Universal structure of the edge states of the fractional quantum Hall states

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(Received 15 October 1998; revised manuscript received 22 January 1999)

We present an effective theory for the bulk fractional quantum Hall states on the Jain sequences on closed surfaces and show that it has a universal form whose structure does not change from fraction to fraction. The structure of this effective theory follows from the condition of global consistency of the flux attachment transformation on closed surfaces. We derive the theory of the edge states on a disk that follows naturally from this globally consistent theory on a torus. We find that, for a fully polarized two-dimensional electron gas, the edge states for all the Jain filling fractions $\nu=p/(2np+1)$ have only one propagating edge field that carries both energy and charge, and two nonpropagating edge fields of topological origin that are responsible for the statistics of the excitations. Explicit results are derived for the electron and quasiparticle operators and for their propagators at the edge. We show that these operators create states with the correct charge and statistics. It is found that the electron tunneling density of states for all the Jain states scales with frequency as $|\omega|^{(1-\nu)/\nu}$. [S0163-1829(99)02124-4]

I. INTRODUCTION

The fractional quantum Hall (FQH) states have been studied extensively and many of their properties are quite well understood. Much of this understanding came from Laughlin's microscopic theory,[1] and its generalizations,[2–4] as well as by Chern-Simons field theory approaches.[5,6] The result of this theoretical effort was a classification of all possible abelian FQH states in the form of effective low-energy theories that capture their universal features.[7–10] Much of the present understanding of the universal properties of the Abelian FQH states is encoded in the effective Lagrangian of Wen and Zee,[8] which is given by

$$\mathcal{L}_{\rm eff}=\sum_{I,I'}\frac{1}{4\pi}K_{II'}\epsilon_{\mu\nu\lambda}a_{I\mu}\partial_\nu a_{I'\lambda}-\sum_I \ell_I j^v_\mu a^\mu_I$$

$$-\sum_I \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu t_I a_{I\lambda}, \qquad (1.1)$$

where, for the Jain states, i.e., for states with $\nu=p/(2np+1)$, the matrix K and the charge vector t take the values

$$K=1_{p\times p}+2nC,\quad t^T=(1,\ldots,1), \qquad (1.2)$$

where C is a $p\times p$ matrix with all its elements equal to one. In Eq. (1.1) j^v_μ is the vortex current, ℓ_I is the vector of vortex (quasiparticle) charges and t_I is the electromagnetic charge vector. A key topological property of the Abelian QH fluids is the topological degeneracy of the Hilbert space (not just the ground state) which is equal to $|\det K|^g$, where g is the genus (i.e., the number of handles) of the surface (manifold) on which the fluid moves.[11] Wen[9] has emphasized the concept of topological order of the fluid and characterized it by the degeneracy of the Hilbert space.

This general classification of the Abelian FQH states is very powerful. In addition to giving a compact and universal description of all Abelian FQH states, it also classifies the possible behaviors of their associated edge states.[9,12,13] The central point of this classification is that for a *given* effective theory of the bulk incompressible state there is a *natural* construction of the edge in terms of the conformal field theory of chiral bosons.[12] A fundamental feature of this construction is the existence of a one-to-one correspondence between the quasiparticle states of the bulk and the primary fields that build the spectrum of the edge states.

Wen's construction further assumes that the behavior of the electron gas near the edge is essentially simple, which may or may not be realized if edge reconstruction actually takes place.[14] Whether or not this happens depends on many nonuniversal microscopic issues, such as edge potentials, interactions, and impurities, which complicate matters and which may be quite relevant for a precise understanding of the edge tunneling experiments away from the middle of the bulk Hall plateaus.[15] Thus, in realistic situations, the detailed microscopic physics near the edge may actually mask the robust physics of the bulk. However, once the details of edge reconstruction are sorted out, it is expected that the physics of the edge should be universal even though its connection with the physics of the bulk is no longer so simple.

Microscopically, the bulk states can be constructed by implementing the idea of flux-attachment, by coupling particle currents to a suitable set of Chern-Simons gauge fields.[5,6] These ideas, which are at the root of the construction of fractional (or braid) statistics,[16,17] play a central role in the field-theoretic descriptions of the fractional quantum Hall effect for both single-layer systems,[5,6,18–21] bilayers,[22–24] and for spin-singlet or partially polarized FQH states.[24,25] Since the effective action of any incompressible state of a system of charged particles in two dimensions in the presence of a strong-magnetic field, by general hydrodynamic arguments,[26,27] must be of the Chern-Simons form, the effective action that is actually derived from the field theoretic descriptions is a Chern-Simons gauge theory and it fits the K-matrix classification.

However, a number of puzzles implicit in this picture have not been fully resolved. In particular, the *K*-matrix classification assumes that the number of physically distinct stable quasiparticles is equal to the rank of the matrix. This feature is quite puzzling since the only known conservation law in this system is just charge conservation, and it is unclear what the dynamical origin of the additional conservation laws may be. Conceptually, the multicomponent structure of the *K*-matrix classification reflects the underlying hierarchical construction of the FQH states. However, although the different hierarchies are useful pictures that organize the FQH states, there is no physical reason to regard the condensates used in these pictures to be literally physical entities.

These problems are also connected with two important features of the *K*-matrix classification. One is the fact that for a given filling fraction there are many possible physically distinct theories, characterized by different charges and statistics of the quasiparticles. It has been argued by Wen that these states have topological orders that tell these different states apart. For example, the $\nu=2/5$ single layer, fully polarized state is given by a 2×2 K matrix and it seems to be closely related to the $\nu=2/5$ fully polarized QH state in bilayers. From a hydrodynamic point of view, in the absence of interlayer tunneling, it is obvious that the bilayer system should have two conserved currents and hence a 2×2 K matrix. However, the physical meaning of these two separately conserved hydrodynamic currents *in single-layer systems* is unclear since there is no known conservation law to insure their stability.

