Statistical properties of a localization-delocalization transition in one dimension

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(Received 11 February 1999)

We study a one-dimensional model of disordered electrons (also relevant for random spin chains), which exhibits a delocalization transition at half-filling. Exact probability distribution functions for the Wigner time and transmission coefficient are calculated. We identify and distinguish those features of probability densities that are due to rare trapping configurations of the random potential from those which are due to the proximity to the delocalization transition. [S0163-1829(99)01220-5]

The Anderson transition in dimensions $D<3$ has recently attracted a renewed interest in relation to such systems as random antiferromagnetic spin chains[1] and high mobility Si metal-oxide-semiconductor field-effect transistors (MOSFET's).[2] Experiments on both systems could not be accounted for in the standard scaling theory of localization.[3]

The simplest disordered model known to exhibit metallic behavior, in spite of being one dimensional, is the random-hopping model

$$H_{rh}=\sum_n t_n(c_n^\dagger c_{n+1}+\text{H.c.}), \tag{1}$$

where $t_n>0$ are random variables, with n-independent average, $\langle t_n\rangle=t$, and c_n annihilates a spinless fermion at site n. This one-dimensional (1D) model has a single delocalized state at the middle of the band $\epsilon=0$ and is an interesting and simple example of delocalization in $D<3$. Moreover, this model has many common features with a wide class of random spin chains, such as the spin-1/2 random Heisenberg chain $H=\Sigma_n J_n \mathbf{S}_n\cdot\mathbf{S}_{n+1}$, where $J_n>0$ are randomly distributed. [The XX version of the latter model is, in fact, exactly equivalent to Eq. (1), upon the Jordan-Wigner transformation.]

For the random-hopping model, a great deal is known about such self-averaging quantities as the total density of states. More recently, some of the correlation functions have also been calculated. However, the behavior of probability distributions in the proximity of the delocalization transition is virtually unexplored. It is the purpose of this work to address this issue.

In the continuum limit, Eq. (1) becomes what is known as the random-mass Dirac model (see, e.g., Refs. 4 and 5):

$$\mathcal{H}=-i[R^\dagger\partial_x R-L^\dagger\partial_x L]-im(x)[R^\dagger L-L^\dagger R], \tag{2}$$

where R and L are the chiral components of the electron field operator. The derivation of the continuum limit assumes weak disorder such that $t_n=t+\delta t_n$ (the Fermi velocity, v_F, associated with t is set to 1). It is the staggered component of the random hopping, $\delta t_n\to(-1)^n m(x)$, which enters into the continuum theory. In field-theoretic language, this corresponds to a random mass.

It was found by Dyson in 1953, (Ref. 6) that the average electron density of states for a model equivalent to Eq. (1) diverges at the middle of the band: $\rho(\epsilon)\sim 1/(\epsilon|\ln\epsilon|^3)$. By the Thouless relation,[7] such a density of states implies a divergent localization length $\lambda_\epsilon\sim|\ln\epsilon|$. The criticality of the model at half-filling was established by Gogolin and Mel'nikov.[8] In particular, they found that model (2) has a finite conductivity in contrast with the Mott law in the standard localized regime. Calculations of Ref. 9 first indicated that it is not the Thouless length λ_ϵ, but rather the length $l_\epsilon\sim\ln^2\epsilon$, which governs the correlation functions. The role of the length l_ϵ was later clarified by means of the real-space renormalization-group method by Fisher.[10] The crossover for the electron Green function between the spatial regions $x\ll l_\epsilon$ (critical regime) and $l_\epsilon\ll x$ (localized regime) was discussed in recent publications[4,5] by means of the Berezinskii technique and the Efetov supersymmetry method, respectively.

It seems natural to undertake the next step and investigate the probability distribution functions for the above systems. To this end, we adopt a simple mathematical technique, based on the recent observation by Shelton and Tsvelik[11] that, at $\epsilon=0$, the random-mass Dirac model can be formulated as a one-particle quantum-mechanical problem. Indeed, if we introduce the combinations $\chi_\pm=(R\mp L)/\sqrt{2}$ of the chiral components of the electron field, then the Dirac equation becomes

$$[\partial_x\mp m(x)]\chi_\pm=i\epsilon\chi_\mp .$$

This equation decouples at $\epsilon=0$, thus admitting the (unnormalized) solutions of the form

$$\chi_\pm(x)\sim e^{\pm V(x)},\quad V(x)=\int_0^x dy\, m(y). \tag{3}$$

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As is customary in the literature, we assume that the random-mass $m(x)$ is δ correlated in the real space and Gaussian: $P_{m_0}[m(x)] \sim \exp\{-[m(x)-m_0]^2/2g\}$. Here g characterizes the disorder strength and the mass has a mean value m_0. Thus, solution (3) for the wave-functions is nothing but an exponential of a random walk with a drift term $m_0 x$.

