From microscopic theory to Boltzmann kinetic equation: Application to vortex dynamics

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We show how to lift the problem of calculating the force acting on a topological defect in a superfluid from the microscopic to the semiclassical level: Starting from the microscopic kinetic equations for a clean superconductor, we derive a Boltzmann equation for the quasiparticle distribution function in and around the defect. The velocity $\dot{q}$ and force $\dot{p}$ appearing in this Boltzmann equation are given through the Hamiltonian equations $\dot{q}=\partial_p E_n(p,q)$ and $\dot{p}=-\partial_q E_n(p,q)$, where $E_n(p,q)$ denotes the (nth branch in the) spectrum of the quasiparticles in the vicinity of the defect. Second, we reformulate the microscopic expression for the force acting on the defect in terms of the total momentum transfer of the quasiparticles from the heat bath to the vortex core. We illustrate our result with an application to vortices in *s*-wave superconductors, where we derive the vortex equation of motion and identify the Magnus, Hall, and dissipative forces. [S0163-1829(99)11321-3]

I. INTRODUCTION

The microscopic nonstationary theory of superconductivity based on the Green function technique in the real-time representation[1,2] has established itself as a powerful method to describe the vortex dynamics both in dirty (for a review, see Ref. 3) and in clean superconductors,[4–6] as well as in other superfluid Fermi systems.[7,8] However, a practical disadvantage of the method is its mathematical complexity, which tends to hide the physical picture of the phenomenon. For a clean system, where the excitation spectrum is well defined, an alternative way to deal with dynamical processes is based on the semiclassical (or quasiclassical) Boltzmann kinetic equation. For normal metals, the equivalence of the quasiclassical Green function approach to the Boltzmann equation has been demonstrated by Keldysh.[9] The quasiclassical approximation usually applies well to superconductors since, typically, the coherence length ξ is much larger than the quasiparticle wave length p_F^{-1}. Generalized sets of kinetic equations have been derived for dirty (for a review, see Ref. 3) and clean superconductors (see, for example, Refs. 10 and 11). In the review[10] (see also references therein), the kinetic equation has been derived under the assumption of a nearly constant magnitude of the order parameter Δ and of the superflow velocity $\mathbf{v}_s$ (more precisely, their gradients have been assumed to be slow such that $k\xi \ll 1$, where k is the characteristic wave vector of the variations in $|\Delta|$ and $\mathbf{v}_s$). It has been shown[10] that this kinetic equation can be written in the form of a Boltzmann transport equation, and some applications of this method have been considered. Unfortunately, the approach of Ref. 10 could not be applied to the dynamics of vortices, because the basic assumption of slow spatial variations is not justified near the vortex core. A more general scheme of deriving the kinetic equations from the microscopic theory for clean superconductors has been developed in Ref. 11. Using the quasiclassical approximation in the small parameter $(p_F\xi)^{-1}\ll 1$, the equations for the distribution function f have been obtained without assuming the spatial variations of $|\Delta|$ and $\mathbf{v}_s$ to be small; however, the resulting equations did not have the canonical form

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \mathbf{q}}\cdot\frac{\partial E_n}{\partial \mathbf{p}}-\frac{\partial E_n}{\partial \mathbf{q}}\cdot\frac{\partial f}{\partial \mathbf{p}}=\left(\frac{\partial f}{\partial t}\right)_{\rm coll}. \tag{1}$$

Here, $(\partial f/\partial t)_{\rm coll}$ is the collision integral and $E_n(\mathbf{q},\mathbf{p})$ is the quasiclassical excitation spectrum, characterized by the canonically conjugated generalized ''coordinate'' $\mathbf{q}$ and ''momentum'' $\mathbf{p}$ of the excitation (n denotes the set of other quantum numbers). It is this kinetic equation in canonical form which was successfully applied by Stone[12] to the vortex dynamics in clean superconductors and which produced results consistent with those of a whole-scaled Green function calculation.

In this paper we present a microscopic derivation of Eq. (1); i.e., we demonstrate that the kinetic equations for the generalized distribution function as derived from the quasiclassical Green function version of the microscopic nonstationary theory can be further transformed into the simple and physically transparent canonical form of Eq. (1). We restrict ourselves to the particular example of vortex dynamics; the calculation can be easily generalized to include the dynamics of other topological defects in superfluid Fermi systems (e.g., see Ref. 13).

In a second step, we derive the corresponding expressions for the force and the torque acting on a moving vortex: we show that, within the quasiclassical approximation, the force $\mathbf{F}$ can be represented as the momentum transfer from the heat bath, via the localized quasiparticle excitations, to the vortex,

$$\mathbf{F}=-\sum_n \int \frac{d^dq\,d^dp}{(2\pi)^d} f(\mathbf{q},\mathbf{p})\frac{\partial \mathbf{p}_n}{\partial t}, \tag{2}$$

where $\partial_t \mathbf{p}_n = -\partial_{\mathbf{q}} E_n(\mathbf{q},\mathbf{p})$. Our analysis thus provides a microscopic verification of the phenomenological approach to the vortex dynamics based on the concept of semiclassical particles obeying the Boltzmann kinetic equation. In this latter approach, the nonequilibrium distribution of the quasiparticles in the vortex core follows from the solution of a Boltzmann equation, where the quasiclassical spectrum E_n plays the role of the Hamiltonian, and the force acting on the vortex results from the force $\partial \mathbf{p}_n/\partial t$ the quasiparticles exert on the vortex core.

We illustrate our result with an application to *s*-wave superconductors, where we make use of the semiclassical formalism to reproduce the force on a moving vortex. In particular, we show how the total force acting from the environment on the vortex can be decomposed into the Magnus, Hall, Iordanskii, and dissipative forces. The consistent derivation of the total force rather than its various elements sheds light on a recent controversy[14] regarding the nature of the transverse force acting on a vortex line in fermionic superconductors and superfluids.

II. KINETIC EQUATION

The time-dependent state of a superconductor can be described by the total Green functions introduced by Keldysh[9] or, equivalently, by Eliashberg,[1] on the basis of the analytical continuation of the Matsubara Green functions onto the real-frequency domain. We define the matrices (in Nambu space)

$$\check{\mathcal{G}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\begin{pmatrix} G & F \\ -F^\dagger & \overline{G}\end{pmatrix}, \tag{3}$$

$$\check{\mathcal{G}}^{R(A)}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\begin{pmatrix} G^{R(A)} & F^{R(A)} \\ -F^{\dagger R(A)} & \overline{G}^{R(A)}\end{pmatrix}, \tag{4}$$

representing the Keldysh and the retarded and advanced Green functions, respectively, and the matrix operators

$$\check{\mathcal{G}}_0^{-1}(t,\check{\mathbf{p}})=\begin{pmatrix} -i\partial_t+\epsilon(\check{\mathbf{p}})-E_F & 0 \\ 0 & i\partial_t+\epsilon(\check{\mathbf{p}})-E_F\end{pmatrix} \tag{5}$$

and

$$\check{\mathcal{H}}(\mathbf{r},t;\check{\mathbf{p}})=\begin{pmatrix} h(\mathbf{r},t;\check{\mathbf{p}}) & -\Delta_{\check{\mathbf{p}}}(\mathbf{r},t) \\ \Delta^*_{\check{\mathbf{p}}}(\mathbf{r},t) & h(\mathbf{r},t;-\check{\mathbf{p}})\end{pmatrix}, \tag{6}$$

where $\epsilon(\check{\mathbf{p}})$ is the quasiparticle spectrum in the normal state, $\Delta_{\check{\mathbf{p}}}(\mathbf{r},t)$ is the order parameter, $\check{\mathbf{p}}=-i\nabla$, and

$$h(\mathbf{r},t;\check{\mathbf{p}})=-\frac{e}{2c}(\check{\mathbf{v}}\cdot\mathbf{A}+\mathbf{A}\cdot\check{\mathbf{v}})+\frac{e^2A^2}{2m^*c^2}+e\varphi.$$

