superconductivity is a lowering of the kinetic energy.[5,8]

In one dimension, even at zero temperature, states with a broken continuous symmetry are destabilized by quantum fluctuations. However, there are states with quasi-long-range order which can be characterized by the existence of ''quasi Goldstone modes,'' i.e., gapless collective modes of the system with a soundlike spectrum. The canonical example of a quasi-Goldstone mode is the longitudinal sound mode of a harmonic chain.[9] For a simple 1DEG, the relevant continuous symmetries are spin rotation invariance [SU(2)], global gauge invariance [U(1)] associated with charge conservation, and Galilean invariance. The latter is not an exact symmetry for the electron gas on a lattice but, so long as the electron density is incommensurate with the crystal, the low-energy dynamics possess an exact translational (chiral) symmetry.

A. Classification of phases

It is a remarkable feature of 1DEG's that all the properties of such systems, including fermionic correlation functions, can be expressed in terms of bosonic fields (bosonization) corresponding to the quasi-Goldstone modes. Thus it is possible to classify all thermodynamically distinct ground-state phases of any multicomponent 1DEG by specifying (1) any spontaneously broken discrete symmetries, such as the lattice translation symmetry or parity, and (2) the number and quantum numbers of the fundamental gapless modes. The minimal quantum numbers of the gapless modes are charge, spin, and (crystal) momentum. Our convention will be to focus on spin and charge modes with the smallest nonzero momentum. Here ''spin modes'' have spin 1 and charge 0, and ''charge modes'' have spin 0 and charge $2me$, where m is an integer. Simply counting gapless modes is insufficient; for instance, a state with one gapless charge and one spin mode with the *same* momentum (which we label $[c,s;2k_F]$, or $[cs]$ for short) is distinct from the state in which they have different momenta, $[c;2k_F^{(1)}][s;2k_F^{(2)}]$ (or $[c][s]$ for short).

This scheme differs from the traditional method[10] of classifying phases of the 1DEG in terms of the most divergent susceptibilities, which is appropriate when the goal is to understand the properties of *quasi*-one dimensional systems, since, in most cases, weak higher-dimensional couplings will stabilize a true broken symmetry state with the corresponding order at finite temperature. However, in the context of the 1DEG *per se*, the Luttinger exponents, and hence the exponents governing the divergence as $T\rightarrow 0$ of the various susceptibilities, vary continuously with parameters. For example, one can pass from a region in which the superconducting susceptibility is the most divergent to one in which the CDW susceptibility is the most divergent without encountering any thermodynamic singularities. Thus, *in the strictly one-dimensional context*, the present classification is more appropriate and, in this respect, it extends and corrects the ground-breaking work of Lin, Balents, and Fisher[11] on this subject, while expanding on our earlier work on the general problem of a ''1DEG in an active environment.''[5]

B. Fixed-point strategy

The concept of a fixed point of the renormalization-group equations of a field-theory Hamiltonian was introduced by Wilson[12] for the study of critical phenomena. This idea ultimately made its way into many-body theory, where the renormalization group had been used for some time. The main point is that the low-energy, long-distance physics of a given model is controlled by the properties of the relevant stable fixed point or critical fixed point of the renormalization-group flows. A particularly effective way of determining this behavior is to identify an exactly solvable field-theory model that starts in the neighborhood of an unstable fixed point and flows to the same fixed point as the model in question. This strategy justifies the use of the ''Toulouse limit'' to solve the single-channel Kondo problem,[13] and of field-theory solutions of the 1DEG with attractive backward scattering or umklapp scattering.[10] It is important to note that a field-theory model that does not exhibit spin-rotation invariance may flow to a spin-rotation-invariant fixed-point Hamiltonian. The flexibility in the choice of solvable models allowed by this behavior is frequently exploited, and it will be used later in this paper.

A fixed-point strategy is more difficult to implement when there are many degrees of freedom, as in multicomponent systems. There may be several stable or critical fixed points, and it is necessary to carry out a ''global renormalization''[14] in order to determine which one controls the low-energy physics of a given model. Usually, such a procedure must be carried out numerically. Also, it is necessary to do a different stability analysis for each fixed point separately. The scaling dimension of any given operator is generally different at different fixed points, so an interaction that is irrelevant at one fixed point may become relevant at another.

One of our major findings is that ''the spin-gap proximity effect''[5] is a powerful force for *destabilizing* many putative fixed points of multicomponent one-dimensional systems, and enhancing superconducting correlations. The physics, which is driven by singlet pair tunneling, is analogous to the proximity effect in superconductivity. It serves to lock the superconducting phases of two subsystems, and either generates a spin gap in both subsystems, or transfers a spin gap from one subsystem to another. We have proposed this effect as a mechanism of high-temperature superconductivity. A more detailed discussion is given in Sec. III.

C. Outline of paper

The paper is organized as follows: In Sec. II the theory of the 1DEG is reviewed from a statistical mechanical point of view, and the perspective required for the analysis of the general problem is developed. In Sec. III it is shown how this analysis can be extended to multicomponent systems. Specifically, the classification of the possible fixed points in terms of their spectrum and broken symmetries is discussed, and the conditions for perturbative stability are derived. This section also contains a discussion of generalizations of Luttinger's theorem to one dimension, including a review of a recent proof[15] of the existence of a gapless neutral collective excitation with momentum $2k_F^*$, and a theorem concerning the existence of a gapless charge $2me$ mode, where m is an appropriate integer. The classification of phases and the proof of this theorem makes use of the continuum representation of the so-called η-pairing operator, which is a product of fermion creation operators with the *same* momentum.[16]

Readers who are familiar with the theory of the 1DEG and are primarily interested in the illustrative examples may prefer to proceed directly to Sec. IV, where this general scheme is applied to the analysis of the global phase diagram of the multichain Hubbard model and the Kondo-Heisenberg model. Specifically, it is shown that (1) many of the proposed partially gapped phases[11] of the multichain Hubbard ladder are destabilized by the ''spin-gap proximity effect;''[5] (2) the charge-ordered ''stripe'' structures which have been observed[4] in numerical studies of multichain t-J models have a (possibly nontrivial) relation to the value of $2k_F^*$ derived from the generalized Luttinger's theorem;[15] and (3) there are several, thermodynamically distinct spin-gap phases of the Kondo-Heisenberg model.[17]

II. ONE-COMPONENT 1DEG

To begin with, we consider the (well understood) theory of the single component 1DEG with spin from the perspective of its quasi Goldstone modes. All known zero-temperature thermodynamic states of the 1DEG can be identified by (1) spontaneously broken discrete symmetries, such as parity; (2) whether or not there is a gapless spin and/or charge collective mode; (3) the smallest nonzero wave vector at which these modes are gapless, which in analogy with Fermi-liquid theory is called $2k_F$ (equivalently, $2k_F$ characterizes the long-distance oscillatory behavior of appropriate correlation functions); and (4) the velocity u and ''Luttinger exponent'' K of each gapless mode.

A. Definition of terms

The definition of the terms used above requires some discussion since, in much previous work, the identification of modes is derived from a particular calculational scheme rather than from general principles. Because of the absence of spontaneously broken spin-rotational symmetry, it is possible even in the thermodynamic limit to classify all states by their spin quantum numbers. Thus, by a ''spin mode,'' we mean[18] a neutral excitation with spin 1; the existence of a gapless spin excitation will typically show up as an asymptotic power-law behavior of the spin-spin correlation function, $\langle \vec{S}_{2k_F}(t)\cdot\vec{S}_{-2k_F}(0)\rangle \sim |t|^{1-2\delta_S}$. (Here δ_S is the scaling dimension of this operator.) A charge excitation refers to excitations made by adding a small number of electrons to the system. Typically, this means[19] an excitation with spin 0 and charge $2e$ produced by adding a singlet pair of electrons with total momentum $2k_F$, which will show up in the asymptotic form of the ''η-pairing'' operator,[16] defined below. It is implicit in the above classification scheme that a state without a gapless spin mode has a ''spin gap,'' and that a state without a gapless charge mode has a ''charge gap.''

The physics of $2k_F$ is central to the following considerations. A remarkable theorem of Yamanaka, Oshikawa, and Affleck,[15] which we refer to as the ''generalized Luttinger's theorem,'' fixes $2k_F=2\pi n/g$ at the same value as it would have in a noninteracting electron gas; here n is the total electron density and $g=2$ because electrons have spin $\frac{1}{2}$. The theorem, which we will discuss in more detail in Sec. II, implies the existence of a gapless neutral excitation with momentum $2k_F$ as long as the 1DEG is not commensurately locked to the underlying crystalline lattice. We also show that the η-pairing operator, with the exact same value of $2k_F$, creates a gapless excitation in the same circumstances. When there are both gapless spin and charge modes, k_F can also be identified as the location of a nonanalyticity in the single-particle occupation probability, $\langle n_k\rangle$, but when there is a spin or charge gap, there is no sharp structure in the single-particle spectral function at all. Thus one must be careful in thinking of k_F as a Fermi momentum.