Wave function (3) allows us to investigate the distribution of several physical quantities in a relatively simple manner. We start by considering the so-called relaxation time, which was introduced in the context of the scattering theory by Wigner,[12] hence also known as the Wigner time. Physically, this is the time spent by a wave packet inside a scattering region, and it can be formally defined as the momentum derivative of the scattering phase shift. Let the effect of the disorder be confined to the segment $[0,L]$, which is the scattering region in our problem. For the sake of concreteness, we assume that the electrons cannot leave the sample on the left-hand side ($x<0$), so that they are scattered off the segment $[0,L]$ on the right. We thus impose a vanishing boundary condition on the left, $\chi_-(0)=0$ [this condition corresponds to suppressing a site in the lattice formulation, Eq. (1)]. The boundary condition on the right is simply the continuity of the Dirac wave function at $x=L$. In this case, the calculation of the Wigner time involves the scattering phase picked up by the incident left-moving wave in the process of scattering from the disordered segment, so that ($x>L$),

$$\tau_\epsilon = d\theta/d\epsilon, \quad \theta(\epsilon) = -i \ln[e^{-2i\epsilon x} R(x)/L(x)]. \tag{4}$$

We are mainly interested in the relaxation time at half-filling, i.e., τ_0. In order to find this, we perturb the Dirac equation in ϵ around the $\epsilon=0$ solution (3). Performing this simple calculation we found the following exact expression for the Wigner time as a functional of the disorder[13]

$$\tau_0[V] = 2\int_0^L dx\{e^{2[V(x)-V(L)]}-1\}. \tag{5}$$

In accordance with, Ref. 12 the relaxation time is related to the total charge $Q_0=\int_0^L dx(|R_0|^2+|L_0|^2)$, and therefore $\tau_0 = Q_0-2L$.

As such, expression (5) does not supply much information. Indeed, the physical content of the problem is revealed by the probability density of the relaxation time $\mathcal{P}[\tau_0] = \langle \delta(\tau_0-\tau_0[V])\rangle$. We observe that the Laplace transform of this density, $\hat{P}[\lambda]=\int_0^\infty d\tau_0 e^{-\lambda\tau_0}\mathcal{P}[\tau_0]$, can be represented as a path integral

$$\hat{P}[\lambda] \sim \int DV(x) P_{m_0}[V] e^{-\lambda\tau_0[V]}. \tag{6}$$

In the above formula, the integration is taken along all paths starting at $V(x=0)$ and ending at $V(x=L)=V_0$. By construction, see Eq. (3), the starting point is $V(0)=0$, while the ending point V_0 is arbitrary, so the path integral in Eq. (6) involves an additional integration over all possible end points V_0. Fixing the normalization $\hat{P}[0]=1$, and performing a convenient shift of the integration variables, we ultimately obtain

$$\hat{P}[\lambda] = \int_{-\infty}^{\infty} dV_0 K_\lambda(V_0,0;L). \tag{7}$$

(For the time being we have set $m_0=0$, so as to be at the criticality.) Here $K_\lambda(V_1,V_2;x)$ is the imaginary-time propagator for a quantum-mechanical particle characterized by the action

$$S_\lambda[V] = \int_0^L dx\left[\frac{1}{2g}(\partial_x V)^2 + 2\lambda(e^{2V}-1)\right]. \tag{8}$$

The system described by this action is known as Liouville quantum mechanics.[14,15] The Schrödinger equation, corresponding to the action (8), has the following (normalized) solution:[14]

$$\psi_\gamma(V) = \sqrt{2\gamma \sinh \pi\gamma/\pi^2}\, K_{i\gamma}(2\sqrt{\lambda/g}e^V) \tag{9}$$

with the energy $-2\lambda+g\gamma^2/2$ ($K_{i\gamma}$ is the MacDonald function). Constructing the propagator in a standard manner by making use of the complete set (9), integrating over V_0, and performing the inverse Laplace transform, we obtain the exact Wigner time probability density

$$\mathcal{P}[\tau_0] = \frac{2^{1/2} e^{\pi^2/8gL}}{\pi g L^{1/2} Q_0^{3/2}} \int_0^\infty dt \cosh t \cos\left(\frac{\pi t}{2gL}\right) \times \exp[-\cosh^2 t/(gQ_0) - t^2/(2gL)]. \tag{10}$$

It is convenient to represent the result in terms of the positively definite quantity (total charge) $Q_0=\tau_0+2L$. According to the general theory of Ref. 12, the delay time can be negative (for repulsive potentials) but there is a lower bound, which is $-2L$ in our case.