Here $\check{\mathbf{v}}=\partial\epsilon(\check{\mathbf{p}})/\partial\check{\mathbf{p}}$ and m^* is the effective mass. The operators (5) and (6) are combined into

$$\check{\mathcal{G}}^{-1}(\mathbf{r},t;\check{\mathbf{p}})=\check{\mathcal{G}}_0^{-1}(\check{\mathbf{p}})+\check{\mathcal{H}}(\mathbf{r},t;\check{\mathbf{p}}).$$

Interactions with phonons, particle-particle collisions, and scattering by impurities introduce the corresponding self energies; e.g., that for impurity scattering reads

$$\check{\Sigma}(\mathbf{r};t_1,t_2)=\frac{1}{2\pi\nu(0)\tau}\check{\mathcal{G}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}}.$$

The equation of motion for the Keldysh Green functions $\check{\mathcal{G}}$ can be written in the form[1]

$$\check{\mathcal{G}}^{-1}(\mathbf{r}_1,t_1;\check{\mathbf{p}}_1)\check{\mathcal{G}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) -\check{\mathcal{G}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\check{\mathcal{G}}^{-1}(\mathbf{r}_2,t_2;\check{\mathbf{p}}_2)=\check{\mathcal{I}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2), \tag{7}$$

where the collision-integral matrix is

$$\check{\mathcal{I}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\check{\Sigma}^R\check{\mathcal{G}}-\check{\mathcal{G}}\,\check{\Sigma}^A+\check{\Sigma}\check{\mathcal{G}}^A-\check{\mathcal{G}}^R\check{\Sigma}. \tag{8}$$

Here, an integration over internal times and coordinates is implied, e.g.,

$$[\check{\mathcal{G}}_1\check{\mathcal{G}}_2](\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) =\int d^3r'\,dt'\,\check{\mathcal{G}}_1(\mathbf{r}_1,t_1;\mathbf{r}',t')\check{\mathcal{G}}_2(\mathbf{r}',t';\mathbf{r}_2,t_2). \tag{9}$$

The retarded and advanced Green functions satisfy Eq. (7) with the collision integral

$$\check{\mathcal{I}}^{R(A)}=\check{\Sigma}^{R(A)}\check{\mathcal{G}}^{R(A)}-\check{\mathcal{G}}^{R(A)}\check{\Sigma}^{R(A)}. \tag{10}$$

Equation (7) is the starting point for the derivation of the quasiclassical kinetic equations. We briefly review the derivation of the kinetic equations for a clean superconductor; see Ref. 11 for details. We assume that the temporal variations are slow with characteristic frequencies small compared to the order parameter magnitude $\omega\ll\Delta$. In the quasiclassical limit, the relative distances $|\mathbf{r}_1-\mathbf{r}_2|\sim p_F^{-1}$ are shorter than the coherence length ξ determining the scale of spatial variations of the order parameter.

For each Green function $\check{\mathcal{G}}(\mathbf{r}_i,t_i;\mathbf{r}_j,t_j)$ we define the center-of-mass coordinate $\mathbf{r}=(\mathbf{r}_i+\mathbf{r}_j)/2$ and time $t=(t_i+t_j)/2$. The convolutions of the type (9) are expanded in the small coordinate differences $|\mathbf{r}_i-\mathbf{r}_j|$ close to $\mathbf{r}$ as well as in the small time differences $|t_i-t_j|$ near the time t. We go over to Fourier space,

$$\check{\mathcal{G}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\int\frac{d^3p_1d^3p_2d\epsilon_1d\epsilon_2}{(2\pi)^8}\check{\mathcal{G}}_{\epsilon_1,\epsilon_2}(\mathbf{p}_1,\mathbf{p}_2) \times e^{i(\mathbf{p}_1\mathbf{r}_1-\mathbf{p}_2\mathbf{r}_2)-i(\epsilon_1t_1-\epsilon_2t_2)},$$

and introduce the quasiclassical Green functions

$$\check{g}_{\epsilon+\omega/2,\epsilon-\omega/2}(\mathbf{p}_F,\mathbf{k}) \tag{11}$$

$$=\int\frac{d\zeta_p}{\pi i}\check{\mathcal{G}}_{\epsilon+\omega/2,\epsilon-\omega/2}\left(\mathbf{p}+\frac{\mathbf{k}}{2},\mathbf{p}-\frac{\mathbf{k}}{2}\right) \tag{12}$$

integrated over the energy $\zeta_p=\epsilon(\mathbf{p})-E_F$ near the Fermi surface. The matrix

$$\check{g}=\begin{pmatrix} g & f \\ -f^\dagger & \overline{g}\end{pmatrix} \tag{13}$$

is composed of the quasiclassical Green functions in the same way as is the matrix of the usual Green functions (we distinguish between the quasiclassical anomalous Green function f and the distribution function f). Below, we shall also make use of the mixed coordinate-momentum representation

$$\check{g}_{\epsilon}(\mathbf{p}_F;\mathbf{r},t)=\int \frac{d^3k\,d\omega}{(2\pi)^4}\check{g}_{\epsilon+\omega/2,\epsilon-\omega/2}(\mathbf{p}_F,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}.$$

Here, $\check{g}_{\epsilon}(\mathbf{p}_F;\mathbf{r},t)$ depends on the momentum direction $\mathbf{p}_F$ and on the energy ϵ, as well as on the center-of-mass coordinate $\mathbf{r}$ and the time t.

The total quasiclassical Green function factorizes to take the form[3,11]

$$\check{g}_{\epsilon+\omega/2,\epsilon-\omega/2}(\mathbf{p}_F,\mathbf{k})=[\check{g}^R\check{f}-\check{f}\check{g}^A]_{\epsilon+\omega/2,\epsilon-\omega/2}(\mathbf{p}_F,\mathbf{k}), \tag{14}$$

where $\check{f}_{\epsilon}(\mathbf{p}_F;\mathbf{r},t)$ derives from the distribution function $\check{\mathcal{F}}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ as outlined above and the product in Eq. (14) involves the usual integration over internal coordinates. The matrix distribution function $\check{f}_{\epsilon}(\mathbf{p}_F;\mathbf{r},t)$ contains only two independent components and can be written in the form

$$\check{f}_{\epsilon}(\mathbf{p}_F;\mathbf{r},t)=(f^{(0)}_{\epsilon}+f_{1,\epsilon})\check{1}+f_{2,\epsilon}\check{\tau}_3.$$

Here $f^{(0)}_{\epsilon}=\tanh(\epsilon/2T)$ is the equilibrium distribution: $f^{(0)}_{\epsilon}=1-2n_{\epsilon}$, where n_{ϵ} is the Fermi function. The functions $f_{1,\epsilon}(\mathbf{p}_F;\mathbf{r},t)$ and $f_{2,\epsilon}(\mathbf{p}_F;\mathbf{r},t)$ describe deviations from equilibrium; $f_{1,\epsilon}$ is odd while $f_{2,\epsilon}$ is even in the energy ϵ and the momentum direction $\mathbf{p}_F$. The nonequilibrium parts f_1 and f_2 of the distribution function are determined by Eq. (7). Taking its trace, we arrive at one of the equations for two unknown distribution functions. The second equation is obtained by taking the trace of Eq. (7) after multiplication with the matrix $\check{\tau}_3$.