Throughout this paper we will distinguish between the ''excitations'' and the ''modes'' of the system. By a gapless ''excitation'' with given quantum numbers we mean a set of excited states with energies which approach that of the ground state in the thermodynamic limit. A ''mode'' refers to an elementary excitation, which therefore has a well-defined dispersion relation. For example, above, we talked about the spin-1, charge-0 excitations of the system, although, in one dimension, under a broad range of circumstances, the elementary excitations are, in fact, ''spinons'' with spin $\frac{1}{2}$ and charge 0. What this means is that the spin dynamic structure factor will exhibit a branch cut, corresponding to a continuum of two spinon excitations, rather than a pole corresponding to a magnon mode. When classifying states in terms of gapless excitations, we have chosen not to distinguish which are elementary and which are multiparticle excitations.

With this distinction in mind, the definition of the collective-mode velocities is obvious. The existence of Luttinger exponents K_s and K_c is one of the triumphs of bosonization; they dictate the relation between correlation functions expressed in terms of the original electronic variables, and eigenmodes of the underlying bosonic free-field theory. For any given lattice model, the quantum critical exponents K_α must be determined by carrying out a global renormalization[14] to the appropriate fixed point, and matching to the continuum theory. In general, this procedure must be implemented numerically, either by studying the long-distance behavior of correlation functions, or by studying the finite-size scaling behavior of energy levels.

Two of the physically most important low energy fluctuations of the 1DEG are the $2k_F$ charge-density-wave (CDW) fluctuations and the zero-momentum (BCS-like) pairing fluctuations. CDW fluctuations are neutral and spinless. For example, for the repulsive U Hubbard model with a half-filled band, the $2k_F$ density-density correlation function has the same asymptotic (power-law) form as the $2k_F$ spin-spin correlation function, although there is manifestly a charge gap.[20,21]

B. Formal implementation (bosonization)

Formally, the above discussion is equivalent to the statement that the low-energy properties of the 1DEG are directly related to the properties of two independent bosonic field theories with Hamiltonian densities

$$\mathcal{H}_\alpha=\frac{u_\alpha}{2}[(\partial_x\theta_\alpha)^2+(\partial_x\phi_\alpha)^2]+V_\alpha\cos(\beta_\alpha\phi_\alpha), \qquad (1)$$

where $\alpha=c$ and s for the charge and spin fields, respectively, θ_α is the dual field to ϕ_α, or equivalently $\partial_x\theta_\alpha$ is the momentum conjugate to ϕ_α. V_α can be set equal to zero[22] at the gapless fixed point. Otherwise, when V_α is nonzero and relevant; i.e., when $\beta_\alpha<\sqrt{8\pi}$, it sets the scale of the gap,[23] according to the scaling relation $\Delta_\alpha\sim V_\alpha(\Lambda/V_\alpha)^{1-\beta_\alpha^2/8\pi}$, where Λ is an ultraviolet cutoff parameter. The Luttinger exponents K_α determine the value of β_α to be $\sqrt{8\pi K_\alpha}$, and also specify the relationship between correlation functions expressed in terms of the bosonic field operators and physical correlation functions, expressed in terms of the original electronic field operators,

$$\psi_{\lambda,\sigma}(x)=\mathcal{N}_\sigma\exp[i\lambda k_F x-i\Phi_{\lambda,\sigma}(x)], \qquad (2)$$

where $\mathcal{N}_\sigma$ contains both a normalization factor (which depends on the ultraviolet cutoff) and a ''Klein'' factor[10] (which can be implemented in many ways) so that $\psi_{\lambda,\sigma}(x)$ anticomutes with $\mathcal{N}_{\sigma'}$ for $\sigma\neq\sigma'$ and commutes with it for $\sigma=\sigma'$. Also

$$\Phi_{\lambda,\sigma}=\sqrt{\pi/2}\,[(\tilde{\theta}_c+\lambda\tilde{\phi}_c)+\sigma(\tilde{\theta}_s+\lambda\tilde{\phi}_s)], \qquad (3)$$

where $\lambda=\pm1$ refers to left- and right-moving electrons, and $\sigma=\pm1$ refers to the spin polarization. In Eq. (3), we have expressed the fermion operators in terms of ''bare'' bosonic fields, $\tilde{\phi}_\alpha$, which are related to the interaction-shifted normal fields[10] that appear in Eq. (1) by the canonical (Bogoliubov) transformation

$$\tilde{\phi}_\alpha=\phi_\alpha\sqrt{K_\alpha}, \quad \tilde{\theta}_\alpha=\theta_\alpha/\sqrt{K_\alpha}. \qquad (4)$$

This transformation brings the Hamiltonian into a canonical form, so that the Luttinger exponents appear only in the relation between the fermionic and bosonic fields, and implicitly in the values of β_α.

From Eq. (2), it is a straightforward (and standard[10]) exercise to obtain bosonic representations of all interesting electron bilinear and quartic operators. Physically ϕ_c and ϕ_s are, respectively, the phases of the $2k_F$ CDW and spin-density-wave fluctuations, and θ_c is the superconducting phase. The long-wavelength components of the charge (ρ) and spin (S_z) densities are given by

$$\rho(x)=\sum_{\lambda,\sigma}\psi^\dagger_{\lambda,\sigma}\psi_{\lambda,\sigma}=2k_F/\pi+\sqrt{2K_c/\pi}(\partial_x\phi_c),$$

$$S_z(x)=\tfrac{1}{2}\sum_{\lambda,\sigma}\sigma\psi^\dagger_{\lambda,\sigma}\psi_{\lambda,\sigma}=\sqrt{2K_s/\pi}(\partial_x\phi_s).$$

We also explicitly bosonize the η-pairing operator,[16] whose correlations are sensitive to the presence or absence of a charge gap,

$$\eta_\lambda = \psi^\dagger_{\lambda,\uparrow}\psi^\dagger_{\lambda,\downarrow} \sim \exp[i\sqrt{2\pi/K_c}(\theta_c + K_c\phi_c) + 2ik_Fx]. \quad (5)$$

This operator is not usually studied as its scaling dimension $\delta_\eta = (K_c + K_c^{-1})/2$ is greater than 1 for $K_c > 0$, so the corresponding susceptibility is never divergent. However, it is interesting in the present context as it has finite momentum and is independent of the spin fields. It is easy to see from Eq. (2) that spatial translation by x_0 is equivalent to the chiral transformation $\phi_c \to \phi_c - k_F x_0\sqrt{2/\pi K_c}$, which must be a symmetry of the Hamiltonian for an incommensurate system. Similarly, gauge invariance implies that the Hamiltonian is invariant under $\theta_c \to \theta_c + \text{const}$. As a consequence, the Hamiltonian must depend only on derivatives of θ_c and ϕ_c, so the η operator defined above must always create a gapless excitation.

A final comment is in order at this point. The Abelian representation favored in the present paper is not manifestly spin rotationally invariant. This is an advantage whenever spin-rotational symmetry is broken at the Hamiltonian level, as there is no need for special treatment of symmetry-breaking terms. Spin-rotational invariance implies a specific value of the Luttinger exponent K_s, which may be obtained by comparing the spin-spin correlation functions for different spin directions. For example, where there is no spin gap, spin-rotation invariance can easily be seen to imply $K_s = 1$ at the fixed point. (Slow flows as K_s approaches 1 also can give logarithmic corrections to various correlation functions). So far as we know, in all cases studied to date, the fixed point value of K_s in a spin-gap phase is $K_s = \frac{1}{4}$ (or in other words, $\beta_s = \sqrt{2\pi}$), at which point the spin correlations are asymptotically equivalent to those of a dimerized spin-$\frac{1}{2}$ Heisenberg model, but it is conceivable that other discrete values could occur in other circumstances.

III. MULTICOMPONENT 1DEG

This section begins with a formal bosonized description of the multicomponent 1DEG, continues with a discussion of the generalized Luttinger theorems for this problem, and concludes with a detailed analysis of the specific example of the two-component 1DEG. In particular, in this latter part, it will be shown how the perturbative stability of each potential fixed point Hamiltonian can be assessed.

A. Bosonizing the multicomponent system

First consider a system of N distinct 1DEG's, which may be bosonized as in Eq. (2):

$$\psi_{b,\lambda,\sigma}(x) = \mathcal{N}^b_\sigma \exp[i\lambda k^b_F x + i\Phi^b_{\lambda,\sigma}(x)],$$
$$\Phi^{(b)}_{\lambda,\sigma} = \sqrt{\pi/2}\,[\tilde{\theta}^{(b)}_c + \lambda\tilde{\phi}^{(b)}_c] + \sigma\sqrt{\pi/2}[\tilde{\theta}^{(b)}_s + \lambda\tilde{\phi}^{(b)}_s], \quad (6)$$

where $b=1$ to N labels the different subsystems, the Klein factors $N^{(b)}_\sigma$ anticommute for $(a,\sigma) \neq (b,\sigma')$, and the bosonic fields satisfy canonical commutation relations

$$[\tilde{\phi}^{(a)}_\alpha(y), \partial_x\tilde{\theta}^{(a)}_\beta(x)] = i\,\delta_{a,b}\,\delta_{\alpha,\beta}\,\delta(x-y). \quad (7)$$

The tilde field variables appearing here are the bare fields, unshifted by interactions.