A closely related problem is the associated composite picture of the edge states with a number of branches equal to the rank of the K matrix. In practice, these structures have their origin in the multiple condensates of the hierarchical construction of the FQH states. In contrast, in the fermion Chern-Simons theory description of the two-dimensional electron gas (2DEG) on a disk, which should be equivalent to the *K*-matrix picture, there is only one gauge field and only one quasiparticle whereas the *K*-matrix classification typically will have several gauge fields and the associated quasiparticles.

It is natural to ask what is the minimal universal structure required to describe the Abelian FQH states in the bulk. A similar issue also arises in the description of the edge states, which require a composite structure (even in the absence of edge reconstruction) with a quite complex behavior as a function of the filling factor. In particular, the contribution of the edge states to the specific heat of the 2DEG (in fact, the total contribution) looks like a number-theoretic function, which diverges at the compressible fractions. However, this picture holds only if the currents that define the structure of the K matrix for the composite edges are actually conserved.

In addition, the problem of the consistency of the standard mean-field theory constructions of the bulk states for a system on a closed surface, such as a torus, has not been fully resolved. Although superficially this may seem a technical issue, it is an important problem since the degeneracy of the Hilbert space on a closed surface is a universal feature of these topological fluids, and it is closely related to the puzzles mentioned above. The usual flux-attachment transformation[5,6,21] is regarded as a process in which a particle is physically and *locally* glued to a certain number of (statistical) flux quanta. For an electron gas on an *open* simply connected disk with fixed boundary conditions, there is no problem with this procedure. However, if we were to carry out this procedure on a closed surface, such as a torus or a sphere, this approach is not consistent as it is usually described. Similar difficulties arise if the disk is replaced by an annulus.

The purpose of this paper is to construct a theory of the FQH fluids for a single layer, fully polarized 2DEG, which satisfies the following criteria: (1) it must contain the minimal possible structure consistent with the existence of only one conservation law (charge conservation) and (2) it must be consistent on a closed surface (such as a torus). In order to do that, we reformulate our approach of our previous work[6] on the FQH Jain states on a disk, to satisfy these criteria. To do so, it is necessary to modify the flux-attachement procedure so as to make it globally consistent. The key issue is the invariance (or lack of) of the Chern-Simons action under large gauge transformations for a theory defined on a closed manifold with a number of handles g. (In practice we will only be interested on describing the behavior of the system on a torus, and so we will use $g=1$.) It has been shown by a number of authors[28–30] that invariance under large gauge transformations of the path integral (not the action) requires that the coupling constant of any Chern-Simons theory (both Abelian and non-Abelian) *must* be quantized. However, the standard Chern-Simons constructions of the flux-attachment transformation violate this principle. Here, we make use of the results of Ref. 31 to derive an effective theory that is globally consistent. The resulting effective theory has a *K*-matrix form but with a different and much simpler structure than the usual one. We find that this structure requires only a fixed (and small) number of gauge fields and their number, i.e., the rank of the K matrix, is independent of the filling fraction. As a result, we find that for each Jain state there is only one quasiparticle. Furthermore, the effective theory that we derive here for a system on a closed surface, has a related and unique minimal universal edge structure on an open surface, in the abscence of edge reconstruction. This edge structure is particularly simple.

As this paper was being prepared we became aware of a very recent work by Lee and Wen[35] where they argued that the effective theory of the edge states for the Jain fractions contains just two independent chiral bosons, one of which does not propagate. Our final result in essence agrees with the picture advocated by Lee and Wen. However, the underlying philosophy is quite different since in our work the nonpropagating fields have a purely topological origin.

This paper is organized as follows. In Sec. II, we introduce a general framework for flux attachment that is consistent on closed manifolds such as a torus. Here, we show that the effective theory of the FQH states for $\nu=p/(2np\pm1)$, has a simple and universal structure. In Sec. III, we derive the theory of the edge states that follows from this universal structure. In Sec. IV, we derive the form of the electron and quasiparticle operators at the edge and compute their propagators. Section V is devoted to the conclusions.

II. FLUX ATTACHMENT ON CLOSED SURFACES

In Ref. 31 it was shown that there is a simple and direct way to reformulate the Chern-Simons theory of the (single

layer, fully polarized) FQH state in order to satisfy the requirement of global consistency on a closed surface. For a fermion representation of the FQH system (i.e., composite fermions) it was shown that the exact partition function can be written as a path integral of a theory in which the particles whose worldlines are represented by the currents j_μ, interact with two gauge fields a_μ and b_μ. These interactions are encoded in the following effective action:

$$S_{\rm eff}[a,b,j]=\frac{1}{2\pi}a^\mu\epsilon_{\mu\nu\lambda}\partial^\nu b^\lambda-a^\mu j_\mu-\frac{2n}{4\pi}\epsilon_{\mu\nu\lambda}b^\mu\partial^\nu b^\lambda. \tag{2.1}$$

Therefore, the amplitudes can be written in terms of a path integral over an Abelian Chern-Simons gauge field with a correctly quantized coupling constant equal to $2n/(4\pi)$. Hence, there exists an exact rewriting of the theory involving two gauge fields a_μ and b_μ. These two gauge fields arise quite naturally: the field b_μ arises from the fact that the particle currents (worldlines) are conserved and the field a_μ is the Lagrange multiplier that imposes the hydrodynamic constraint between the current and the curl of b_μ.

The usual form of the flux-attachment transformation is found by integrating out the gauge field b_μ. For vanishing boundary conditions at infinity, this leads to an effective action for the field a_μ of the conventional form[6]

$$S_{\rm eff}[a]=\frac{1}{4\pi 2n}\int d^3x\,\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda. \tag{2.2}$$

However, this form of the effective action is not valid for manifolds with nontrivial topology. Nevertheless, Eq. (2.1) is correct in all cases as it is invariant under both local and large gauge transformations.