In what follows, we concentrate on clean superconductors. For a momentum-independent order parameter the derivation of the kinetic equations for clean superconductors with a mean free path $l\gg\xi(T)$ has been carried out in Ref. 11 and the equations have been generalized to include the momentum dependence in Ref. 8. In the gauge invariant representation they take the form

$$\left[e(\mathbf{v}_F\cdot\mathbf{E})g_- +\frac{1}{2}\left(f_-\frac{\hat\partial\Delta^*_{\mathbf{p}}}{\partial t}+f^{\dagger}_-\frac{\hat\partial\Delta_{\mathbf{p}}}{\partial t}\right)\right]\frac{\partial f^{(0)}}{\partial\epsilon}+g_-\frac{\partial f_1}{\partial t}$$
$$+(\mathbf{v}_F\cdot\nabla)(g_-f_2)+\frac{1}{2}\left(f_-\frac{\partial\Delta^*_{\mathbf{p}}}{\partial\mathbf{p}}+f^{\dagger}_-\frac{\partial\Delta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\cdot\nabla f_1$$
$$+\left[\frac{e}{c}[\mathbf{v}_F\times\mathbf{H}]g_- -\frac{1}{2}(f_-\hat\nabla\Delta^*_{\mathbf{p}}+f^{\dagger}_-\hat\nabla\Delta_{\mathbf{p}})\right]\cdot\frac{\partial f_1}{\partial\mathbf{p}}=J \tag{15}$$

and

$$g_-(\mathbf{v}_F\cdot\nabla)f_1=0. \tag{16}$$

Here, g_-, f_-, and $f^{\dagger}_-$ are combinations of stationary retarded and advanced quasiclassical Green functions, e.g.,

$$g_-=\frac{1}{2}(g^R_{\epsilon}-g^A_{\epsilon}),\quad f_-=\frac{1}{2}(f^R_{\epsilon}-f^A_{\epsilon}).$$

We define the gauge-invariant operators $\hat\nabla\Delta=[\nabla-(2ie/c)\mathbf{A}]\Delta$ and $\hat\nabla\Delta^*=[\nabla+(2ie/c)\mathbf{A}]\Delta^*$, as well as $\hat\partial\Delta/\partial t=(\partial/\partial t+2ie\varphi)\Delta$ and $\hat\partial\Delta^*/\partial t=(\partial/\partial t-2ie\varphi)\Delta^*$, where φ is the scalar potential. The electric field is $\mathbf{E}=-(1/c)(\partial\mathbf{A}/\partial t)-\nabla\varphi$ and J is the collision integral (see also Ref. 3),

$$J=\int\frac{d\zeta_p}{4\pi}\mathrm{Tr}[\check{\mathcal{I}}-\check{\mathcal{I}}^R(f^{(0)}+f_1)+(f^{(0)}+f_1)\check{\mathcal{I}}^A].$$

Equations (15) and (16) constitute the full set of equations for the functions f_1 and f_2. Equation (15) differs from the kinetic equation derived for dirty superconductors[3] by the presence of the momentum derivatives of the distribution function and of the order parameter. In the clean limit, we can consistently keep the terms with the momentum derivative in the expansion of Eq. (7) as compared to the collision integral (see Ref. 11). Note that this procedure does not work for dirty superconductors, where the collision integral dominates and the momentum derivatives become small in the quasi-classical parameter $(p_F\xi)^{-1}$. Also, in the dirty limit Eq. (16) changes and the function f_1 is no longer constant.

III. TRANSFORMATION INTO THE BOLTZMANN EQUATION

Let us proceed with a few preparatory steps before transforming the kinetic equation (15) into the Boltzmann equation. First, in a clean superconductor, the odd part of the distribution function f_1 is much larger than the even part f_2. Indeed, using Eq. (15), an order of magnitude estimate gives $J\sim -f_1/\tau$ and $f_2\sim[\xi(T)/l]f_1$. Furthermore, using Eq. (16), we find that the distribution function f_1 is constant along the quasiparticle trajectory. Second, quasiparticles with classically localized trajectories have a distribution qualitatively different from that of delocalized quasiparticles. Delocalized quasiparticles can move distances away from the vortex which are much longer than the mean free path and are practically in equilibrium with the heat bath; their contribution to the force on the vortex is small.[5] As the largest contribution to the vortex dynamics arises from localized excitations, we concentrate on the excitations localized in the vortex core (more generally, in the potential well of the order parameter landscape).

In what follows we concentrate on linear topological defects such as vortices; hence all the functions depend on the coordinates in the plane perpendicular to the vortex axis. The z axis is chosen parallel to the axis of the vortex, with the positive direction along the vortex circulation $\hat{\mathbf{z}}=\mathrm{sgn}(e)\mathbf{H}/H$. We introduce the distance $s=\rho\cos(\phi-\alpha)$ along the quasiparticle trajectory, as well as the impact parameter $b=\rho\sin(\phi-\alpha)$, where ρ and ϕ are the radial distance and the azimuthal angle in the cylindrical frame, respectively, and α is the angle between $\mathbf{v}_\perp$ and the x axis; see Fig. 1. In this representation, the quasiclassical Green function is specified by the momentum projection on the vortex axis $p_z=p_F\cos\theta$, the momentum direction α in the plane perpendicular to the vortex axis, and the impact parameter b,

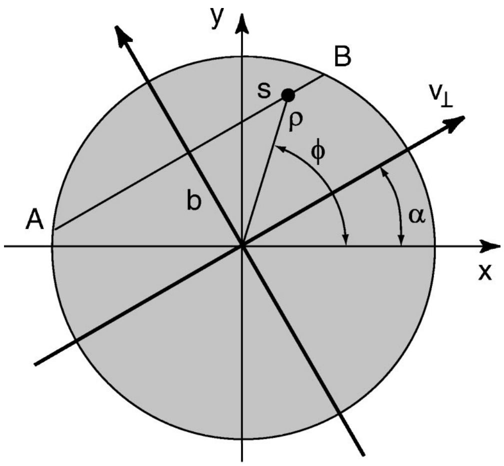

FIG. 1. The coordinate frame used to describe the quasiparticles moving in the vortex core (shaded region) along the localized trajectory AB. Given the velocity $\mathbf{v}_\perp$, the particle position $\mathbf{r}=(\rho,\phi)$ in polar coordinates can alternatively be specified by the distance b of the trajectory from the vortex axis (impact parameter) and the distance s along the trajectory.