We note the following interesting limiting cases of the formula (10).

Consider first the probability of long time delays, $\tau_0\to\infty$ (and therefore $Q_0\to\infty$). The t integral in Eq. (10) is convergent for all Q_0. It is therefore tempting to expand the exponential in the integrand in powers of $1/Q_0$. It is easy to check, however, that all the coefficients of such an expansion identically vanish. It follows that Eq. (10) has an essential singularity at $\tau_0=\infty$. The nature of this singularity can be determined by first neglecting the $1/Q_0$ term in the exponential and then simulating its effect by cutting off the t integration at large times $t_0\sim(1/2)\ln(g\tau_0)$, when the term becomes of the order of unity. Then, within the leading logarithmic accuracy, we obtain

$$\mathcal{P}[\tau_0] \sim \exp[-\ln^2(g\tau_0)/8gL]. \tag{11}$$

This formula is valid when the factor in the exponential is large, i.e., $\ln^2\tau_0 \gg L$. As a simple application to random spin chains, we consider long-time relaxation of the magnetization $M(t)$ inside a finite segment of length L. Due to Eq. (11) we find that $M(t)\sim\exp[-\ln^2(t)/8gL]$: a very slow decay.

Since the applicability of Eq. (11) involves the system size L, the probability distribution for large samples ought to be different. Indeed, for a fixed Q_0, we obtain

$$\mathcal{P}[\tau_0]|_{L\to\infty} \simeq \frac{1}{\sqrt{2\pi g L Q_0}} e^{-1/gQ_0}. \tag{12}$$

The next quantity of interest is the transmission coefficient at $\epsilon=0$, which is also proportional to the Landauer conductance at half-filling. In order to have a finite transmission

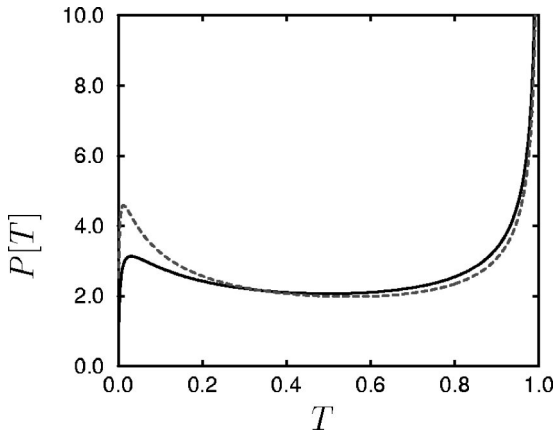

FIG. 1. Distribution of the transmission coefficient with $gL=1.25$ and average mass zero (solid line) and $m_0L=0.5$ (dashed line).

coefficient T, let us open our sample on the left-hand side, so that the electrons can now leave it at $x=0$ (where the boundary condition thus becomes the same as at $x=L$). Upon matching the wave functions in the usual way, one finds $T[V]=1/\cosh^2 V(L)$. Therefore the probability distribution $P[T]$ can be found in an elementary way without using the Liouville mechanics. We obtain

$$P[T]=\frac{\sqrt{1/2\pi gL}}{T\sqrt{1-T}}\exp\left\{-\frac{1}{2gL}[\cosh^{-1}(1/\sqrt{T})]^2\right\}, \quad (13)$$

where $0\leqslant T\leqslant 1$. This function [or rather the distribution function for the related quantity $V(L)$] has, to our knowledge, been first obtained by Mathur.[16] This is an intriguing distribution function, plotted in Fig. 1. It has the following properties. (i) The probability of a small transmission is

$$P[T]\sim \exp[-\ln^2(1/T)/8gL]. \quad (14)$$

(ii) The function $P[T]$ has a low-transmission peak at $T_0\sim e^{-4gL}$. (iii) There is an integrable divergence close to the perfect transmission ($T=1$),