In the continuum limit, the Hamiltonian density of this system consists of three terms:

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s + \mathcal{H}_{int}. \quad (8)$$

Here $\mathcal{H}_c$ includes all the marginal interactions involving the charge degrees of freedom,

$$\mathcal{H}_c = \tfrac{1}{2}\{(\partial_x\underline{\tilde{\theta}}^T_c)\mathbf{W}_c(\partial_x\underline{\tilde{\theta}}_c) + (\partial_x\underline{\tilde{\phi}}^T_c)\mathbf{V}_c(\partial_x\underline{\tilde{\phi}}_c)\}, \quad (9)$$

where $\underline{\tilde{\theta}}_c$ and $\underline{\tilde{\phi}}_c$ are column vectors with N components $\tilde{\theta}_{ci}$ and $\tilde{\phi}_{ci}$ respectively, and $\mathbf{W}_c$ and $\mathbf{V}_c$ are real, symmetric $N\times N$ matrices. So that the spectrum is bounded below it is necessary and sufficient that the all eigenvalues of $\mathbf{W}_c$ and $\mathbf{V}_c$ be positive. $\mathcal{H}_s$ is similarly defined for the spin degrees of freedom; however, at the spin rotationally invariant gapless fixed point, $\mathbf{W}_s = \mathbf{V}_s = \mathbf{u}_s$, where $[\mathbf{u}_s]_{a,b} = u^{(b)}_s\delta_{a,b}$ is the spin-velocity matrix. Finally, $\mathcal{H}_{int}$ contains the terms nonlinear in the field variables (the various cosine interactions), which when relevant lead to the opening of gaps in the spectrum, and when irrelevant can be ignored.

For the case in which the nonlinear interactions are perturbatively irrelevant, $\mathcal{H}$ is the fixed point Hamiltonian for a system with N gapless charge and N gapless spin modes. In this case, as in the single-component problem, we perform a Bogoliubov transformation to normal coordinates.[24] A more detailed derivation is given in Appendix A. First define the column vector $\underline{\eta}_i$ such that

$$\mathbf{W}_c\mathbf{V}_c\underline{\eta}_i = u^2_{ci}\underline{\eta}_i \quad (10)$$

where u_{ci} are the normal mode velocities, and

$$\underline{\eta}^T_i\mathbf{W}^{-1}_c\underline{\eta}_j = \delta_{ij}. \quad (11)$$

With these definitions, it is straightforward to show that

$$\mathbf{W}_c = \sum_i \underline{\eta}_i\underline{\eta}^T_i \quad (12)$$

and

$$\mathbf{V}_c = \mathbf{W}^{-1}_c\sum_i u^2_{ci}\underline{\eta}_i\underline{\eta}^T_i\mathbf{W}^{-1}_c. \quad (13)$$

Then the Hamiltonian may be diagonalized by a canonical transformation to new fields ϕ'_i and their conjugate momenta $\partial_x\theta'_i$:

$$\phi_i = u^{1/2}_i\underline{\eta}^T_i\mathbf{W}^{-1}_c\underline{\tilde{\phi}}_c,$$
$$\theta_i = u^{-1/2}_i\underline{\eta}^T_i\underline{\tilde{\theta}}_c. \quad (14)$$

In transformed variables, the Hamiltonian consists of N decoupled acoustic normal modes,

$$\mathcal{H}_c = \tfrac{1}{2}\sum_i u_{ci}\{\partial_x\theta^2_{ci} + [\partial_x\phi_{ci}]^2\}. \quad (15)$$

The relation between the fermionic fields and the normal mode coordinates is easily derived from this expression and Eq. (6).

Finally, correlation functions of the untransformed fields can be expressed in terms of the transformed fields using Eq. (14). A typical operator has the form

$$\hat{O}(x) \equiv \exp\{i[\underline{a}^T \underline{\tilde{\phi}}(x) + \underline{b}^T \underline{\tilde{\theta}}(x)]\}, \tag{16}$$

where $\underline{a}$ and $\underline{b}$ are N component real vectors, and its zero-temperature equal-time correlation function is given by

$$\langle \hat{O}(x) \hat{O}^\dagger(0) \rangle = [\Lambda |x|]^{-2\delta} \tag{17}$$

where Λ is an ultraviolet cutoff, and the scaling dimension

$$\delta = \frac{1}{4\pi}[\underline{a}^T \mathbf{M}^{-1} \underline{a} + \underline{b}^T \mathbf{M} \underline{b}], \tag{18}$$

where

$$\mathbf{M} \equiv \mathbf{W}^{-1/2} \mathbf{N} \mathbf{W}^{-1/2}, \tag{19}$$

with

$$\mathbf{N}^2 \equiv \mathbf{W}^{1/2} \mathbf{V} \mathbf{W}^{1/2}. \tag{20}$$

The perturbative stability of the free-boson fixed point can be readily analyzed by studying the scaling dimension of the various operators which enter into $\mathcal{H}_{int}$. As usual, the stability of the fixed point turns on whether there are any physically allowed vectors $\underline{a}$ and $\underline{b}$ that lead to a scaling dimension less than 1, which would imply that the operator is relevant. If any of these interactions is relevant, it is necessary to identify the new fixed point to which the system flows, and to study its properties. Typically, the effect of a relevant interaction in $\mathcal{H}_{int}$ is to freeze out certain fluctuations (i.e., to gap some modes) and at the same time produce a renormalization of the matrices **V** and **W**. This leads to a new fixed-point Hamiltonian, whose stability must be reassessed, since operators that were irrelevant at the original fixed point could become relevant at the new fixed point. This stability analysis will be performed more explicitly in the two-component example discussed below.

B. Generalization of Luttinger's theorem

The generalized Luttinger's theorem[15] imposes an important constraint on the allowed momenta at which gapless neutral excitations occur. No matter how complex the system (e.g., no matter how many bands cross the Fermi surface), unless there is an even-integer number of electrons per unit cell, there must be a zero-energy excited state with charge 0 and (for the case of zero net magnetization) with crystal momentum $2k_F^* = \pi n_T$, where n_T is the total electron density, including all bands. Thus if a multicomponent system has gapless modes at only one crystal momentum, it must be $2k_F^*$, and in a system with multiple values of $2k_F^{(b)}$, there must be a set of integers m_b such that

$$2k_F^* = \sum_{b=1}^{N} m_b 2k_F^{(b)} + (\text{reciprocal lattice vector}). \tag{21}$$

(If some modes are charged, there is an obvious further constraint on the integers implied by the neutrality of the composite mode at $2k_F^\star$.)

There is a second general constraint governing the existence of a gapless charge excitation, which to our knowledge is discussed here for the first time. This argument generalizes our earlier discussion of the η-pairing mode. Consider the generalized η operator, which creates $2N_c$ right-moving electrons with spin 0:

$$\eta_{T,1} = \prod_{b=1}^{N_c} \psi^\dagger_{b,1,\uparrow} \psi^\dagger_{b,1,\downarrow} = (\ ) \exp[i\sqrt{2\pi}(\tilde{\theta}_c + \tilde{\phi}_c) + i 2k_F^c x], \tag{22}$$

where N_c is the number of ''extended'' charge modes and

$$\tilde{\theta}_c = \sum_b \tilde{\theta}_c^{(b)} / \sqrt{N_c} \quad \text{and} \quad \tilde{\phi}_c = \sum_b \tilde{\phi}_c^{(b)} / \sqrt{N_c} \tag{23}$$

are the global superconducting phase and the dual CDW phase. Global gauge invariance implies that the Hamiltonian is invariant under the transformation $\tilde{\theta}_c \to \tilde{\theta}_c + \text{const}$. Operationally, ''extended'' charge modes are defined to be those modes that acquire a nonzero phase under a global gauge transformation. Similarly, spatial translation is equivalent to the phase shift, $\tilde{\phi}_c \to \tilde{\phi}_c + \text{const}$. As a consequence of these invariances, $\mathcal{H}_{int}$ must dependent only on derivatives of $\tilde{\theta}_c$ and (so long as the system is incommensurate) on derivatives of $\tilde{\phi}_c$. Thus the associated modes must be gapless. This implies that $\eta_{T,1}$ (and of course, $\eta_{T,-1}$ as well) must create a gapless, spin-0 charge-$2N_c e$ excitation with crystal momentum $2k_F^c$, and that the η correlations must fall like a power law with distance. In many cases, $2k_F^c = 2k_F^*$ and $N_c = N$, the number of ''bands'' which cross the Fermi surface, but we will encounter cases, such as the Kondo-Heisenberg model discussed in Sec. III B, in which $N_c < N$.

This proof relies on the field-theoretic representation of operators; it is desirable to generalize it to the actual lattice system, but we have not yet succeeded in doing so. Since a gapless, spinless, neutral excitation with momentum $2k_F^*$ always exist on general grounds, it need not be listed when classifying phases.