which is related to the angular momentum $\mu=\mp b p_\perp$, with $p_\perp$ the momentum projection on the plane perpendicular to the vortex axis. The upper sign is for particles with $\mathbf{p}_\perp$ parallel to $\mathbf{v}_\perp$ while the lower sign is for holes with $\mathbf{p}_\perp$ antiparallel to $\mathbf{v}_\perp$ (the terms ''particles'' and ''holes'' refer to the normal-state spectrum). Up to corrections of order $(p_F\xi)^{-1}$ we can assume straight trajectories for the quasiparticles and thus the angular momentum μ is a conserved quantity even for a nonaxisymmetric vortex. The Green functions then can be written as sums over the energy spectrum of bound states,[15,16]

$$\check{g}_-=\sum_n \check{g}_n(\alpha,b,p_z;s)\,\delta[\epsilon-E_n(\alpha,b,p_z)],$$

and, similarly, the collision integral can be presented as a sum over the quasiclassical states,

$$J=\sum_n J_n(\alpha,b,p_z;s)\,\delta(\epsilon-E_n).$$

Next, we transform the operators in Eq. (15). The momentum derivative $\partial\mathbf{p}$ in Eq. (15) is taken at a constant position vector $\mathbf{r}=(\rho,\phi)$ with respect to variations in the momentum direction, with the magnitude of the momentum being fixed at the Fermi surface. The planar projection can be written as

$$\left(\frac{\partial}{\partial\mathbf{p}_\perp}\right)_{\mathbf{r}}=\pm\hat{\mathbf{v}}_\perp\frac{\partial}{\partial p_\perp}\pm\frac{[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]}{p_\perp}\left(\frac{\partial}{\partial\alpha}\right)_{\mathbf{r}},$$

with $\hat{\mathbf{v}}_\perp$ the unit vector in the direction of $\mathbf{v}_\perp$ and the upper (lower) sign again applying for particles (holes). Changing to variables s and b, the derivative with respect to α becomes

$$\left(\frac{\partial}{\partial\alpha}\right)_{\mathbf{r}}=b\frac{\partial}{\partial s}-s\frac{\partial}{\partial b}+\frac{\partial}{\partial\alpha}$$

and the spatial gradient in the (s,b) frame is

$$\nabla=\hat{\mathbf{v}}_\perp\frac{\partial}{\partial s}+[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]\frac{\partial}{\partial b}. \tag{17}$$

In the presence of a vortex, the order parameter has the form $\Delta_{\mathbf{p}}(\rho,\phi)=\Delta_{p_\perp}(\alpha,s,b)e^{i\alpha}$. Here $\Delta_{p_\perp}(\alpha,s,b)$ is the order parameter expressed in the coordinate frame (s,b), where the azimuthal angle is measured from the momentum direction. For a nonaxisymmetric vortex and/or in a d-wave superconductor, $\Delta_{p_\perp}(\alpha,s,b)$ can have an explicit dependence on the angular coordinate α.

We then are ready to proceed with the transformation of the various terms in the kinetic equation (15). Keeping in mind that f_1 is independent of s [see Eq. (16)], we rewrite the terms in the second line of Eq. (15) in the form

$$\begin{aligned}(f_-\nabla\Delta^*_{\mathbf{p}}+f^\dagger_-\nabla\Delta_{\mathbf{p}})\cdot\frac{\partial f_1}{\partial\mathbf{p}}&\\ =\pm\frac{1}{p_\perp}\left(f_-\frac{\partial\Delta^*}{\partial b}+f^\dagger_-\frac{\partial\Delta}{\partial b}\right)\left(\frac{\partial f_1}{\partial\alpha}-s\frac{\partial f_1}{\partial b}\right)&\\ \pm\left(f_-\frac{\partial\Delta^*}{\partial s}+f^\dagger_-\frac{\partial\Delta}{\partial s}\right)\frac{\partial f_1}{\partial p_\perp}&\end{aligned} \tag{18}$$

and

$$\begin{aligned}\left(f_-\frac{\partial\Delta^*_{\mathbf{p}}}{\partial\mathbf{p}}+f^\dagger_-\frac{\partial\Delta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\cdot\nabla f_1&\\ =\pm\frac{1}{p_\perp}\left[\left(f_-\frac{\partial\Delta^*}{\partial\alpha}+f^\dagger_-\frac{\partial\Delta}{\partial\alpha}\right)-s\left(f_-\frac{\partial\Delta^*}{\partial b}+f^\dagger_-\frac{\partial\Delta}{\partial b}\right)\right]\frac{\partial f_1}{\partial b}&\\ \pm\frac{1}{p_\perp}\left[\left(f_-\frac{\partial\Delta^*}{\partial s}+f^\dagger_-\frac{\partial\Delta}{\partial s}\right)b\right]\frac{\partial f_1}{\partial b}.&\end{aligned} \tag{19}$$

Moreover, since $\mathbf{A}$ does not depend on the momentum direction, we can subtract the zero term

$$\frac{e}{c}v_{\perp i}\left(\frac{\partial A_i}{\partial\mathbf{p}}\right)_{\mathbf{r}}\cdot\nabla f_1=0$$

from the last line of Eq. (15). Next, we integrate Eq. (15) along the quasiclassical trajectory, using $\partial f_1/\partial s=0$ and one of the Eilenberger equations,

$$i(\mathbf{v}_F\cdot\nabla)g_-=\Delta f^\dagger_- -\Delta^* f_-. \tag{20}$$

After some algebra, the second line on the left-hand side (LHS) of Eq. (15) takes the form

$$\int ds\left[\frac{e}{c}[\mathbf{v}_F\times\mathbf{H}]g_- - \frac{1}{2}(f_-\hat{\nabla}\Delta^*_{\mathbf{p}}+f^\dagger_-\hat{\nabla}\Delta_{\mathbf{p}})\right]\frac{\partial f_1}{\partial\mathbf{p}}+\int ds\left[-\frac{e}{c}v_{\perp i}\frac{\partial A_i}{\partial\mathbf{p}}g_- \nabla f_1+\frac{1}{2}\left(f_-\frac{\partial\Delta^*_{\mathbf{p}}}{\partial\mathbf{p}}+f^\dagger_-\frac{\partial\Delta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\right]\nabla f_1$$

$$=\mp\frac{1}{p_\perp}\int ds\left[\frac{1}{2}\left(f_-\frac{\partial\Delta^*}{\partial b}+f^\dagger_-\frac{\partial\Delta}{\partial b}\right)-\frac{e}{c}v_\perp\frac{\partial A_s}{\partial b}g_-\right]\frac{\partial f_1}{\partial\alpha}\pm\frac{1}{p_\perp}\int ds\left[\frac{1}{2}\left(f_-\frac{\partial\Delta^*}{\partial\alpha}+f^\dagger_-\frac{\partial\Delta}{\partial\alpha}\right)-\frac{e}{c}v_\perp\frac{\partial A_s}{\partial\alpha}g_-\right]\frac{\partial f_1}{\partial b}. \quad (21)$$