As usual,[6] the mean-field theory in the composite fermion language proceeds by first spreading out the field and constructing an effective integer Hall effect of the partially screened magnetic field. The result is a description of the states in the generalized Jain hierarchies with filling fractions in the series $\nu_\pm(n,p)=p/(2np\pm1)$, where $p,n\in\mathbf{Z}$ and $\pm$ stands for an electron and holelike FQH state, respectively.

The effective action in the composite fermion picture is found by integrating out the local particle-hole fluctuations of the fermions about the uniform mean-field state. At long distances and low energies the effective Lagrangian once again involves a 2×2 K matrix and it has the usual Wen-Zee form

$$\mathcal{L}_{\rm eff}=\frac{1}{4\pi}K^{IJ}\epsilon^{\mu\nu\lambda}a^I_\mu\partial_\nu a^J_\lambda \tag{2.3}$$

with

$$K^{IJ}=\begin{pmatrix}\pm p & 1\\ 1 & -2n\end{pmatrix}. \tag{2.4}$$

We now notice that this effective theory is globally well defined since the Chern-Simons coupling constants are correctly quantized. Indeed, if we integrate out the gauge field $b_\mu=a^2_\mu$, we find the same effective action for a_μ of our previous work.[6] Since the absolute value of the determinant is $|\det K|=|2np\pm1|$, we find that the generalized Jain states are $|2np\pm1|$-fold degenerate on the torus, which is the correct result.

In what follows, we will consider the effective Lagrangian of Eq. (2.4) expanded to include the the quantum dynamics of the quasiparticles. The effective Lagrangian now reads

$$\mathcal{L}_{\rm eff}=\frac{p}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda+\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda-\frac{2n}{4\pi}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda+\frac{1}{4\pi}\epsilon_{\mu\nu\lambda}e_\mu\partial_\nu e_\lambda+\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu A_\lambda-(a_\mu+e_\mu)j^\mu_{qp}. \tag{2.5}$$

The current j^μ_{qp} in the last term of Eq. (2.5) represents the effects of quasiparticles. However, in the fermionic picture the bare quasiparticles are composite *fermions* whose statistics is modified by the Chern-Simons gauge fields. Thus, the statistics of all excitations that we will compute is defined relative to *fermions*. A simple way to keep track of the underlying statistics is to introduce, as we did in Eq. (2.5), an additional Chern-Simons gauge field e_μ, which couples only to the quasiparticle current j^μ_{qp}. From now on, and in order to simplify the notation, we shall call $p=\pm p$. This effective Lagrangian, includes the coupling to a weak external gauge field A_μ.

We can write the effective Lagrangian in a more compact form if we define $a^1_\mu=b_\mu$, $a^2_\mu=a_\mu$, $a^3_\mu=e_\mu$, the charge vector $t_I=(1,0,0)$ and the flux vector $\ell_I=(0,1,-1)$, as

$$\mathcal{L}=\frac{1}{4\pi}K_{IJ}\epsilon_{\mu\nu\lambda}a^\mu_I\partial^\nu a^\lambda_J+\frac{1}{2\pi}t_I\epsilon_{\mu\nu\lambda}a^\mu_I\partial^\nu A^\lambda+\ell_I a^\mu_I j^\mu_{qp}, \tag{2.6}$$

where the coupling-constant matrix is

$$K_{IJ}=\begin{pmatrix}-2n & 1 & 0\\ 1 & p & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.7}$$

whose determinant is $|\det K|=2np+1$. Hence, we get the correct degeneracy on closed surfaces.

Following Wen[9] we can compute the filling fraction, which is given by

$$\nu=|t^TK^{-1}t|=\frac{p}{2np+1}. \tag{2.8}$$

The quantum numbers of the quasiparticles are

$$Q_{qp}=-et^TK^{-1}l=\frac{-e}{2np+1}, \tag{2.9}$$

$$\frac{\theta_{qp}}{\pi}=l^TK^{-1}l=\frac{2n}{2np+1}+1 \tag{2.10}$$

for the charge and the statistics, respectively. For the special case of the Laughlin states, $p=\pm1$, the gauge field a^2_μ can be integrated out and the effective action is now identical to the dual action found by Wen.[9]

Therefore, the theory defined by Eq. (2.6) gives the correct quantum numbers for the quasiparticle as well as the correct Hall conductance. This bosonic representation gives an alternative effective theory of the Jain states and it does not involve a hierarchy of condensates, as in Wen's construction. This picture also suggests that the effective theory of the edge states for the FQH states in the Jain sequence does not necessarily require a composite structure of the edge states.

III. EDGE THEORY FOR THE JAIN STATES

In this section we use the effective theory for all the states in the Jain sequence, derived in the previous section, to extract the effective theory for the edge states. The effective Lagrangian of Eq. (2.6) is globally well defined (on closed surfaces), yields the correct ground-state degeneracy on the torus as well as excitations with the correct fractional charge and statistics. However, unlike the standard hierarchical construction of the effective theory of the Jain states,[8] for a generic state in the Jain sequence, the effective Lagrangian of Eq. (2.6) contains the same number of gauge fields, for all filling fractions on the Jain sequences, and it can be reduced to just a single gauge field for the special case of the Laughlin states. In a sense, the Lagrangian of Eq. (2.6) is the *minimal* effective theory. This effective Lagrangian has the standard form introduced by Wen and Zee[8] and, following the general arguments of Wen,[9,12] it is straightforward to extract a theory for the edge states, which we do in this section. Clearly, since the effective theory of the bulk in general contains just three gauge fields, the number of edges does not grow from one state in the hierarchy to the next. In particular this implies that the specific heat of the system does not grow without limit as one goes up in the hierarchy. Consequently, the changes in the thermodynamic properties of the system that occur as the system becomes compressible is not due to a proliferation of edges but to a physical collapse of the gap in the spectrum and the resulting failure to separate the edge from the bulk.