C. Classification of fixed points

The essential steps in extending the above analysis are to identify the possible fixed points of a multicomponent system, and then examine their perturbative stability. As for the single-component system, *states are identified by their discrete broken symmetries and by the ''irreducible'' or minimal set of charge and spin-carrying gapless excitations*. The gapless, spinless excitation at $2k_F^*$ implied by the generalized Luttinger's theorem may be left implicit. In contrast to the single-component 1DEG, it is necessary to specify not only the modes but also their momenta (aside from 0), which are no longer completely determined by the generalized Luttinger's theorem. For instance, it will be seen that it is possible to encounter a state with a gapless charge excitation at crystal momentum $2k_F$, and a gapless spin mode at crystal momentum $2k_F' \neq 2k_F$. Such a state will be labeled $[2k_F : c][2k_F' : s]$ or, leaving the values of the crystal momentum implicit, $[c][s]$. In the canonical ordering to be adopted here, the modes with the larger momentum, $2k_F > 2k_F'$, will be listed first. An interesting feature of the multicomponent system which is obviated by the generalized Luttinger's theorem for the single-component case is that, when there are multiple values of k_F, their values can (and generally will) shift continuously as a function of interactions.

Of course, it is always implicit that, if there exists a gapless excitation at crystal momentum $2k_F$, then one can make gapless excitations with integer multiples of $2k_F$, as well. However, it is clearly not sufficient to specify the number of gapless modes, as proposed by Lin, Balents, and Fisher.[11] For example, the state with gapless charge and spin modes with the same crystal momentum, $[cs]$, is thermodynamically distinct from the states $[c][s]$ and $[s][c]$ in which they occur at distinct crystal momenta.

D. Two-component 1DEG

To make the discussion more concrete, and in particular to illustrate the nature of the stability analysis, we now consider the case of a two-component 1DEG, $N=2$. Two independent, decoupled, and generally inequivalent 1DEG's are separately described by an appropriate fixed-point (free-boson) Hamiltonian. Clearly no coupling between the two subsystems can be generated by any reasonable renormalization-group transformation. Thus the fixed points may be specified for each subsystem separately. The discussion will be restricted to the spin rotationally invariant case, although this is easily generalized.

The next step is to determine the circumstances in which each fixed point is stable with respect to weak interactions between the two 1DEG's. In general, whenever a given fixed point is stable for some range of parameters, there is no more to say. If the fixed point is unstable, the character of the stable fixed point to which the Hamiltonian flows under renormalization must be determined. The new fixed point could, in principle, have only gapless modes, although, usually some modes that were gapless become gapped.

If the two subsystems are mutually incommensurable $[k_F^{(1)}/k_F^{(2)}=(\text{irrational})]$, the only potentially perturbatively relevant couplings are those that do not transfer momentum between the two systems. The most relevant terms are quartic in fermion operators and are of three types. The interaction piece of the Hamiltonian density is given by

$$\mathcal{H}'=\mathcal{H}_1'+\mathcal{H}_2'+\mathcal{H}_3', \tag{24}$$

where

$$\mathcal{H}_1'=V\rho^{(1)}\rho^{(2)}+V'j^{(1)}j^{(2)}, \tag{25}$$

with

$$\rho^{(b)}(x)=\sum_{\lambda,\sigma}\psi^{\dagger}_{b,\lambda,\sigma}\psi_{b,\lambda,\sigma}, \tag{26}$$

$$j^{(b)}(x)=\sum_{\lambda,\sigma}\lambda\,\psi^{\dagger}_{b,\lambda,\sigma}\psi_{b,\lambda,\sigma}, \tag{27}$$

$$\mathcal{H}_2'=J\vec{S}^{(1)}\cdot\vec{S}^{(2)}+J'\vec{j}^{(1)}\cdot\vec{j}^{(2)}, \tag{28}$$

with

$$\vec{S}^{(b)}(x)=\sum_{\lambda,\sigma,\sigma'}\psi^{\dagger}_{b,\lambda,\sigma}\vec{\tau}_{\sigma,\sigma'}\psi_{b,\lambda,\sigma'}, \tag{29}$$

$$\vec{j}^{(b)}(x)=\sum_{\lambda,\sigma,\sigma'}\lambda\,\psi^{\dagger}_{b,\lambda,\sigma}\vec{\tau}_{\sigma,\sigma'}\psi_{b,\lambda,\sigma'}, \tag{30}$$

and

$$\mathcal{H}_3'=[\mathcal{J}_s\Delta^{(1)\dagger}\Delta^{(2)}+\text{H.c.}]+[\mathcal{J}_t\vec{\Delta}^{(1)\dagger}\cdot\vec{\Delta}^{(2)}+\text{H. c.}], \tag{31}$$

where

$$\Delta^{(b)}(x)=\sum_{\lambda}\psi^{\dagger}_{b,\lambda,\uparrow}\psi^{\dagger}_{b,-\lambda,\downarrow}, \tag{32}$$

$$\vec{\Delta}^{(b)}(x)=\sum_{\sigma,\sigma'}\psi^{\dagger}_{b,1,\sigma}\vec{\tau}_{\sigma,\sigma'}\psi^{\dagger}_{b,-1,\sigma'}. \tag{33}$$

Here $\vec{\tau}$ are the Pauli matrices. Of these interactions, the charge and current-density interactions in $\mathcal{H}_1'$ are marginal, i.e., they are quadratic in boson variables, and so must (and can) be absorbed into the definition of the fixed-point Hamiltonian density and treated exactly, as in Sec. III A. The perturbative stability analysis is then performed with respect to the remaining interactions $\mathcal{H}_2'$ and $\mathcal{H}_3'$ by computing the scaling dimensions of these operators, as in Eq. (18).

This stability analysis was carried out previously for the two chain problem in Refs. 24 and 5. The results are algebraically complicated, but are simplified, without significant loss of physical insight, by considering systems in which

$$V'=-(v_c^{(2)}/v_c^{(1)})(K_c^{(1)}K_c^{(2)})V, \tag{34}$$

where $K_\alpha^{(b)}$ and $v_\alpha^{(b)}$ are the Luttinger exponent and velocity at the decoupled fixed point of subsystem $b=1$ and 2, with $\alpha=c$ and s for charge and spin modes, respectively. In this case,

$$\begin{aligned}
\delta_{(\vec{S}_1\cdot\vec{S}_2)}&=\tfrac{1}{4}(K_s^{(1)}+K_s^{(2)}+1/K_s^{(1)}+1/K_s^{(2)}),\\
\delta_{(\vec{j}_1\cdot\vec{j}_2)}&=\delta_{(\vec{S}_1\cdot\vec{S}_2)},\\
\delta_{(\Delta_1\Delta_2)}&=\tfrac{1}{4}(A/K_c^{(1)}+B/K_c^{(2)}+K_s^{(1)}+K_s^{(2)}),\\
\delta_{(\vec{\Delta}_1\cdot\vec{\Delta}_2)}&=\tfrac{1}{4}(A/K_c^{(1)}+B/K_c^{(2)}+1/K_s^{(1)}+1/K_s^{(2)}),
\end{aligned} \tag{35}$$

with

$$A=\sqrt{1+\frac{4VV'}{(\pi v_c^{(2)})^2}},$$

$$B=\left(1-\frac{2VK_c^{(1)}}{\pi v_c^{(1)}}\right)^2\left(1+\frac{4VV'}{(\pi v_c^{(1)})^2}\right)^{-1/2}. \tag{36}$$

The weak-coupling fixed point of two incommensurate 1DEG's is stable if all of these dimensions are greater than 1, and is unstable otherwise. For weak interactions, any gapless charge modes have $K_c^{(b)}$ near 1, and $K_c^{(b)}$ generally increases with increasingly strong repulsive interactions. When the fixed-point Hamiltonian is spin rotationally invariant, $K_{sj}=1$.

The expressions in Eq. (35) were derived for the gapless fixed-point Hamiltonian, but it is relatively easy to deduce how these scaling dimensions are altered at a strong-coupling fixed point in which certain fluctuations are frozen out by the presence of relevant interactions of the form $\cos[\beta\phi_c^{(b)}]$ and/or $\cos[\beta\phi_s^{(b)}]$ which open a gap. This is equivalent to replacing $K_c^{(b)}$ by an effective Luttinger exponent, $K_c^{(b)}\to 0$ and/or $K_s^{(b)}\to 0$. Conversely, if the fluctuations of the dual phases are suppressed by a relevant interaction of the form $\cos[\beta\theta_c^{(b)}]$ and/or $\cos[\beta\theta_s^{(b)}]$, these expressions should be evaluated in the limit $K_c^{(b)}\to\infty$ and/or $K_s^{(b)}\to\infty$. Other types of strong-coupling fixed points can be analyzed in the same fashion.