The term $\partial A_s/\partial\alpha$ accounts for the explicit α dependence picked up by the vector potential when expressed in the coordinate frame (s,b). The s derivatives present in Eqs. (18) and (19) disappear in Eq. (21), as can be seen from the identity derived in Ref. 5,

$$\int ds\,\mathrm{Tr}\,[(\hat{\nabla}\check{\mathcal{H}})\check{g}_-]=\int ds\,\mathrm{Tr}\,[(\nabla\check{\mathcal{H}})\check{g}_-]$$

$$=2\pi[\hat{\mathbf{z}}\times\mathbf{v}_\perp]\sum_n\frac{\partial E_n}{\partial b}\delta(\epsilon-E_n), \quad (22)$$

and valid for localized states. Here, the quasiclassical "field matrix" takes the form

$$\check{\mathcal{H}}=\begin{pmatrix}-(e/c)(\mathbf{v}_F\cdot\mathbf{A}) & -\Delta_{\mathbf{p}}\\ \Delta^*_{\mathbf{p}} & (e/c)(\mathbf{v}_F\cdot\mathbf{A})\end{pmatrix} \quad (23)$$

and the gauge-invariant field gradient is

$$\hat{\nabla}\check{\mathcal{H}}=\begin{pmatrix}(e/c)[\mathbf{H}\times\mathbf{v}_F] & -\hat{\nabla}\Delta_{\mathbf{p}}\\ \hat{\nabla}\Delta^*_{\mathbf{p}} & -(e/c)[\mathbf{H}\times\mathbf{v}_F]\end{pmatrix}. \quad (24)$$

In particular, taking the $\mathbf{v}_\perp$ projection of Eq. (22) we find

$$\int ds\,\mathrm{Tr}\left[\frac{\partial\check{\mathcal{H}}}{\partial s}\check{g}_-\right]=0.$$

It is this relation which serves to eliminate the s derivatives from Eq. (21).

The first term on the RHS of Eq. (21) has the form

$$\mp\frac{1}{2p_\perp}\frac{\partial f_1}{\partial\alpha}\int ds\,\mathrm{Tr}\left[\check{g}_-\frac{\partial\check{\mathcal{H}}}{\partial b}\right],$$

which can be transformed with help of Eq. (22). Using the relation

$$\int ds\,\mathrm{Tr}\left[\frac{\partial\check{\mathcal{H}}}{\partial\alpha}\check{g}_-\right]=2\pi v_\perp\sum_n\delta(\epsilon-E_n)\frac{\partial E_n}{\partial\alpha} \quad (25)$$

derived in Appendix A, the second term

$$\pm\frac{1}{2p_\perp}\frac{\partial f_1}{\partial b}\int ds\,\mathrm{Tr}\left[\check{g}_-\frac{\partial\check{\mathcal{H}}}{\partial\alpha}\right]$$

is transformed in a similar way. As a result, the RHS of Eq. (21) takes the simple form

$$\mp\frac{1}{p_\perp}\sum_n\left[\frac{\partial E_n}{\partial b}\frac{\partial f_1}{\partial\alpha}-\frac{\partial E_n}{\partial\alpha}\frac{\partial f_1}{\partial b}\right]\pi v_\perp\delta(\epsilon-E_n). \quad (26)$$

Next, we concentrate on the first line of Eq. (15). The driving term $\propto\partial_\epsilon f^{(0)}$ is the source of the nonequilibrium state as produced by time variations of the order parameter together with the electric field. In the present context, the vortex moving with a velocity $\mathbf{u}$ induces the time variations through the vortex displacement,

$$\Delta(\mathbf{r},t)=\Delta_0(\mathbf{r}-\mathbf{u}t)+\Delta_1,$$

$$\mathbf{A}(\mathbf{r},t)=\mathbf{A}_0(\mathbf{r}-\mathbf{u}t)+\mathbf{A}_1,$$

where Δ_1 and $\mathbf{A}_1$ are corrections proportional to the vortex velocity. Restricting ourselves to the linear response in $\mathbf{u}$ we can approximate the time derivative $\partial/\partial t=-(\mathbf{u}\cdot\nabla)$. Integrating by parts, we see that the scalar potential drops out; the resulting kinetic equation does not depend on the scalar potential, a result which holds only for clean superconductors. In the dirty case, the scalar potential plays an important role through the (so-called) charge imbalance phenomenon. Combining the elements in the above discussion we can cast the first term of the kinetic equation (15) into the form

$$-\frac{1}{2}\frac{\partial f^{(0)}}{\partial\epsilon}\int ds\,\mathrm{Tr}[(\mathbf{u}\cdot\nabla)\check{\mathcal{H}}g_-],$$

which then can be further transformed using Eq. (22).

The second term in the first line of Eq. (15) vanishes after integration over ds. The next term can be transformed with help of the identity

$$\int ds\,\mathrm{Tr}[\check{\tau}_3\check{g}_-]=2\pi v_\perp\sum_n\delta(\epsilon-E_n) \quad (27)$$

(see Appendix A), and we finally obtain the equation

$$([\hat{\mathbf{v}}_\perp\times\mathbf{u}]\cdot\hat{\mathbf{z}})\frac{\partial f^{(0)}}{\partial E}\frac{\partial E_n}{\partial b}\pm\frac{1}{p_\perp}\frac{\partial f_1}{\partial\alpha}\frac{\partial E_n}{\partial b}\mp\frac{1}{p_\perp}\frac{\partial E_n}{\partial\alpha}\frac{\partial f_1}{\partial b}-\frac{\partial f_1}{\partial t}$$

$$+\frac{1}{\pi v_\perp}\int_{-\infty}^{\infty}ds\;J_n=0, \quad (28)$$

which is nothing but the Boltzmann equation. Indeed, since

$$\frac{\partial E}{\partial b}=\mp p_\perp\frac{\partial E}{\partial\mu},$$

we find

$$([\mathbf{p}_\perp\times\mathbf{u}]\cdot\hat{\mathbf{z}})\frac{\partial f^{(0)}}{\partial E}\frac{\partial E_n}{\partial\mu}+\frac{\partial f_1}{\partial t}+\frac{\partial f_1}{\partial\alpha}\frac{\partial E_n}{\partial\mu}-\frac{\partial E_n}{\partial\alpha}\frac{\partial f_1}{\partial\mu}$$

$$=\frac{1}{\pi v_\perp}\int_{-\infty}^{\infty}ds\;J_n, \quad (29)$$

with $E_n=E_n(\mu,\alpha;p_z)$ and $\mu=\mp bp_\perp$. Equation (29) can be rewritten into the generic form (1) for the total distribution function $f = f^{(0)}(E_n) + f_1$ with the canonical variables $q = \alpha$ and $p = \mu$ and the collision integral

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll} = \frac{1}{\pi v_\perp}\int_{-\infty}^{\infty} ds\; J_n . \tag{30}$$

The driving term is obtained from Eq. (1) if we assume that the energy E_n contains a time dependence through the impact parameter $b(t) = \mp \mu(t) p_\perp^{-1}$, where $\mu(t) = [(\mathbf{r} - \mathbf{u}t) \times \mathbf{p}] \cdot \hat{\mathbf{z}}$:

$$\frac{\partial f}{\partial t} = \frac{\partial f^{(0)}}{\partial E}\frac{\partial E_n}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial f^{(0)}}{\partial E}\frac{\partial E_n}{\partial \mu}([\mathbf{p}_\perp \times \mathbf{u}] \cdot \hat{\mathbf{z}}). \tag{31}$$

Equation (1) deals with the localized states. As we already have mentioned above, delocalized quasiparticles are in equilibrium with the heat bath and their contribution to the force is small. However, in the case of a vortex in a charged superfluid, the associated magnetic field introduces a magnetic quantization and quasiparticles with energies above the gap become localized at cyclotron orbits. As a result, they give a finite contribution to the force acting on a moving vortex which can also be calculated within the framework of Eq. (1).