It is quite well known from the work of Wen,[9] in turn based on the work of Witten,[28] that a Chern-Simons theory on a disk Ω is equivalent to a theory of chiral bosons whose Hilbert space has support at the boundary $\partial\Omega$ of the disk. Here we will only need the salient features of the derivation and we will omit the details, which can be found in Ref. 9. The key idea is that, in the gauge $a_J^0=0$, the Chern-Simons theory reduces to a set of commutation relations among the spacial components of the gauge fields, and to the constraint (i.e., Gauss's law)

$$\mathcal{J}_I^0 = -K_{IJ}\epsilon_{ij}\partial^i a_J^j, \tag{3.1}$$

where the current $\mathcal{J}_I^i$ is defined by

$$\mathcal{J}_I^\mu \equiv t_I \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda + 2\pi \not{\ell}_I j_{qp}^\mu. \tag{3.2}$$

The solution of Gauss's law is

$$a_i^I = \partial_i \phi^I \tag{3.3}$$

where ϕ^I are three multivalued scalar fields, i.e., singular gauge transformations. If the quasiparticles and the external fluxes are quasistatic bulk perturbations of the condensate, of quasiparticle number N_{qp} and flux $\Phi = 2\pi N_\phi$, the scalar fields ϕ^I at the boundary $\partial\Omega$ must satisfy the conditions

$$\Delta\phi_I = 2\pi (K^{-1})_{IJ} \begin{bmatrix} N_\phi \\ N_{qp} \\ -N_{qp} \end{bmatrix}_J, \tag{3.4}$$

where $\Delta\phi_I \equiv \oint_{\partial\Omega} dx_i \partial_i \phi_I$ is the change of the field ϕ_I once around the boundary $\partial\Omega$. In components we get

$$\Delta\phi_1 = \frac{2\pi}{2np+1}(N_{qp} - pN_\phi),$$
$$\Delta\phi_2 = \frac{2\pi}{2np+1}(N_\phi + 2nN_{qp}), \quad \Delta\phi_3 = -2\pi N_{qp}. \tag{3.5}$$

In particular, if there is just one quasiparticle in the bulk $N_{qp}=1$ and no extra flux, $N_\phi=0$, we get $\Delta\phi_1 = 2\pi/(2np+1)$, $\Delta\phi_2 = 2\pi 2n/(2np+1)$, and $\Delta\phi_3 = -2\pi$. Conversely, for $N_\phi=1$ and $N_{qp}=0$, we get instead $\Delta\phi_1 = -2\pi\nu$, $\Delta\phi_2 = 2\pi/(2np+1)$ and $\Delta\phi_3 = 0$. Likewise, if we add an electron to the bulk, $N_{qp} = 2np+1$ but no flux $N_\phi = 0$, we get $\Delta\phi_1 = 2\pi$, $\Delta\phi_2 = 2\pi 2n$ and $\Delta\phi_3 = -2\pi(2np+1)$. These conditions will play an important role below.

Once the constraint Eq. (3.1) is solved, it is immediate to show that the result is a theory of chiral bosons at the boundary $\partial\Omega$, whose action is

$$S = \frac{1}{4\pi} K_{IJ} \int dx_0 \oint_{\partial\Omega} dx_i \partial^i \phi_I \partial^0 \phi_J. \tag{3.6}$$

However, as emphasized by Wen,[9] as they stand these bosons do not propagate. The reason is that the Chern-Simons gauge theory is actually a topological field theory, and in addition to being gauge invariant, it is independent of the metric of the surface where the electrons reside. In particular, this means that the Hamiltonian of the Chern-Simons theory is zero, which is just the statement that this is an effective theory for the degrees of freedom below the gap of the incompressible fluid. There are no local degrees of freedom left in the bulk of the system, and the physical degrees of freedom are present only at the boundary, which also carries energy.

There are many ways to represent this physics in the effective theory. For example, Wen[9] chooses the gauge condition $a_0 + v a_1 = 0$, where v is chosen to be the velocity of noninteracting electrons at the edge, i.e., $v = cE/B$, with c the speed of light and E the electric field of the confining potential at the edge. In the context of the construction that we are pursuing here, only the gauge field a_μ^1 couples to the electromagnetic field and thus it is the only one that will represent propagating degrees of freedom, the charge fluctuations at the edge.

Another option, which we will use here, is to keep the gauge condition $a_0 = 0$, which does not break topological invariance, but to add boundary terms to the effective action to represent the effect of the propagating modes at the edge. By power counting the boundary term with the smallest scaling dimension, one can add to the effective action the form

$$S_{\text{boundary}} = -\int dx_0 \oint_{\partial\Omega} dx_i \frac{\tilde{g}}{2}[a_i^1(x)]^2$$

$$= -\int dx_0 \oint_{\partial\Omega} dx_i \frac{\tilde{g}}{2}[\partial_i \phi^1(x)]^2. \qquad (3.7)$$