It is worth commenting, briefly, on the physical implications of the dependence of these various scaling dimensions on the parameters in the fixed-point Hamiltonian. The spin interactions, $\mathcal{H}_2'$, are manifestly unimportant ($\delta_{(\vec{S}_1\cdot\vec{S}_2)}$ and $\delta_{(\vec{\Delta}_1\cdot\vec{\Delta}_2)}$ are infinite) if either subsystem has a spin gap. This makes good physical sense. If neither system has a spin gap, then the constraints of spin-rotation invariance imply that these interactions are marginal ($\delta_{(\vec{S}_1\cdot\vec{S}_2)}=1$); further analysis (i.e., carrying out the perturbative analysis to order J^2), following on the work of Ref. 7 on the Kondo-Heisenberg problem, shows that for antiferromagnetic couplings ($J>0$), these interactions are marginally relevant while for ferromagnetic couplings ($J<0$), they are marginally irrelevant. The authors of Ref. 7 speculated that in the antiferromagnetic case, the system scales to a strong-coupling fixed point with J large and a total spin gap. This conclusion is supported by numerical studies carried out by these same authors, and by additional analytic work by one of us.[17]

The singlet pair tunneling interaction $\mathcal{H}_3'$ has its scaling dimension significantly reduced if either or both subsystems have a spin gap, since then the effective $K_s^{(b)}$ in Eq. (35) is zero. For instance, if subsystem 1 has a spin gap, and subsystem 2 does not, then, from Eq. (35)

$$\delta_{(\Delta_1\Delta_2)}=\tfrac{1}{4}(A/K_c^{(1)}+B/K_c^{(2)})+\tfrac{1}{4}. \tag{37}$$

The underlying physics is analogous to the proximity effect in superconductivity, and we have named it[5] the "spin gap proximity effect": because subsystem 1 has a spin gap, it is already substantially superconducting,[25] so it can readily infect any coupled subsystem with its superconducting character. From this point of view, one would expect a relevant pair tunneling interaction to induce pairing correlations in subsystem 2 (i.e., to open a spin gap) and to lock the superconducting phases of the two subsystems (i.e., to gap the out-of-phase CDW mode). We have confirmed the validity of this expectation by an exact solution of this problem in a particular solvable limit.[5] When $\mathcal{J}_s$ is relevant, i.e., whenever $\delta_{(\Delta_1\cdot\Delta_2)}<1$, the stable fixed-point behavior is characterized by a total spin gap, and a locking of the charge degrees of freedom of the two subsystems. Indeed, this effect is very efficient at destabilizing any fixed point with a partial spin gap; for instance, any $[cs][c]$ fixed point is most often unstable for nonzero $\mathcal{J}_s$, and flows to the $[c]$ fixed point. The effect of a relevant triplet pair tunneling interaction $\mathcal{J}_t$ has not yet been thoroughly investigated.

When the two subsystems are mutually commensurate, or nearly commensurate, the above stability analysis becomes more complicated. We defer detailed discussion of this problem to a later date. However, a few interesting features of the problem can be understood on the basis of very general considerations. In the first place, the decoupled fixed point of two 1DEG's with the same values of $2k_F$ is always perturbatively unstable, unless at least one system has a fully gapped spectrum. This follows directly from the observation that in all the known phases of the 1DEG, at least one susceptibility is enhanced relative to noninteracting electrons.[10] What this also implies is that, if we start with two decoupled 1DEG's with $2k_F^{(1)}$ nearly equal to $2k_F^{(2)}$, and then gradually turn on interactions between them, there is a strong tendency to induce transfer of electrons between the two subsystems, with a cost of unperturbed energy but a gain of energy from the relative commensurate locking of the two subsystems. As a result, one expects a *relative* incommensurate to commensurate transition as a function of increasing interaction strength in such systems. As for the 1DEG itself, the situation is somewhat more complicated for higher-order *relative* commensurabilities, since newly allowed interactions are generally irrelevant when the interactions in the 1DEG are weak, and relevant only when they are sufficiently strong and sufficiently long ranged.[26]

IV. APPLICATION TO SPECIFIC MODEL PROBLEMS

To demonstrate the utility of this analysis, we conclude with a discussion of four specific problems that have been of considerable recent theoretical interest.

A. Critique of the perturbative RG analysis of the *N*-chain Hubbard model

There have been a number of recent papers concerning the phases of the *N*-chain Hubbard model, following the early work of Varma and Zawadowski.[27] In particular, in two interesting papers, Balents, Fisher, and Lin[11] analyzed the renormalization-group (RG) flows in the neighborhood of the *noninteracting fixed point*, by computing the β function to lowest order in powers of U/t, where U is the on-site repulsion between electrons, and t is the intersite hopping matrix. (More generally, they allowed for possibly different values of the hopping amplitudes, t and t', parallel and perpendicular, respectively, to the chain direction.) Specifically, they identified which interactions are perturbatively most relevant for various geometries of the chains, and as a function of the electron concentration per site and the ratio t'/t. They then conjectured a phase diagram by analyzing the nature of the fixed point, obtained by bosonizing the model with the relevant interactions taken to infinity, and all others neglected.

This analysis has, we believe, three flaws, which lead to significant errors in the resulting phase diagram and other conclusions. (1) The only rigorous conclusion that can be drawn from a perturbative RG analysis when there are relevant interactions is that the initial fixed point is unstable, and that therefore the asymptotic physics is controlled by another fixed point. (Tracing the effects of the perturbatively relevant interactions to strong coupling is, *at best*, suggestive of the character of the new fixed point.) (2) Even if we accepted the perturbative analysis of the nature of the interactions which are important at the strong-coupling fixed point to which the Hamiltonian flows, it is essential to perform a perturbative stability analysis at the new conjectured strong-coupling fixed point to make certain that it is, in fact stable. This was not done by Balents, Fisher, and Lin. Specifically, their analysis ignored the possibility that interactions that are marginal or irrelevant at the weak-coupling fixed point, can become relevant at the strong-coupling fixed point. (3) Because of the presence of marginal interactions, e.g., interactions which shift the various collective-mode velocities, it is an incorrect procedure to perturb about the *noninteracting* fixed point. The proper perturbative RG analysis should include the effects of the marginal interactions *exactly*, as in Sec. III A, and should be performed about the appropriate free boson fixed point. Since the various collective mode velocities enter the one-loop RG equations, this makes small (but at times important) differences in the character of the weak-coupling flows.

Stability of phases

How do these general observations affect the conclusions of Lin, Balents, and Fisher[11] concerning the phase diagram? Where they find no relevant interactions in their approach, and conclude that all of the modes of the noninteracting system remain ungapped (e.g., where they find phases of the form $[cs]^n$, or CnSn in their notation), or where they find completely gapped phases, or phases with only a single, ungapped charge mode (e.g., phases labeled $[c]$ or $C1S0$ in their notation) the only differences between our analysis and theirs are $O(U/t)$ shifts of the locations of various phase boundaries due to the effect of marginal interactions. However, all of the partially spin-gapped phases, (such as the phase $[cs][c]$, which, in their notation is $C2S1$), are destabilized by the spin-gap proximity effect. To see this, note that the pair tunneling interaction, like all other interactions, has scaling dimension $\delta = 1 + O(U/t)$ at the noninteracting fixed point; the opening of a spin gap, with all other interactions held small (of order U/t) reduces this dimension by a finite amount, e.g., to $\delta = \frac{3}{4} + O(U/t)$ in the example in Eq. (37). Thus, of all the conjectured phases in their phase diagrams, only the familiar phases with one or fewer gapless charge and spin modes, and the totally gapless phases $[cs]^m$, are stable at weak interactions.

Of course, as is implied by point (1) above, it cannot be ruled out that during the flow to strong coupling, interactions other than the ones identified in perturbation theory will become large, and this could alter the stability of the partially gapped phases. However, we suspect that the partially gapped phases are not, generically, stabilized for large U/t, because strong interactions can produce significant shifts in the values of $k_F^{(b)}$, which then permits the commensurate locking of different subsystems; indeed, this conclusion was reached, previously, by Schulz.[28] In this regard it is worth noting that there have been extensive numerical[29] studies of various N-leg Hubbard ladders for intermediate to strong U/t, in none of which has evidence for these exotic phases been reported.

B. Example of the spin-gap proximity effect: The asymmetric three-chain Hubbard model

In Ref. 5, where the spin-gap proximity effect was first elucidated, the simplest model system to which it was applied was the asymmetric three-leg Hubbard ladder. The predictions made there were later confirmed in numerical experiments on the symmetric three-leg t-J ladder.[3] For the reasons outlined above, these results are in clear disagreement with the predictions based on the perturbative RG analysis of Lin, Balents, and Fisher.[11] We briefly review the analysis here, as an illustrative example.

As in Ref. 5, consider an asymmetric system, with a two-leg Hubbard ladder weakly coupled to a one-leg Hubbard ''chain,'' with a difference in site energy ϵ. For concreteness, take U to be large, and consider the phase diagram as a function of electron density, although in Ref. 5 it was constructed as a function of ϵ. The method of analysis, as presented above, first neglects the coupling between the ladder and the chain, and then assesses its effect on the final result.

When the electron density per site is $n=1$, the system is manifestly insulating. The two-leg ladder, on its own, has a spin gap while the chain has a gapless spinon mode. Because of the spin gap, this fixed point is stable for weak chain-ladder couplings, so the phase is $[s]$.