To summarize the results of this section, we have started with the exact microscopic description of the nonstationary processes in terms of the Green function technique. Using the quasiclassical approximation, we have been able to reduce the problem of finding the nonequilibrium state of the superconductor with a moving vortex to the problem of solving the canonical Boltzmann equation for the distribution of nonequilibrium excitations localized in the vortex core. The only information needed to find the distribution function is the energy spectrum of the equilibrium excitations in the vortex core.

In the next section, we demonstrate that the knowledge of the energy spectrum is also sufficient to calculate the force acting on the moving vortex. The full problem of the vortex dynamics thus reduces to several much easier and more transparent steps which are finding the equilibrium energy spectrum of the excitations in the vortex core, solving the Boltzmann equation, and, finally, calculating the momentum transfer from the localized excitations.

IV. FORCE ON A MOVING VORTEX

In order to calculate the force exerted by the environment on the moving vortex, we start again with the Green function formalism. Consider the thermodynamic potential Ω of the superconductor, which is a function of the order parameter and the magnetic field, as well as of temperature, volume, and the chemical potential. In a nonstationary case, the variation of the thermodynamic potential of the superconductor with respect to Δ and $\mathbf{A}$ can be written as[3,7]

$$\int d^3r\, \delta\Omega_s(t) = \int d^3r \int \frac{d\epsilon}{4}\int \frac{dS_F}{(2\pi)^3 v_F}\mathrm{Tr}\,[\check{g}^{(\mathrm{nst})}\delta\check{\mathcal{H}}], \tag{32}$$

where the operator $\check{\mathcal{H}}$ has been defined in Eq. (23), the function $\check{g}^{(\mathrm{nst})}$ denotes the nonstationary part of the total quasiclassical Green function [see Eq. (34) below], and the integration

$$\int \frac{dS_F}{(2\pi)^3 v_F}$$

is taken over the Fermi surface. The force acting on the vortex is obtained through the variation $\delta\check{\mathcal{H}} = (\mathbf{d}\cdot\nabla)\check{\mathcal{H}}$, with $\mathbf{d}$ an arbitrary constant vector. Note that the gauge-invariant representation $\delta\check{\mathcal{H}} = (\mathbf{d}\cdot\hat{\nabla})\check{\mathcal{H}}$ produces the same result [see Eq. (22)]. The force per unit length exerted by the environment on the moving vortex takes the form[3,17]

$$\mathbf{F}_{\rm env} = -\int d^2r \int \frac{d\epsilon}{4}\int \frac{dS_F}{(2\pi)^3 v_F}\mathrm{Tr}\,[\check{g}^{(\mathrm{nst})}\hat{\boldsymbol{\nabla}}\mathcal{H}], \tag{33}$$

where the spatial integration is taken over the area occupied by the vortex. With the distribution function in the form $\check{f}_{\epsilon_1,\epsilon_2} = f^{(0)}_\epsilon \delta(\epsilon_1 - \epsilon_2) + f_1 \check{1} + f_2 \check{\tau}_3$ we obtain the nonequilibrium Green function[3] [cf. Eq. (14)]

$$\check{g}^{(\mathrm{nst})} = -\frac{i}{2}\frac{\hat{\partial}(\check{g}^R + \check{g}^A)}{\partial t}\frac{\partial f^{(0)}}{\partial \epsilon} + (\check{g}^R - \check{g}^A) f_1 + (\check{g}^R\check{\tau}_3 - \check{\tau}_3\check{g}^A) f_2, \tag{34}$$

where the first term arises from an expansion in frequency $\omega = \epsilon_1 - \epsilon_2 = i\partial_t$ (we drop the term $[\check{g}^R_{\epsilon+\omega/2,\epsilon-\omega/2} - \check{g}^A_{\epsilon+\omega/2,\epsilon-\omega/2}] f^{(0)}_\epsilon$, which is even in ω and does not enter the expression for the force acting on the vortex; see Ref. 16). The operator $\hat{\partial}/\partial t$ has the form $\hat{\partial}/\partial t = \partial/\partial t \pm 2ie\varphi$ when applied to f and $f^\dagger$, respectively, and reduces to $\partial/\partial t$ when applied to the functions g and $\underline{g}$. The first term on the RHS of Eq. (34) is of the order of $(u/\xi)(\partial f^{(0)}/\partial\epsilon)$, much smaller than f_1, where an order-of-magnitude estimate gives [see Eq. (15)] $f_1 \sim (p_F u)(\partial f^{(0)}/\partial \epsilon)$, with $\mathbf{u}$ the vortex velocity. Since also $f_2 \ll f_1$, the main contribution to Eq. (33) comes from the part containing f_1.

As delocalized quasiparticles are in equilibrium with the heat bath and hence $f_1 = 0$ they do not contribute to the force. Using the identity (22), the localized states produce the contribution

$$\mathbf{F}_{\rm env} = -\int db\; ds \int \frac{d\epsilon}{2}\int \frac{dS_F}{(2\pi)^3 v_F}\mathrm{Tr}\,[\check{g}_- \hat{\boldsymbol{\nabla}}\check{\mathcal{H}}] f_1$$
$$= -\pi \sum_n \int db \int \frac{dS_F}{(2\pi)^3 v_F}[\hat{\mathbf{z}} \times \mathbf{v}_\perp]\frac{\partial E_n}{\partial b} f_1 . \tag{35}$$

This expression can be rewritten as the momentum transfer from the localized excitations to the vortex. Indeed, with the Fermi-surface area element

$$dS_F = dp'\, dp_\perp = \frac{v_F}{v_\perp} dp_z\, p_\perp\, d\alpha,$$

where dp' is the increment in the direction perpendicular to $\mathbf{v}_F$ and $d\mathbf{p}_\perp$, we obtain

$$\mathbf{F}_{\rm env}=-\frac{1}{2}\sum_n \int \frac{dp_z}{2\pi}\frac{d\alpha d\mu}{2\pi}\frac{\partial E_n}{\partial b}[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]f_1$$

$$=\frac{1}{2}\sum_n \int \frac{dp_z}{2\pi}\frac{d\alpha d\mu}{2\pi}\frac{\partial \mathbf{p}_n}{\partial t}f_1. \quad (36)$$

Here, we make use of the Hamilton equation [see Eq. (17)]

$$\frac{\partial \mathbf{p}_n}{\partial t}=-\nabla E_n=-\frac{\partial E_n}{\partial b}[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]. \quad (37)$$

Note that, in our notation, the deviation from the equilibrium distribution is $f_1=-2\delta n_\epsilon$, where δn_ϵ is the deviation from the Fermi distribution, thus explaining the positive sign in Eq. (36). The normalization in Eqs. (35) and (36) is chosen such that the sum over the two spin states enters as $(1/2)\Sigma_s$.