This is a marginal operator. Here $\tilde{g}$ is a coupling constant whose physical meaning we discuss below. Notice that this term only affects the field a_μ^1. Terms of this form arise microscopically as follows. Within the framework of the fermionic Chern-Simons theory,[6] in addition to the bulk states there are edge states. A realistic description of these states requires an understanding of the problem of edge reconstruction. At the level of a Hartree-Fock approximation for the fermions in the Chern-Simons picture a theoretical description was given in Refs. 32 and 33. Although it is not clear if such descriptions are reliable for systems as quantum mechanical as the 2DEG in the lowest Landau level, it is clear that an effective edge must exist even if many of the modes predicted by the mean-field theory were to be an artifact of the approximation. In any event there should be at least one edge mode that will carry the correct Hall current at the edge. At the level of the mean-field theory these states are fermionic, as they are in the bulk before the Gaussian fluctuations are integrated out.[6] These edge fermionic states will couple to the *boundary* component of the bulk gauge field a_μ^1. We can now proceed to integrate out the fermions, as we did before. If the fermions where nonchiral, the result of integrating out the fermions is equivalent to conventional bosonization. Their contribution to the effective action is calculated from the determinant of the Dirac operator coupled to gauge fields. This is a very standard result[34] and the effective action is

$$S_{\text{edge}} = -\frac{p}{8\pi}\int dx_0 \oint_{\partial\Omega} dx_1 (a_\mu^1)^2, \qquad (3.8)$$

which holds in the continuum limit, i.e., infinite bandwidth, and for an infinitesimally narrow edge. Since the edge theory is actually chiral, we need to keep only the right moving piece of Eq. (3.8), and S_{edge} becomes

$$S_{\text{edge}} = -\frac{p}{4\pi}\int dx_0 \oint_{\partial\Omega} dx_1 (a_R^1)^2, \qquad (3.9)$$

where a_R^1 is

$$a_R^1 = \frac{1}{\sqrt{v}} a_0^1 + \sqrt{v} a_1^1 \equiv \sqrt{v} a_1^1 \qquad (3.10)$$

with $v = eE/B$ the speed of the edge excitations, and we have used the gauge condition $a_0^1 = 0$ at the boundary. Hence, we find

$$S_{\text{edge}} = -\frac{pv}{4\pi}\int dx_0 \oint_{\partial\Omega} dx_1 (\partial_1 \phi^1)^2, \qquad (3.11)$$

which has the form of Eq. (3.7), with $\tilde{g} = pv/(2\pi)$.

The electron-electron interaction term becomes

$$S_{\text{int}} = \int_\Omega d^3x \int_\Omega d^3x' \frac{1}{2}[\rho(x) - \overline{\rho}] V(x-x')[\rho(x') - \overline{\rho}]$$

$$= \int_\Omega d^3x \int_\Omega d^3x' \frac{1}{8\pi^2} \epsilon_{ij} \partial^i a_1^j(x) V(x-x') \epsilon_{kl} \partial^k a_1^l(x')$$

$$\equiv \int dx_0 \oint_{\partial\Omega} dx_i dx_i' \frac{t_I t_J}{8\pi^2} \partial_i \phi_I(x) V(x-x') \partial_i \phi_J(x'), \qquad (3.12)$$

where we have only retained the boundary contribution since the bulk excitations have a finite (and for present purposes large) energy gap. Notice that since $t_I = (1,0,0)$ this term of the action only affects the mode ϕ_1. Likewise, the interaction with an external potential with support at the boundary becomes

$$S_{\text{ext}} = \int d^3x [\rho(x) - \overline{\rho}] A_0(x)$$

$$= \int dx_0 \oint_{\partial\Omega} dx_i \frac{t_I}{2\pi} \partial_i \phi_I(x) A_0(x) \qquad (3.13)$$

and it involves only ϕ_1.

Thus, as expected, the effective action is in fact a theory of edge modes, in agreement with Wen's general arguments. The effective action involves just three chiral bosons ϕ_I (with $I = 1,2,3$) and takes the form

$$S = \frac{1}{4\pi}\int_{\partial\Omega\times\mathbf{R}} dx_0 dx_1 (K_{IJ} \partial_1 \phi_I \partial_0 \phi_J + U_{IJ} \partial_1 \phi_I \partial_1 \phi_J), \qquad (3.14)$$

where $U_{IJ}(x-x') = t_I t_J [pv + (1/2\pi) V(x-x')]$, and its only effect is to determine the velocity of the edge modes. Notice that, as it is well known, the actual velocity of the edge modes is the sum of two terms, one of which is determined by the interactions. In what follows, we will work with an effective edge velocity v, which includes both the effects of the edge electric field and of the Coulomb interactions. Implicitly, and for simplicity, we assume here a short-range interaction. In reality, a strict $1/r$ Coulomb interaction gives a well-known logarithmic correction to the dispersion of the excitations and hence it is not just equivalent to a redefinition of the velocity. However, this is a well-understood phenomenon that does not affect the main physics of this system and hence, we will work with an effective velocity v. Notice that the only mode with a nonvanishing velocity is ϕ_1, which is the only mode that couples to perturbations due to an external electromagnetic field. Thus, we identify ϕ_1 as the *charge mode*. The two remaining modes do not propagate. Their effect is to fix the statistics of the states.

Finally, we need to relate these fields to edge charge density. The local bulk charge and current density $J_\mu(x)$ is given by

$$J_\mu(x) = \frac{\delta S}{\delta A_\mu} = \frac{1}{2\pi} t_I \epsilon_{\mu\nu\lambda} \partial^\nu a_I^\lambda. \qquad (3.15)$$

The edge currents and densities are integrals of the bulk currents and densities across the edge. Let λ be the physical width of the edge, which we will denote by Λ and it is perpendicular to the edge. (Here, $\lambda \approx \ell_0$, the magnetic length.) For an edge along the direction x_1, the edge density is given by

$$j_0 \equiv \int_\Lambda dx_2 J_0(x) = \frac{\lambda}{2\pi}\partial_1 \overline{a_1^2} - \frac{1}{2\pi}\Delta a_1^1, \tag{3.16}$$

where $\overline{a_1^2}$ is the average of the gauge potential a_1^2 across the edge and Δa_1^1 is the difference of the gauge potential a_1^1 across the edge. We will take the width of the edge to be infinitesimal $\lambda \to 0$, and since the potential a_j^2 is regular at the edge, the first term in Eq. (3.16) vanishes. With a fixed number of electrons and at fixed total magnetic field, we can also choose the gauge potentials to vanish outside the system. Thus, $\Delta a_1^1 = -a_1^1 = -\partial_1 \phi_1$ where a_j^1 is measured *inside* the system, at the edge. The edge charge density then becomes

$$j_0 = \frac{1}{2\pi}\partial_1 \phi_1, \tag{3.17}$$

which is the standard result.[9] It is straightforward to check that if N_{qp} quasiparticles are added to the bulk at constant magnetic field ($N_\phi = 0$), the edge acquires a charge

$$Q_{\text{edge}} = \int dx_1 j_0(x_1) = \frac{N_{qp}}{2np+1}, \tag{3.18}$$

which is equal to the extra charge added to the bulk, as it should.