For $\epsilon > 0$, when the density of electrons is reduced slightly to $n = 1 - x$ with the number of ''doped holes'' $x \ll \frac{1}{3}$, the added holes go onto the chain. Because the interactions in the chain are repulsive and spin rotationally invariant, and the electron density is incommensurate, the chain will form a Luttinger-liquid state with gapless charge and spin modes. Pair tunneling between the chain and the ladder induces an effective attraction between spinons, but because there is an energy denominator $2\epsilon^*$ (where ϵ^* is the renormalized energy to transfer a singlet pair of electrons from the ladder to the chain), the bare repulsion between electrons on the chain is the dominant interaction, so the decoupled fixed point is stable; this phase is $[cs]$.

Finally, with increasing doping, although still in the regime $x < 1/3$, the value of ϵ^* decreases steadily due to the repulsion between doped holes so, if $|\epsilon|$ is not too large, we reach a regime in which the pair tunneling between the chain and the ladder becomes significant. Now, via the spin-gap proximity effect, the chain becomes infected with the spin gap of the ladder. The result is a phase which has a total spin gap, and only the charge-$2e$ and neutral spinless modes implied by the generalized Luttinger's theorem; this phase is $[c]$.

The system studied numerically[3] is, in fact, the three-chain t-J ladder. The differences between the t-J and Hubbard models for intermediate to strong U are not believed to be very significant in the present context. Because of the boundary conditions, the central chain of the ladder is physically distinct from the two edge chains. Even though the bare difference in site energy $\epsilon = 0$, there is manifestly a nonzero value of ϵ^*. Thus there is, in fact, a very close (although not entirely quantified) relation between the system we analyzed theoretically, and that studied in the numerical experiments, so it is not surprising that the reported phase diagrams agree. In the numerical experiments, with $J/t \sim 0.5$, the critical value of doping at which the transition from $[cs]$ to $[c]$ occurs is $x \approx 0.06$; the very small value of this critical density reveals the robustness and strength of the spin-gap proximity effect. The perturbative RG analysis of Lin, Balents, and Fisher leads to a phase diagram in which the undoped system is (correctly) in the $[s]$ phase, but has the doped system exhibiting the $[s][cs]$ phase over the relevant range of x; this is a specific case of a partially spin-gapped phase which, as we argued above, should be generally unstable to the formation of a fully spin-gapped $[c]$ phase due to pair tunneling interactions.

C. Concerning the $N=6$ t-J cylinder

Recent important advances in the numerical evaluation of the ground-state properties of correlated systems have allowed the study of much larger t-J and Hubbard clusters than before. White and Scalapino[4] considered six-component t-J systems with cylindrical boundary conditions, i.e., periodic boundary conditions in the finite direction and open boundary conditions along the chains. To draw conclusions concerning the 2D t-J model, it is necessary to perform a two-dimensional finite-size scaling analysis of these results in order to extrapolate to the thermodynamic limit. So far, it has not been possible to do so, and the conclusions of White and Scalapino disagree with those of other studies of comparably large systems[30] which did do a finite-size scaling analysis. Consequently, it is still unclear to what extent these results are representative of the actual ground state of the two-dimensional system. However, we may imagine that the results of White and Scalapino are representative of the ground-state properties of an infinite-length six-leg cylinder, which itself is an example of a multicomponent one-dimensional system.

The principal finding of White and Scalapino (obtained for $J/t=0.35$) is that the ground state exhibits ''stripe'' correlations in the expectation value of the charge-density operator. For the six-leg cylinder at a small density of doped holes, x, the period of the observed density oscillations is $\lambda_6 = 2/3x$. (Here units of length are chosen so that the lattice constant is equal to 1.) Of course, since this is one dimension, the density-wave order observed by White and Scalapino on finite length systems should be interpreted as the period of power-law CDW correlations in the infinite system. Now, the value of $2k_F^* = G + 2\pi/\lambda^*$, where G is a reciprocal-lattice vector, corresponds to density-wave correlations with a wavelength $\lambda_6^* = 1/3x$, so the period found by White and Scalapino is twice that required by the generalized Luttinger's theorem. In other words, the fundamental gapless, spinless neutral CDW mode of the system occurs at a wave number $\frac{1}{2}2\pi/\lambda^*$, and the excitation at $2k_F^*$ is thus a second harmonic.

It is also easy to see from the present analysis that these cylinders are not good candidates for high-temperature superconductors, since there are no gapless charge-$2e$ excitations of this system. At present, it is not clear to us whether the system supports gapless excitations with charge $4e$, corresponding to the injection of an additional ''stripe'' into the system, or whether because there is a tendency for spin correlations to suffer a π phase shift across a stripe, it is necessary to inject charge $8e$ corresponding to a pair of ''stripes.''

D. Kondo-Heisenberg array

The Kondo-Heisenberg model is the simplest example of a metallic system (here, a 1DEG) coupled to an insulating antiferromagnet (here, a spin-$\frac{1}{2}$ Heisenberg chain). The large (or infinite) charge gap in the spin chain implies that charge-transfer interactions, such as pair tunneling, are unimportant. The dominant interactions involve the spin density, i.e., they are the Kondo interaction between the conduction electron spin $\vec{s}(x)$ and localized ''impurity'' spins $\vec{\tau}_j$,

$$H_K = J_K \sum_j \vec{\tau}_j \cdot \vec{\mathbf{s}}(x_j), \tag{38}$$

where x_j are the positions of the localized spins and the Heisenberg interaction between nearest-neighbor localized spins,

$$H_0^{Heis} = J_H \sum_j \vec{\tau}_j \cdot \vec{\tau}_{j+1}. \tag{39}$$

Also it will be assumed that $J_H \ll E_F$, which is typically true in physical applications. The resulting Kondo-Heisenberg Hamiltonian is

$$H = H_0^{1DEG} + H_0^{Heis} + H_K, \tag{40}$$

where the subscript 0 refers to the Hamiltonian of the decoupled system with

$$H_0^{1DEG} = -iv_F \sum_{\sigma,\lambda} \lambda \int dx\, \psi_{\lambda,\sigma}^{\dagger} \partial_x \psi_{\lambda\sigma}. \tag{41}$$

It will be assumed that the relative concentration of localized spins is $c=1/b<1$, i.e., $x_j = jb$, that the two systems are relatively incommensurate, and that $2k_F$ is incommensurate with the underlying lattice. The effective Fermi wave number (in the sense of the generalized Luttinger's theorem) for the 1DEG and the spin chain are $2k_F$ and $2k_F^{Heis} = \pi/b$, respectively, so, for the coupled system,[15] there must be a gapless neutral excitation with wave number $2k_F^* = 2k_F + (\pi/b)$, and a charge-$2e$ excitation with momentum $2k_F$.

The determination of the phase diagram of this model provides a further example of the application of the methods developed above. We shall give a brief physical description of the origin of the various phases—the theoretical manipulations may be found in Appendix B and in the published and unpublished literature.[6,7,17]

1. Decoupled Luttinger liquid: $J_H \gg -J_K \geq 0$

As discussed above, spin-rotation invariance implies that, to lowest order, the spin coupling between two gapless systems is marginal, while to second order, as pointed out by Sikkema, Affleck, and White,[31] the interactions are perturbatively irrelevant for $J_K<0$ (ferromagnetic interactions) and relevant for $J_K>0$. Thus, for ferromagnetic Kondo coupling, the decoupled fixed point is perturbatively stable. This phase has, trivially, one gapless spin excitation at momentum $2k_F^{Heis}$, and gapless spin and charge excitations at momentum $2k_F$. Since $2k_F > 2k_F^{Heis}$, this phase is labeled $[c,s;2k_F][s;2k_F^{Heis}]$, and has no discrete broken symmetries. In shorthand notation, the decoupled fixed point is classified as $[c,s][s]$. However, for future reference, it is important to note that there is also a gapless, odd-parity charge-$2e$ composite pairing excitation at $2k_F^{Heis}$, as discussed in Appendix B. Because this is a composite excitation, its existence is already implied by the existence of the other gapless modes. However, as we shall now show, when the Kondo coupling produces a spin gap, this composite mode can still remain gapless.

2. Antiferromagnetic Kondo coupling: $J_K>0$

a. Odd-parity singlet pairing: $J_H \gg J_K > 0$. If $J_K>0$, the decoupled fixed point is unstable, and the low-energy physics is governed by a strong-coupling fixed point with a spin gap.[31] This phase has several unexpected features. Of course, as required by the generalized Luttinger's theorem, there is a neutral, spinless gapless excitation of this system with a minimal momentum $2k_F^*$, as pointed out by Yamanaka, Oshikawa, and Affleck.[15] Remarkably, the effective Fermi sea ''knows'' about the localized electrons as well as the itinerant ones. In addition, this system clearly has a gapless spinless charge-$2e$ excitation created by the η-pairing operator of the 1DEG; in this case, since the localized spins are unaffected by a global gauge transformation, our theorem implies that this mode carries momentum $2k_F$. Of course these two statements, taken together imply that there exists a gapless, spinless charge-$2e$ excitation with momentum $2k_F^{Heis}$. Because $4k_F^{Heis}=2\pi/b$ is a reciprocal-lattice vector, $2k_F^{Heis}$ and $-2k_F^{Heis}$ are equivalent; as a consequence, excitations with momentum $2k_F^{Heis}$ can simultaneously be characterized by their parity. As discussed in Appendix B, it may be shown that, in the present case, the only gapless charge-$2e$ excitation with momentum $2k_F^{Heis}$ has odd parity.[32] This phase is labeled $[c][c]$, or more fully as $[c;2k_F][(c,odd);2k_F^{Heis}]$ (where we put the modes in parentheses when an additional descriptive element, such as even/odd parity, must be noted).