We emphasize that the force $\mathbf{F}_{\rm env}$ is defined as the response of the whole environment to the vortex displacement. It is therefore the *total* force acting on the vortex from the ambient system, including all partial forces such as the longitudinal friction force and the nondissipative transverse force. The transverse force, in turn, includes various parts which can be identified as the Iordanskii force, the spectral flow force, and the part of the Magnus force containing the vortex velocity.

For a moving vortex, the force from the environment should be balanced by the Lorentz force:[3,17] $\mathbf{F}_{\rm L}+\mathbf{F}_{\rm env}=0$, where the Lorentz force is

$$\mathbf{F}_{\rm L}=\frac{\Phi_0}{c}[\mathbf{j}_{\rm tr}\times\hat{\mathbf{z}}]{\rm sgn}(e),$$

with the flux quantum $\Phi_0=\pi c/|e|$. The force balance equation then determines the transport current in terms of the vortex velocity and thus allows to find the flux flow conductivity tensor. This conductivity was calculated in Refs. 4,5 and 8 using the Green function technique. On the other hand, the semiclassical description in terms of the Boltzmann equation was employed in Refs. 6 and 12. The above derivation then demonstrates the equivalence of these two methods.

In order to understand the structure of the total force more clearly, we review, by way of example, the solution of the Boltzmann equation for a vortex in a *s*-wave superconductor in the limit of low fields when $\omega_c\tau\ll 1$ and no localized above-gap states complicate the situation. This limit also applies to an uncharged *s*-wave Fermi superfluid.

V. VORTICES IN *s*-WAVE SUPERCONDUCTORS

We consider normal particles with a parabolic spectrum and a spherical Fermi surface. The excitation spectrum of bound states in the vortex core consists of an anomalous chiral branch[15] with the radial quantum number $n=0$; this branch has the energy $E_0(\mu)=-\omega_0\mu$ for $|E_0|\ll\Delta_0$ and saturates at $E_0=-\Delta_0\,{\rm sgn}(\mu)$ for $\mu\to\pm\infty$. Here Δ_0 denotes the modulus of the order parameter at large distances away from the vortex core. The other branches with $n\neq 0$ are separated from the $n=0$ spectrum by energies of the order of Δ_0 and are even functions of μ.

Using the relaxation time approximation for the collision integral

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll}=\frac{1}{\pi v_\perp}\int_{-\infty}^{\infty}ds\; J_n\approx-\frac{f_1}{\tau_n}, \quad (38)$$

the solution of Eq. (1) is simplified considerably (in our estimates below we will usually use $\tau_n\sim\tau$ with τ the normal state impurity scattering time and ignore the specific dependences in the radial quantum number). For an axisymmetric *s*-wave vortex the energies E_n do not depend on α and the term $\partial E/\partial\alpha$ vanishes. Using the ansatz

$$f_1=-\frac{\partial f^{(0)}}{\partial E}\{([\mathbf{u}\times\mathbf{p}_\perp]\cdot\hat{\mathbf{z}})\gamma_{\rm O}+(\mathbf{u}\cdot\mathbf{p}_\perp)\gamma_{\rm H}\}$$

for the distribution function, the Boltzmann equation Eq. (29) gives[5]

$$\gamma_{\rm O}=\frac{\omega_n\tau_n}{\omega_n^2\tau_n^2+1},\qquad \gamma_{\rm H}=\frac{\omega_n^2\tau_n^2}{\omega_n^2\tau_n^2+1},$$

where we remind that $\omega_n=p_\perp^{-1}(\partial E_n/\partial b)$. The force splits into the two terms $\mathbf{F}_{\rm env}=\mathbf{F}_\parallel+\mathbf{F}_\perp$, with the friction $\mathbf{F}_\parallel$ and transverse $\mathbf{F}_\perp$ forces given by

$$\mathbf{F}_\parallel=-\pi N\left\langle\sum_n\int\frac{d\mu}{2}\omega_n\gamma_{\rm O}\frac{\partial f^{(0)}}{\partial E}\right\rangle_F\mathbf{u}, \quad (39)$$

$$\mathbf{F}_\perp=\pi N\left\langle\int\frac{d\mu}{2}\omega_0\gamma_{\rm H}\frac{\partial f^{(0)}}{\partial E}\right\rangle_F[\hat{\mathbf{z}}\times\mathbf{u}], \quad (40)$$

where $p_z=p_F\cos\theta$ and N is the electron density; $\langle\cdots\rangle$ is the average over the Fermi surface with the weight $p_\perp^2$,

$$\langle\cdots\rangle_F=\frac{3}{4}\int d\theta\sin^3\theta\,(\cdots).$$

Only the spectral branch with $n=0$ contributes to the transverse force $\mathbf{F}_\perp$ as all ω_n with $n\neq 0$ are odd functions of μ and thus drop out of the sum over n in Eq. (40). A simple estimate gives $\omega_n\sim\omega_0\approx\Delta_0^2/E_F$, where E_F denotes the Fermi energy.

The *longitudinal force* $\mathbf{F}_\parallel$ defines the friction coefficient in the vortex equation of motion and determines the Ohmic component of the conductivity $\sigma_{\rm O}$. Expressing the vortex velocity $\mathbf{u}$ through the average electric field $\mathbf{E}$, $\mathbf{u}=c[\mathbf{E}\times\hat{\mathbf{z}}]/B\,{\rm sgn}(e)$, we find

$$\sigma_{\rm O}=\frac{N|e|c}{B}\left\langle\sum_n\int\frac{d\mu}{2}\frac{\omega_n^2\tau_n}{\omega_n^2\tau_n^2+1}\frac{\partial f^{(0)}}{\partial E}\right\rangle_F.$$

In the moderately clean limit where $\omega_0\tau\ll 1$, the conductivity roughly follows the Bardeen-Stephen expression at low temperatures, but exhibits an additional temperature-dependent factor Δ_0/T_c on approaching T_c,[5]

$$\sigma_{\rm O}\sim\sigma_n\frac{H_{c2}}{H}\frac{\Delta_0}{T_c}$$

(we recall that Δ_0 denotes the temperature-dependent gap parameter far away from the vortex core).