In summary, in this picture there are three edge modes, one propagating mode associated with charge fluctuations and the other two nonpropagating modes associated with the global topological consistency of flux-attachment. As we will see in the next section the only effect of these nonpropagating topological modes is to give the correct statistics to the excitations. Since $|\det K| = 2np+1$ this effective theory reproduces the correct topological degeneracy of the Hilbert space. Notice that from the point of view of this effective theory there is no particular difference between the electronlike FQH states and the holelike FQH states apart from the value of the filling fraction. In addition, Eq. (3.5) requires that the chiral bosons ϕ_I satisfy the boundary conditions

$$\Delta\phi_I = \frac{2\pi}{2np+1}\begin{pmatrix} -p & 1 & 0 \\ 1 & 2n & 0 \\ 0 & 0 & 2np+1 \end{pmatrix}_{IJ} \begin{pmatrix} N_\phi \\ N_{qp} \\ -N_{qp} \end{pmatrix}_J, \tag{3.19}$$

where N_{qp} and N_ϕ are the total number of quasiparticles (or charge) in the bulk and the extra magnetic flux in the bulk, both with respect to the middle of the plateau, respectively.

IV. ELECTRON AND QUASIPARTICLE OPERATORS

We will now seek a new basis of modes in which the quantum numbers of the excitations are more transparent. We will use this representation to construct the electron and quasiparticle operators at the edge.

Let us consider a generic operator that creates excitations, which can be written as

$$\Psi(x) = e^{i(m_1\phi_1 + m_2\phi_2 + m_3\phi_3)}. \tag{4.1}$$

The values that the coefficients m_I take depend on the quantum numbers of the particular quasiparticle that the operator $\Psi(x)$ creates. Recall that for physical states $m_2 = -m_3$. It can be shown (see, for instance, Ref. 9) the quantum numbers of the operator $\Psi(x)$ are given by the following expressions:

$$\begin{aligned} \frac{Q}{e} &= \sum_{IJ} m_I K_{IJ}^{-1} t_J = \frac{-m_1 p + m_2}{2np+1}, \\ \frac{\theta}{\pi} &= \sum_{IJ} m_I K_{IJ}^{-1} m_J = -\frac{Q^2}{\nu e^2} + \frac{m_2^2}{p} + m_2^2, \end{aligned} \tag{4.2}$$

where Q is the quasiparticle charge and θ is its statistics.

We have already identified the mode ϕ_1 as the charge mode and we will denote it as $\phi_1 \equiv \phi_C$. Equation (4.2) shows that an operator Ψ with $m_2 = p m_1$ creates neutral solitons. Although these states are not in general part of the Hilbert space, we can nevertheless construct linear combinations of the chiral bosons with these quantum numbers. We will refer to these fields as the ''neutral modes.'' In particular, it will be useful to rewrite the effective theory in terms of the following linear combinations of the fields

$$\begin{aligned} \phi_C &= \phi_1, \\ \phi_N &= \frac{1}{\sqrt{p}}\phi_1 + \sqrt{p}\,\phi_2, \\ \phi_{N'} &= \phi_3, \end{aligned} \tag{4.3}$$

where we have introduced the ''neutral'' modes ϕ_N and $\phi_{N'}$.

The edge effective Lagrangian of Eq. (3.14) in terms of the charged and neutral modes is diagonal,

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4\pi\nu}(\partial_1\phi_C\partial_0\phi_C - v\,\partial_1\phi_C\partial_1\phi_C) \\ &+ \frac{1}{4\pi}(\partial_1\phi_N\partial_0\phi_N + \partial_1\phi_N\partial_0\phi_{N'}). \end{aligned} \tag{4.4}$$

We see that only the charge mode ϕ_C propagates. The role of the two remaining modes is to give the correct quantum numbers to the quasiparticles, in particular their statistics.

The new fields ϕ_C, ϕ_N, and $\phi_{N'}$ obey the boundary conditions

$$\begin{aligned} \Delta\phi_C &= \frac{2\pi}{2np+1}(N_{qp} - pN_\phi), \\ \Delta\phi_N &= \frac{2\pi}{\sqrt{p}}N_{qp}, \\ \Delta\phi_{N'} &= -2\pi N_{qp}. \end{aligned} \tag{4.5}$$

In the new basis of Eq. (4.3) the most general quasiparticle operator of Eq. (4.1) can be written as

$$\Psi(x)=e^{i(\alpha_C\phi_C+\alpha_N\phi_N+\alpha_{N'}\phi_{N'})}, \tag{4.6}$$

where

$$\alpha_C=m_1-\frac{m_2}{p},$$

$$\alpha_N=\frac{m_2}{\sqrt{p}},$$

$$\alpha_{N'}=-m_2. \tag{4.7}$$

Hence, the coefficients satisfy

$$\frac{Q}{e}=-\nu\alpha_C, \tag{4.8}$$

$$\frac{\theta}{\pi}=-\nu\alpha_C^2+\alpha_N^2+\alpha_{N'}^2, \tag{4.9}$$

where Q is the quasiparticle charge and θ its statistics.

The coefficients for the *quasiparticle* operator should be such that they satisfy $Q=-e/(2np+1)$ and $\theta/\pi=2n/(2np+1)+1$. Therefore, we find

$$\alpha_C^{qp}=\frac{1}{p},\quad \alpha_N^{qp}=-\frac{1}{\sqrt{p}},\quad \alpha_{N'}^{qp}=1, \tag{4.10}$$

which is consistent with setting $m_1=0$ and $m_2=-1$.