There are some remarkable features of the charge fluctuations in this state.[32] Whereas the decoupled system had gapless CDW modes at $2k_F$ and $2k_F^*=2k_F+2k_F^{Heis}$, the spin-gap phase retains only the composite CDW mode at $2k_F^*$ (see Appendix B). As a result, because of the mismatch between $2k_F^{Heis}$ and $2k_F$, there are no spinless charge-$2e$ gapless excitations at momentum zero, and the system cannot be a conventional superconductor with pairing at total momentum zero induced by spin fluctuation exchange. An intuitive feeling for the origin of the spin gap can be obtained by considering the strong-coupling limit of the model, although care is always needed in identifying the specific strong-coupling fixed point to which a given weak-coupling Hamiltonian flows. In the present case, the natural candidate is a model in which the conduction electrons form singlets with the localized spins, and any remaining localized spins form singlets with each other.[31]

b. Staggered pairing: $J_K \gg J_H > 0$. An exact solution of the model,[6] also with a spin gap, may be obtained from a field theory with explicitly broken spin-rotational symmetry and in a special ''Toulouse limit,'' in which one component of the Kondo coupling takes a specific value $\sim E_F$. The renormalization-group strategy behind this solution was described in Sec. I. The long-distance behavior of the system in the Toulouse limit is spin rotationally invariant,[13,33] which implies that spin-rotational symmetry-breaking terms are irrelevant at the fixed point, so it is unlikely that the behavior we found is an artifact of the model. This phase may be distinguished[32] from the weak-coupling ($J_H \gg J_K$) spin-gap state by classifying its gapless excitations.

Clearly, as before, there must exist a gapless neutral excitation with a minimal momentum $2k_F^*$, a gapless charge-$2e$ excitation, produced by the η-pairing operator, with momentum $2k_F$, and as a consequence of these two general statements, a gapless charge-$2e$ excitation at momentum $2k_F^{Heis}$. However, we find[6] that there exist both even and odd-parity gapless charge-$2e$ excitations at momentum $2k_F^{Heis}$, so that there is, in fact, one more finite-momentum gapless charge mode in this state than in the weak-coupling spin-gap state. This phase is labeled $[c][cc]$, or more fully as $[c;2k_F][(c,\mathrm{odd})(c,\mathrm{even});2k_F^{Heis}]$. This solution also provides an example of the fact that an analysis of the relevant operators at weak coupling does not necessarily tell us the character of the stable fixed point. It is long-distance physics (forward scattering) that destabilizes the weak-coupling fixed point, but the character of the strong-coupling fixed point is determined by short-distance physics.

An elaborate comparison[32] of the two spin-gap phases reveals that the additional gapless mode in the Toulouse limit phase may be associated with an additional hidden broken translation symmetry. A further distinction between the states may be made by considering the origin of the spin gap and its consequences for enhanced pairing correlations; an intuitive, strong-coupling picture of the origin of the spin gap in the Toulouse limit phase involves pairing of the spins in each subsystem, separately. The existence of the two spin-gap fixed points of the one dimensional Kondo-Heisenberg model underscores the need to consider the explicit solutions of the strong coupling fixed points, which do not follow from simply establishing the existence of a spin gap based on the weak coupling perturbative renormalization group analysis.

c. Strong coupling: $J_K \gg E_F \gg J_H$. Finally, for completeness, it is important to remark that there exists a direct ''strong-coupling'' limit of the model $J_K \gg E_F \gg J_H$, which has distinct physics from either of the spin-gap phases discussed above. In particular, via a mapping to the t-J model, it has been shown that in this limit the system is governed by a Luttinger-liquid fixed point with no spin gap.[34,35] (Note that if $J_H \sim E_F$ there is also the possibility of a strong-coupling spin gap phase.[31])

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APPENDIX A: MULTICOMPONENT FREE BOSON FIXED POINT

This appendix gives a derivation of Eqs. (12), (13), (18), (19), and (20). First of all, Eq. (10) may be rewritten as the eigenvalue equation of a real symmetric matrix $\mathbf{W}_c^{1/2}\mathbf{V}_c\mathbf{W}_c^{1/2}$, with eigenvalues u_{ci}^2 and eigenvectors $\mathbf{W}_c^{1/2}\underline{\eta}_i$. It follows that

$$\mathbf{W}_c^{1/2}\mathbf{V}_c\mathbf{W}_c^{1/2}=\mathbf{W}_c^{-1/2}\sum_i u_{ci}^2\underline{\eta}_i\underline{\eta}_i^T\mathbf{W}_c^{-1/2}. \tag{A1}$$

Equation (12) may now be obtained by premultiplying this equation by $\mathbf{W}_c^{1/2}$ and postmultiplying by $\mathbf{W}_c^{-1/2}\mathbf{V}_c^{-1}$, and using Eq. (10). Similarly Eq. (13) may be obtained by premultiplying and postmultiplying Eq. (A1) by $\mathbf{W}_c^{-1/2}$.

To derive Eq. (18), first use Eq. (12) to rewrite Eqs. (14) as

$$\underline{a}^T\underline{\tilde{\phi}}=\sum_i u_i^{-1/2}\phi'_{ci}\underline{a}^T\underline{\eta}_i \tag{A2}$$

and

$$\underline{b}^T\underline{\tilde{\theta}}=u_i^{1/2}\theta'_{ci}\underline{b}^T\mathbf{W}_c^{-1}\underline{\eta}_i. \tag{A3}$$

Then, if Eq. (16) is written as

$$\hat{O}(x)=\sum_i (f_i\phi'_i+g_i\theta'_i), \tag{A4}$$

where

$$f_i=u_i^{-1/2}\underline{a}^T\underline{\eta}_i \tag{A5}$$

and

$$g_i=u_i^{1/2}\underline{b}^T\mathbf{W}_c^{-1}\underline{\eta}_i \tag{A6}$$

the critical exponent δ is given by

$$\delta=\frac{1}{4\pi}\sum_i (f_i^2+g_i^2), \tag{A7}$$

from which Eq. (18) follows with the aid of Eqs. (11) and (12), and

$$M=\mathbf{W}_c^{-1}\sum_i u_i\underline{\eta}_i\underline{\eta}_i^T\mathbf{W}_c^{-1}. \tag{A8}$$

This equation may be rearranged to give Eqs. (19) and (20) by using Eq. (A1).

APPENDIX B: GAPLESS MODES OF THE DECOUPLED KONDO-HEISENBERG ARRAY

The study of the different stable phases of the Kondo-Heisenberg array begins with an analysis of the gapless excitations of the decoupled fixed point. From there, as usual, we sort the phases by determining which of these excitations become gapped, and which remain gapless in the presence of the (Kondo) couplings between the 1DEG and the Heisenberg chain. In this appendix, then, we analyze the system in the absense of any Kondo interactions. However, since our ultimate goal is to study the coupled system, we will consider the character of gapless excitations constructed of composites of the gapless modes of the two subsystems, as well as the excitations of each, separate subsystem.

TABLE I. Gapless SDW excitations.

Operator	Wave number
$\vec{n}_1$	$2k_F$
$\vec{n}_\tau$	$\frac{\pi}{b}$

The low energy spin currents of the 1DEG, $\vec{s}(x)$, can be decomposed into two parts;

$$\vec{s}(x)=\vec{J}_1(x)+[\vec{n}_1(x)e^{i2k_Fx}+\text{H.c.}] \tag{B1}$$

where $\vec{J}_1=\frac{1}{2}\Sigma_{\lambda,\sigma,\sigma'}\psi^\dagger_{\lambda,\sigma}\vec{\sigma}_{\sigma,\sigma'}\psi_{\lambda,\sigma'}$ and $\vec{n}_1=\frac{1}{2}\Sigma_{,\sigma,\sigma'}\psi^\dagger_{1,\sigma}\vec{\sigma}_{\sigma,\sigma'}\psi_{-1,\sigma'}$ are the $k=0$ and $k=2k_F$ components of the SDW (charge 0, spin 1) of the 1DEG, respectively.

The Heisenberg chain spin current, $\vec{\tau}_j$, may be similarly decomposed into a $k=0$ part $\vec{J}_\tau$, and a finite momentum $k=\pi/b$ part $(-1)^j\vec{n}_\tau$ (where $2\pi/b$ is the reciprocal-lattice vector of the Heisenberg chain);

$$\vec{\tau}_j=\vec{J}_\tau(x_j)+(-1)^j\vec{n}_\tau(x_j). \tag{B2}$$

As explained in Sec. I, we count only the number of finite-momentum excitations. It follows from time-reversal symmetry that, for finite momentum, if there is a gapless mode at momentum q than there is also a gapless mode with momentum $-q$. We count them as one mode. To summarize, the gapless spin-1 excitations of the 1DEG and the Heisenberg spin chain, and the operator whose correlation function is most directly sensitive to it are listed in Table I.