$$P[T]\simeq \sqrt{8/\pi gL(1-T)}. \quad (15)$$

(iv) The mean transmission coefficient is given by

$$\langle T\rangle \simeq \sqrt{2/\pi gL}. \quad (16)$$

This formula is asymptotically exact as $L\to\infty$ (we also verified this result by an independent calculation using the Berezinskii technique[17]). This result, pointed out by Mathur,[16] is surprising. Since $T\sim\sigma/L$, σ being the dc conductivity, the $1/\sqrt{L}$ behavior of the average transmission coefficient suggests that the system is even more metallic than what one would expect in a hypothetical case of a weakly disordered 1D metal (hypothetical because any weak disorder is supposed to lead to localization). It is worth noting that the transmission coefficient is not a good scaling variable in the limit $L\to\infty$. So, from Eq. (13), one finds that the variance of the transmission coefficient $\delta^2=\langle(T-\langle T\rangle)^2\rangle\sim 1/\sqrt{L}$, implying that the width of the distribution, normalized to the mean transmission $\delta/\langle T\rangle\sim L^{1/4}$, diverges when $L\to\infty$. On the other hand, the logarithm of the transmission coefficient is a good scaling variable in the above sense, with average

$$\left\langle \ln\left(\frac{1}{T}\right)\right\rangle = \alpha\sqrt{gL},$$

α being a positive numerical coefficient. This conclusion is generic in 1D disordered systems, as was shown in a seminal paper by Anderson *et al.* in 1980.[18] They introduced the *scale conductance*, $T_{\rm typ}=e^{\langle \ln T\rangle}$, instead of the average conductance. However, as a result of the criticality of our model and unlike the case discussed in Ref. 18, we find that $T_{\rm typ}=e^{-\alpha\sqrt{gL}}$ behaves quite differently from the average transmission, and both are different from the peak in the probability distribution $T_0\sim e^{-4gL}$.

There is a difference between our $1/\sqrt{L}$ result for the Landauer conductance and the finite conductivity found for an infinite system in Ref. 8. A possible physical explanation is that this difference is due to resonant scattering processes, which enhance the probability of near-perfect transmission in finite samples but are absent in infinite systems (this is characteristic for a critical system; otherwise, the $T\to 1$ divergence of $P[T]$ is exponentially suppressed in the sample length). However, this question requires further study.

Both calculations for $\mathcal{P}[\tau_0]$ and $P[T]$ can be generalized to the off-critical case when m_0 is nonzero. The details will be given in an extended article.[17] Here we only quote the results. Formulas (11) and (14) are not affected. The power law in the denominator of Eq. (12) changes to $1/Q_0^{1+m_0/g}$. Both formulas (15) and (16) acquire a suppressing factor, exponentially small in the parameter m_0L.

Note that the probability of long time delays, Eq. (11), as well as the probability of a small transmission, Eq. (14), follows the so-called log-normal law. To our knowledge, the log-normal tails of the distribution functions in one-dimensional disordered systems were first obtained in Ref. 18 by means of a scaling argument. Their existence was rigorously established by Mel'nikov via the Bereziskii technique.[19] In $2+\epsilon$ dimensions, the log-normal tails were found in Ref. 20, which was thought to be a signature of the Anderson transition. The discussion of the log-normal tails was recently revived by Muzykantskii and Khmelnitskii[21] (see also Ref. 22) who gave a simple derivation based on a specific saddle-point approximation in the supersymmetric σ model. (For a recent collection of results on the log-normal distributions see Ref. 23.)

We have shown that the random-mass Dirac model does possess log-normal tails at the criticality. Moreover, these tails are unaffected when one moves away from the criticality. This is consistent with the interpretations[21,23] that these tails are due to the so-called ''anomalously localized'' electronic states, which occur in rare trapping disorder configurations. Indeed our result (11) can be understood in terms of the ''optimal fluctuation'' concept, discussed in this context in Ref. 21. The optimal fluctuation in our case corresponds to having a constant mass m_0 within the sample. This would accumulate charge $Q_0\propto\exp(2m_0L)$. The probability to have such a potential and therefore such a charge is $\sim\exp(-m_0^2L/2g)\sim\exp(-\ln^2 Q_0/8gL)$.