Likewise, to create an *electron* is equivalent to create $2np+1$ quasiparticles and hence, it is defined by the choice

$$\alpha_C^{e}=\frac{1}{\nu},\quad \alpha_N^{e}=-\frac{\sqrt{p}}{\nu},\quad \alpha_{N'}^{e}=\frac{p}{\nu}. \tag{4.11}$$

It is immediately shown that this operator creates a state that has a charge of $Q=-e$ and statistics πk, where $k=(2np+1)[2n(p+1)+1]$ is an *odd* integer.

Thus, in this new basis, the *quasiparticle* operator is

$$\Psi_{qp}=e^{i[(1/p)\,\phi_C-(1/\sqrt{p})\,\phi_N+\phi_{N'}]} \tag{4.12}$$

and the *electron* operator has the form

$$\Psi_e=e^{i[(1/\nu)\phi_C-(\sqrt{p}/\nu)\phi_N+(p/\nu)\phi_{N'}]}. \tag{4.13}$$

It is straightforward to show that if an integer number of *electrons* ΔN_e is added to the bulk of the system, the electron operator is not affected by the twist in the boundary conditions Eq. (4.5) since the exponent shifts by $2\pi s$, where $s=[2n(p+1)+1](2np+1)\Delta N_e$ is an integer.

Finally, we will compute the propagators for the electron and the quasiparticle operators. We will need the propagators of the chiral bosons ϕ_C, ϕ_N, and $\phi_{N'}$. Since the Lagrangians for ϕ_N and $\phi_{N'}$ are identical their propagators are the same. Furthermore the chiral bosons ϕ_N and $\phi_{N'}$ do not propagate (i.e., their velocity is zero).

The propagator of the charged mode ϕ_C, in imaginary time, is

$$\langle\phi_C(x,t)\,\phi_C(0,0)\rangle=-\frac{\nu}{2}\ln\left(1-\frac{z^2}{\epsilon^2}\right)+\frac{\nu}{2}\mathrm{sgn}(t)\ln\left(\frac{\epsilon+iz}{\epsilon-iz}\right), \tag{4.14}$$

where $z=x+iv t$ and ϵ is an ultraviolet cutoff. As $\epsilon\to 0$ we find

$$\langle\phi_C(x,t)\,\phi_C(0,0)\rangle=-\nu\ln\frac{iz}{\epsilon}+i\nu\frac{\pi}{2}\mathrm{sgn}(t). \tag{4.15}$$

Notice that the (regulated) propagator obeys $\langle\phi_C(0,0)\,\phi_C(0,0)\rangle=0$. The same applies to the propagator of the neutral modes discussed below.

Likewise, the propagator (in imaginary time) of the neutral modes ϕ_N and $\phi_{N'}$, in the same limit $\epsilon\to 0$, becomes

$$\langle\phi_N(x,t)\,\phi_N(0,0)\rangle=\langle\phi_{N'}(x,t)\,\phi_{N'}(0,0)\rangle=-i\frac{\pi}{2}\mathrm{sgn}(t). \tag{4.16}$$

Using the propagators of Eqs. (4.14) and (4.16), we find that the *electron* propagator is given by

$$\begin{aligned}\langle\Psi_e^{\dagger}(x,t)\Psi_e(0,0)\rangle&=\exp\left[\frac{1}{\nu^2}\langle\phi_C(x,t)\,\phi_C(0,0)\rangle+\frac{p}{\nu^2}\langle\phi_N(x,t)\,\phi_N(0,0)\rangle\right.\\&\quad\left.+\frac{p^2}{\nu^2}\langle\phi_{N'}(x,t)\,\phi_{N'}(0,0)\rangle\right]\\&\equiv\frac{1}{|t|^{1/\nu}}e^{[i(\pi/2)[1/\nu-(p+p^2)/\nu^2]\mathrm{sgn}(t)}=\frac{1}{|t|^{1/\nu}}e^{-i(\pi s/2)\mathrm{sgn}(t)},\end{aligned} \tag{4.17}$$

where we have analytically continued to real time t and taken the limit $x\to 0$.

Equation (4.17) shows clearly that the electron operator of the Jain states with filling fraction ν has scaling dimension $(2\nu)^{-1}$. This result implies that the tunneling density of states for electrons at this edge obeys the law $|\omega|^{(1-\nu)/\nu}$. Notice that the nonpropagating modes are responsible for the fermionic statistics of the electron.

Equation (4.17) agrees with the work by Lee and Wen,[35] who have found independently the same result as this paper was being written. However, in Ref. 35 the neutral modes have a very different physical origin and they result from considering the role of the microscopic structure of the edge and edge reconstruction. Instead, in the approach that we present in this paper the neutral modes originate from global topological consistency requirements for flux attachement and are a remnant of the topological invariance of the Chern-Simons theory. Our result also agrees with the recent work of Zülucke and MacDonald[36] who calculated the electron tunneling spectral function using a variational approach. These authors found that although they could account for the correct spectral function, their electron operator did not obey Fermi statistics, in contrast with the result of Eq. (4.17).

The construction of the electron operator that we just derived also has the following interesting interpretation. The electron operator, as given by Eq. (4.13), is a product of the operator $\exp[(i/\nu)\phi_C]$, which carries the charge, and the operators $\exp[-(i/\sqrt{p})\phi_N]$ and $\exp(i\phi_{N'})$, which combined fix the statistics. In fact this is the only role of these latter operators since the fields Φ_N and $\phi_{N'}$ do not propagate. Essen-

tially, the combined operator $\exp[-(1/\sqrt{p})\phi_N + i\phi_{N'}]$ must be regarded as an effective Klein factor. In particular, it also means that in a local probe of the edge, such as in electron tunneling, only the charge mode plays a dynamical role. We will discuss this problem elsewhere.