The incommensurate 1DEG has one charge-0, spin-0 CDW excitation with momentum $2k_F$, created by the operator

$$\hat{O}_{CDW}=\frac{1}{2}\sum_{\lambda,\sigma}\psi^\dagger_{\lambda,\sigma}\psi_{-\lambda,\sigma}. \tag{B3}$$

The reader may notice an apparent conflict with the ‘‘prediction’’ of the generalized Luttinger theorem that there must be a gapless CDW mode at $2k_F^*=2k_F+(\pi/b)$. This conflict is resolved by realizing the existence of composite-CDW (Refs. 32 and 33) order parameters which are formed by combining a spin-1 SDW of the 1DEG with a spin-1 SDW of the Heisenberg chain into a composite singlet $\hat{O}_{c\text{-}CDW}$,

$$\hat{O}_{c\text{-}CDW}=\vec{s}\cdot\vec{\tau}=\vec{J}_1\cdot\vec{J}_\tau+\vec{J}_1\cdot\vec{n}_\tau(-1)^j+[\vec{n}_1\cdot\vec{J}_\tau e^{i2k_Fx}+\text{H.c.}]$$
$$+[\vec{n}_1\cdot\vec{n}_\tau e^{i2k_Fx}+\text{H.c.}](-1)^j. \tag{B4}$$

The staggered component $\vec{n}\cdot\vec{n}_\tau$ has momentum $2k_F^*=2k_F+(\pi/b)$, and is thus the CDW excitation required by the generalized Luttinger theorem. There is also a composite-CDW excitation $\vec{J}\cdot\vec{n}_\tau$ at $k=\pi/b$. To summarize, the noninteracting two-chain system of a Luttinger liquid and a Heisenberg spin chain has *gapless CDW modes at three wave vectors* (Table II). Note that the composite-CDW excitations at wave vectors π/b and $2k_F+(\pi/b)$ are not inde-

TABLE II. Gapless CDW excitations.

Operator	Wave number
$\vec{n}_1 \cdot \vec{n}_\tau$	$2k_F + \frac{\pi}{b}$
$\hat{O}_{CDW}$	$2k_F$
$\vec{J}_1 \cdot \vec{n}_\tau$	$\frac{\pi}{b}$

pendent, since they can be related through a multiplication by the 1DEG $\hat{O}_{CDW}$ (which has wave vector $2k_F$).

The charge-$2e$ singlet pairing modes also require careful consideration. In addition to the usual $k=0$ BCS even-parity singlet pairing,

$$\Delta = \frac{1}{\sqrt{2}} \sum_{\lambda} \psi^{\dagger}_{\lambda,\uparrow} \psi^{\dagger}_{-\lambda,\downarrow},$$

we note also the existence of an η-pairing mode, $\psi^{\dagger}_{\lambda,\uparrow}\psi^{\dagger}_{\lambda,\downarrow}$, at momentum $\pm 2k_F$, corresponding to right- and left-going singlet pairs.

As with the CDW modes, in addition to the singlet pairing modes of the 1DEG, it is necessary to consider the *composite* singlet pairing, $O_{c\text{-}SP}$, (a product of a triplet pairing in the 1DEG with a spin-1 mode of the Heisenberg chain) which turns out to be odd parity,[36,37]

$$O_{c\text{-}SP} = -i\tfrac{1}{2}(\psi_1^{\dagger}\vec{\sigma}\sigma_2\psi_2^{\dagger})\cdot\vec{\tau} \tag{B5}$$

where the sum over the spin indices of the spinors is implicit. It is decomposed into two momentum components: a uniform $k=0$ composite odd-parity singlet

$$\hat{O}^{k=0}_{c\text{-}SP}(x) = -i\tfrac{1}{2}(\psi_1^{\dagger}\vec{\sigma}\sigma_2\psi_2^{\dagger})\cdot\vec{J}_\tau \tag{B6}$$

and a $k=\pi/b$, i.e., a *staggered* composite odd-parity singlet

$$\hat{O}^{stagger}_{c\text{-}SP}(x) = -i\frac{1}{2}(\psi_1^{\dagger}\vec{\sigma}\sigma_2\psi_2^{\dagger})\cdot\vec{n}_\tau(-1)^j. \tag{B7}$$

The pairing and CDW modes can be related through the η-pairing modes by using the identities[32]

$$[\hat{O}_{CDW}, \eta^{even}] = \Delta, \tag{B8}$$

$$[\hat{O}_{c\text{-}CDW}, \eta^{odd}] = \hat{O}_{c\text{-}SP}. \tag{B9}$$

where

$$\eta^{even} \equiv \frac{1}{\sqrt{2}} \sum_{\lambda=\pm} \psi^{\dagger}_{\lambda,\uparrow}\psi^{\dagger}_{\lambda,\downarrow} \tag{B10}$$

TABLE III. Gapless charge-$2e$ pairing excitations.

Operator	Wave number
η	$2k_F$
$\hat{O}_{c\text{-}SP}$	$\frac{\pi}{b}$

$$\eta^{odd} \equiv \frac{1}{\sqrt{2}} \sum_{\lambda=\pm} \lambda\, \psi^{\dagger}_{\lambda,\uparrow}\psi^{\dagger}_{\lambda,\downarrow}. \tag{B11}$$

The above relations are a manifestation of the fact that for each gapless pairing mode there is a corresponding gapless CDW mode. Therefore, we adopt the custom of dropping the CDW modes from the explicit notation, (but their ''trivial'' existence should be implicitly understood in any fixed point which has the corresponding gapless charge-$2e$ gapless mode).

In summary, there are two independent gapless, finite-momentum, charge-$2e$ pairing modes.

The gapless modes in Tables I–III characterize the noninteracting fixed point of a two-chain system consisting of a 1DEG Luttinger liquid with Fermi wave number $2k_F$ and a Heisenberg spin chain with a reciprocal-lattice vector π/b. In our compact notation, which counts only the singlet charge-$2e$ and spin-1, charge-0 modes at finite momentum, the decoupled fixed point is denoted by

$$[c,s;2k_F]\left[s;\frac{\pi}{b}\right] \tag{B12}$$

or $[c,s][s]$ for short. However, as emphasized in the above, this description leaves implicit the gapless, odd-parity charged excitation at momentum π/b. Since this excitation is a composite of the excitations already listed, it can be omitted in a minimal labeling scheme. But because of this, it looks a bit mysterious that there appears a a gapless composite pairing mode with momentum π/b in the spin-gap states which appear under the influence of a relevant perturbation—it looks (incorrectly) as if the relevant interaction is generating new gapless excitations. From this viewpoint, one might be tempted to label the decoupled fixed point $[c][c,s]$, and to view the spin-1 excitation of the 1DEG as a composite of the three other modes, however far that is from the actual physics of two decoupled systems. The nonuniqueness of the label associated with each state is an intrinsic feature of the approach taken in the present paper.

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[19] If there exists a gapless, spinless charge-$2e$ mode, one can clearly construct gapless, spinless excitation with any charge $2ne$. One can imagine cases in which there are no gapless charge-$2e$ excitations, but there are gapless spinless excitations with charge $4e$ and multiples thereof. We will adopt the convention that the fundamental charged, spinless excitation is assumed to have charge $2e$, unless otherwise specified.

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[21] Where there is a gapless spin mode, a CDW fluctuation can be viewed as a total spin-zero composite of three spin modes. Where there is a spin gap but no charge gap, the generalized Luttinger's theorem implies that there must be a gapless, zero-momentum charge $2e$ ''BCS mode,'' which can be thought of as a composite of an η-pairing mode and a CDW mode; conversely, the CDW mode can be thought of as a composite of oppositely charged η-pairing and BCS modes. This ambiguity is a reflection of the number-phase duality of the charge modes.

[22] The Abelian boson representation used here is not explicitly SU(2) invariant, and indeed is valid even if there is Ising-Heisenberg symmetry which breaks the spin symmetry down to $U(1)\times Z(2)$. There is a subtlety in the SU(2) invariant case when the spin modes are gapless, in that the cosine term is only marginally irrelevant, so that to compute the asymptotic behavior of correlation functions completely correctly, one cannot simply set $V_s=0$, but must rather let it renormalize to zero. Even when there is a spin gap, SU(2) symmetry constrains the values of the renormalized Luttinger exponents in the fixed-point Hamiltonian, since there are special degeneracies of the gapped excitations implied by this symmetry. For instance, if $K_s=\frac{1}{2}$ (i.e., if $\beta_s=\sqrt{2\pi}$), the lowest breather (or soliton anti-soliton bound state) has the same energy as the single-soliton state, so together they form the triplet (massive-magnon) state; for slightly different values of K_s, this degeneracy, which is required by SU(2), is lifted.

[23] Here we have considered only the case in which a gap results from a relevant interaction that pins the value of the density-wave phase, ϕ. While it is typically forbidden by symmetry in most cases, it is certainly possible to imagine cases in which there is a relevant interaction of the form $\cos(\beta'\theta_\alpha)$ which would pin the dual phase; we do not explore this possibility in the present paper.

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