So, does it follow that the proximity to the delocalization transition (criticality) plays no role? To clarify this point, we

first notice that the log-normal tails are only present in a finite system. (These tails do not follow from and are not directly related to the so-called multi-critical exponents appearing in the wave functions' statistics for the infinite system.[5,11]) The Wigner time distribution function takes a different form in the limit of a large system, Eq. (12). This can be interpreted as a ''limiting'' (''equilibrium'') distribution in terms of the Fokker-Planck equation approach,[13] and it reveals no trace of the log-normal behavior. It is this distribution that bears the signature of the criticality: it ceases to be normalizable for $m_0=0$ thus requiring a long-time cutoff. Similarly, the log-normal tail appears at low T in the transmission coefficient distribution function, while the criticality shows up near the perfect transmission. Indeed, the divergence of $P[T\rightarrow 1]$ is always in place (due to resonant scattering processes), but it is exponentially suppressed in the sample length, unless the system is critical. As a result, at the criticality, the mean transmission coefficient is not any more exponentially small but is given by the power-law formula (16). (Note, though, that the peak of $P[T]$ still exists at the criticality.) It follows that the disorder configurations leading to the log-normal tails and to the delocalization phenomenon act independently. They affect different domains of the parameter space of the problem and show up in the distinct limits for the probability distribution functions.

We are thankful to A.M. Tsvelik, D.E. Khmelnitskii, and B.A. Muzykantskii for interesting conversations. M.F. acknowledges support by INFM, Project No. HTSC. M.S. was supported by the Gottlieb Daimler-und Karl Benz-Stiftung and by the EPSRC, UK.

[1] M. Fabrizio and R. Melin, Phys. Rev. Lett. **78**, 3382 (1997). For an experimental review on spin-Peierls systems, see J. P. Boucher and L. P. Regnault, J. Phys. I **6**, 1936 (1996). For doping effects in spin ladders, see M. Tanako, Nature (London) **377**, 41 (1995); M. Azuma *et al.*, Phys. Rev. B **55**, R8658 (1997).

[2] S. V. Kravchenko, D. Simonian, M. P. Sarachik, W. Mason, and J. E. Furneaux, Phys. Rev. Lett. **77**, 4938 (1996).

[3] E. Abrahams, P. W. Andesron, D. Licciardello, and T. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[4] M. Steiner, M. Fabrizio, and A. O. Gogolin, Phys. Rev. B **57**, 8290 (1998).

[5] L. Balents and M. P. A. Fisher, Phys. Rev. B **56**, 12 970 (1997).

[6] F. J. Dyson, Phys. Rev. **92**, 1331 (1953).

[7] D. J. Thouless, J. Phys. C **5**, 77 (1972).

[8] A. A. Gogolin and V. I. Mel'nikov, Zh. Éksp. Teor. Fiz. **73**, 706 (1977) [Sov. Phys. JETP **46**, 369 (1977)]. See also the review paper A. A. Gogolin, Phys. Rep. **86**, 1 (1982).

[9] J. P. Bouchaud, A. Comtet, A. George, and P. Le Doussal, Ann. Phys. (N.Y.) **201**, 285 (1990).

[10] D. S. Fisher, Phys. Rev. B **50**, 3799 (1994).

[11] D. G. Shelton and A. M. Tsvelik, Phys. Rev. B **57**, 14 242 (1998).

[12] E. Wigner, Phys. Rev. **98**, 145 (1955).

[13] A similar expression was found, using a different method, in the recent paper by A. Comtet and C. Texier, J. Phys. A **30**, 8017 (1997).

[14] E. D'Hoker and R. Jackiw, Phys. Rev. D **26**, 3517 (1982).

[15] We note that, apart from physics literature, such functionals are also of current interest in mathematical literature, see A. Comtet, C. Monthus, and M. Yor, J. Appl. Probab. **35**, 255 (1998).

[16] H. Mathur, Phys. Rev. B **56**, 15 794 (1997).

[17] M. Steiner, M. Fabrizio, and A. O. Gogolin (unpublished).

[18] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980).

[19] V. I. Mel'nikov, Pis'ma Zh. Éksp. Theor. Fiz. **32**, 244 (1980) [JETP Lett. **32**, 225 (1980)].

[20] C. Castellani, C. Di Castro, and L. Peliti, J. Phys. A **19**, L1099 (1986); B. L. Altsuler, V. E. Kravtsov, and I. V. Lerner, Pis'ma Zh. Éksp. Theor. Fiz. **45**, 160 (1987) [JETP Lett. **45**, 199 (1987)]. See also J. S. Caux, I. Kogan, A. Lewis, and A. M. Tsvelik, Nucl. Phys. B **489**, 469 (1997).

[21] B. A. Muzykantskii and D. E. Khmelnitskii, Phys. Rev. B **51**, 5480 (1995); Phys. Rep. **288**, 259 (1997).

[22] V. I. Fal'ko and K. B. Efetov, Phys. Rev. B **52**, 17 413 (1995).

[23] A. D. Mirlin, J. Math. Phys. **38**, 1888 (1997).