Finally, a similar calculation yields the *quasiparticle* propagator, in imaginary time, which is found to be given by the following expression:

$$\begin{aligned}\langle \Psi_{qp}(x,t)\Psi_{qp}(0,0)\rangle &= \exp\left[\frac{1}{p^2}\langle \phi_C(x,t)\phi_C(0,0)\rangle + \frac{1}{p}\langle \phi_N(x,t)\phi_N(0,0)\rangle \right. \\ &\quad \left. + \langle \phi_{N'}(x,t)\phi_{N'}(0,0)\rangle\right] \\ &\equiv \frac{1}{|t|^{\nu/p^2}} e^{i(\pi/2)(\nu/p^2 - 1/p - 1)\mathrm{sgn}(t)} = \frac{1}{|t|^{\nu/p^2}} e^{-i(\theta_{qp}/2)\mathrm{sgn}(t)}\end{aligned} \tag{4.18}$$

again in the limit $x \to 0$. Equation (4.18) shows that the quasiparticle operator has scaling dimension $\nu/(2p^2)$ and the correct statistics.

V. CONCLUSIONS

In this paper, we have derived an effective Chern-Simons theory for the Jain states in a finite geometry that is consistent with global gauge invariance. We showed that this theory can be cast into a *K*-matrix form but with a different and much simpler structure than the usual one. We found that this structure requires only a small number of gauge fields and their number, that is the rank of the *K* matrix, is fixed. We used this effective theory on a closed surface to find a universal minimal structure of the theory on an open surface and determine the structure of the edge states, for all the states in the Jain sequences. We found that, in all cases, there is one and only one propagating mode and hence only one mode that carries electric charge and energy. The role of the remaining (two) modes is to fix the statistics of the excitations. We constructed the electron and quasiparticle operators for these states, which turn out to be uniquely determined and carry the correct charge and statistics.

We calculated the propagators of the excitations at the edge and, in particular, found that the propagator for the electron (which is a fermion as it should be) behaves as a function of time like $|t|^{-1/\nu}$ for all the Jain states, and a tunneling density of states that, as a function of frequency, behaves like $|\omega|^{(1-\nu)/\nu}$. In a separate publication we generalize these results to other quantum Hall effects.[37]

These results in essence agree with the very recent work of Lee and Wen.[35] In particular, they also find only one propagating mode, which carries the charge current in addition to a nonpropagating mode that fixes the statistics. However, the physical origin of this latter mode appears to be quite different from the ones we find here. In the work of Lee and Wen, the nonpropagating mode is one that survived after much of the edge structure of the *K*-matrix theory has been integrated out and this mode does not propagate in the sense that their velocity is much smaller that the velocity of the charge mode. In contrast, in the structure that we find here the nonpropagating modes have a topological origin and that is why they do not propagate (or contribute to the specific heat of the system). Zülicke and MacDonald[36] have also found recently the same result for the tunneling density of states, although the electron operator they use does not have the correct Fermi statistics. This result was actually anticipated by Wen[12] and by Kane and Fisher,[38] who noted that, at the level of the effective theory of the edge states, even though this result would follow if only the charge mode is kept, the statistics of this electron operator is fermionic only for Laughlin states.

The universal structure of this effective theory has a number of potentially important implications. Recent experiments by Grayson and co-workers,[39] on tunneling of electrons into the edges of FQH states have shown that, while the large voltage conductance of these devices does exhibit the FQH effect, the low-voltage scaling exponent fails to show any signature of the existence of plateaus in the bulk, as it is required by the theory of edge states. In fact, even after taking into account the nonlocal effects of charge redistribution due to Coulomb interactions near the edges, as in the work of Shytov, Levitov, and Halperin,[40] the theory still predicts a nontrivial structure in the dependence of the scaling exponent with the magnetic field, which does not agree with the experiment. However, to explain these experiments is roughly equivalent to have a description of the transition between plateaus, but now seen from the edge. It is tempting to speculate that the results of this paper may be used to develop a description of the edge tunneling problem with a linear behavior of the tunnel exponent on the filling factor, precisely because the structure of both bulk and edge does not change dramatically from one Jain state to the next. For such a description to work, it is necessary to assume that no edge reconstruction has taken place. Or, rather, that the only effects of edge reconstruction are equivalent to the transition between plateaus. From this point of view, it is reasonable to expect that such a generalization of this theory is likely to give a smooth dependence of the tunneling density of states with the filling fraction for a continuous range of magnetic fields. Such a description does not exist yet. Finally, it is also interesting to comment that it may be necessary to reexamine under this light the arguments that led to the phase diagram of Kivelson, Lee, and Zhang,[41] since the selection rules for the transitions between plateaus are superficially related, to an extent, to the number of edge states of nearby plateaus.[42]

ACKNOWLEDGMENTS

We thank C. Chamon, J. Jain, S. Kivelson, and C. Nayak for stimulating discussions, and particularly Dung Hai Lee for an early communication of his work with Xiao-GangWen. This work was begun during a visit of E.F. to Universidad de La Plata, Argentina, and to Instituto Balseiro, Bariloche, Argentina. E.F. is very grateful to Fidel Schaposnik for his hospitality in La Plata, and to Andrés García and

Manuel Fuentes for their hospitality in Bariloche. E.F. is a participant at the ITP Program on ''Disorder and Interactions in Quantum Hall and Mesoscopic Systems,'' and thanks D. Gross, Director of Institute for Theoretical Physics of the University of California Santa Barbara, for his kind hospitality. This work was supported in part by NSF Grant No. NSF DMR98-17941 at UIUC, and NSF Grant No. PHY94-07194 at ITP-UCSB (EF), and by CONICET (AL).

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[42] We are grateful to J. Jain for this suggestion.