The *transverse force* determines the Hall conductivity. Within the quasiclassical approximation, the result (40) gives the total transverse force. However, historically, various terms have been calculated separately and we proceed with identifying these contributions within Eq. (40). We first isolate the Magnus and Iordanskii[18] forces $\mathbf{F}_{\rm M}=\pi N_s[(\mathbf{v}_s-\mathbf{u})\times\hat{\mathbf{z}}]$ and $\mathbf{F}_{\rm I}=\pi N_n[(\mathbf{v}_n-\mathbf{u})\times\hat{\mathbf{z}}]$ (we assume, for the moment, that both fluids are at rest, $\mathbf{v}_s=\mathbf{v}_n=0$ in the laboratory frame),

$$\mathbf{F}_\perp=-\pi N_s[\mathbf{u}\times\hat{\mathbf{z}}]-\pi N_n[\mathbf{u}\times\hat{\mathbf{z}}]+\mathbf{F}_{\rm sf}. \tag{41}$$

They represent, respectively, the hydrodynamic forces originating from the Magnus effect due to the superfluid density N_s and the normal quasiparticle density N_n, with $N=N_s+N_n$ the total particle density. The remaining term is the (so-called) spectral flow force

$$\mathbf{F}_{\rm sf}=\pi N[\mathbf{u}\times\hat{\mathbf{z}}]\left\langle\int\frac{d\mu}{2}\frac{\omega_0}{\omega_0^2\tau_0^2+1}\frac{\partial f^{(0)}}{\partial E}\right\rangle_F + \pi N[\mathbf{u}\times\hat{\mathbf{z}}]\left[1-\tanh\frac{\Delta_0}{2T}\right], \tag{42}$$

where, using $\omega_0=-\partial E_0/\partial\mu$, we have accounted for the fact that

$$\int\frac{d\mu}{2}\omega_0\frac{\partial f^{(0)}}{\partial E}=\tanh\frac{\Delta_0}{2T}$$

[we point out that going over to the quasiclassical description in Eq. (12) we lose an additional term arising from broken particle-hole asymmetry; see Refs. 19 and 20: this term can be well ignored in the present discussion of the clean situation; however, it does become relevant in the dirty case where the Hall force is small]. An estimate of the first term in Eq. (42), which is due to states localized in the core, gives $\mathbf{F}_{\rm sf}^{\rm loc}\approx\pi N[\mathbf{u}\times\hat{\mathbf{z}}](\omega_0^2\tau_0^2+1)^{-1}\tanh(\Delta_0/2T)$ (we assume a cylindrical Fermi surface).

For a superfluid moving with the velocity $\mathbf{v}_s$ the first term in Eq. (41) combines with the Lorentz force $\mathbf{F}_{\rm L}=\pi N_s[\mathbf{v}_s\times\hat{\mathbf{z}}]$ into the usual expression $\mathbf{F}_{\rm M}=\pi N_s[(\mathbf{v}_s-\mathbf{u})\times\hat{\mathbf{z}}]$ for the Magnus force involving the relative velocity between the superfluid and the vortex line (similarly, in case of a moving normal fluid the Iordanskii force has to be generalized to involve the relative velocity $\mathbf{v}_n-\mathbf{u}$; here, we always assume that the normal fluid is at rest). The full force balance equation then takes the form

$$\mathbf{F}_{\rm M}+\mathbf{F}_{\rm I}+\mathbf{F}_{\rm sf}+\mathbf{F}_\parallel=0.$$

While the original result (40) involves only localized states from the $n=0$ chiral branch, the introduction of the Magnus and Iordanskii forces $\mathbf{F}_{\rm M}$ and $\mathbf{F}_{\rm I}$ seemingly adds contributions from delocalized states to the result. However, as we will explain below, the corresponding terms mutually cancel each other such that only forces from localized states survive in the end.

A detailed discussion of the spectral flow force is given in Ref. 21. The force $\mathbf{F}_{\rm sf}$ is due to the momentum flow from the Fermi see of normal excitations to the moving vortex via the gapless spectral branch going through the vortex core from negative to positive energies.[22] Due to the time-dependent angular momentum of the excitations $\mu(t)=[(\mathbf{r}-\mathbf{u}t)\times\mathbf{p}]\cdot\hat{\mathbf{z}}$, there appears a flow (with the velocity $\partial\mu/\partial t$) of spectral levels characterized by the angular momentum μ. Each particle on a level carries a momentum p_F. The momentum transfer to the vortex associated with the spectral flow is effective if the quasiparticle relaxation occurs quickly: the factor $(\omega_0^2\tau_0^2+1)^{-1}$ in Eq. (42) accounts for the relaxation rate: the relaxation and hence the momentum transfer is complete for $\omega_0\tau_0\ll 1$, and vanishes in the opposite limit. The first term in Eq. (42) thus describes the disorder-mediated momentum flow along the anomalous chiral branch $E_0(\mu)$ for energies below the gap. For energies above the gap, the spectrum also has a chiral branch[5] which is made from the Landau levels for the states with definite angular momenta. For small magnetic fields such that $\omega_c\tau\ll 1$, the magnetic levels $E(\mu)$ form a continuum. The integral over energies above the gap

$$\int_\Delta^\infty\frac{d\mu}{2}\omega_c\frac{\partial f^{(0)}}{\partial E}=1-\tanh\frac{\Delta_0}{2T},$$

where $\omega_c=-\partial E/\partial\mu$, then provides the second term in Eq. (42).[21]

It is interesting to note that the spectral flow force from above gap states is related to the anomalous contribution to the transverse scattering cross section of delocalized quasiparticles as found in Ref. 23,

$$\sigma_\perp=\frac{\pi}{p_\perp}\left[\frac{\epsilon}{\sqrt{\epsilon^2-\Delta_0^2}}-1\right]. \tag{43}$$

Here, the first term corresponds to the cross section found in a Bose superfluid,[24] $\sigma_\perp=2\pi/mv_g$, where v_g is the group velocity. Inserting this term into the expression for the force exerted on the vortex by scattered excitations,

$$\mathbf{F}_\perp=\int_{|\epsilon|>\Delta_0}\frac{d\epsilon}{4}\frac{\partial f^{(0)}}{\partial\epsilon}\int\frac{dp_z}{2\pi}p_\perp^3\sigma_\perp^{(1)}[\hat{\mathbf{z}}\times\mathbf{u}], \tag{44}$$

we recover the Iordanskii force $\mathbf{F}_{\rm I}=-\pi N_n[\mathbf{u}\times\hat{\mathbf{z}}]$. The second term in Eq. (43) originates from the fact that here, in contrast to the situation in a Bose superfluid, the phase of the single-particle wave function changes by π upon encircling the vortex, while it is the order parameter phase which changes by 2π. It is this singularity, produced by the vortex in the single-particle wave function, which results in the anomalous contribution to the cross section in Eq. (43); inserted into Eq. (44), it exactly reproduces the second term in Eq. (42). Again we see that the spectral flow force is related to a single-particle anomaly associated with the vortex.

Let us confirm that indeed all contributions from delocalized states vanish from expression (41) for the transverse force. We split the spectral force $\mathbf{F}_{\rm sf}$ into the two terms $\mathbf{F}_{\rm sf}^{\rm loc}$ and $\mathbf{F}_{\rm sf}^{\rm deloc}=\pi N[\mathbf{u}\times\hat{\mathbf{z}}][1-\tanh(\Delta_0/2T)]$. As shown above, scattering of delocalized states on the vortex produces the two terms $\mathbf{F}_{\rm scatt}^{\rm deloc}=\mathbf{F}_{\rm I}+\mathbf{F}_{\rm sf}^{\rm deloc}$. On the other hand, we can write the term $\mathbf{F}_{\rm sf}^{\rm deloc}$ as the difference between a contribution from all states (the term $\pi N[\mathbf{u}\times\hat{\mathbf{z}}]$) and one from localized states (the second term $\pi N[\mathbf{u}\times\hat{\mathbf{z}}]\tanh(\Delta_0/2T)]$), $\mathbf{F}_{\rm sf}^{\rm deloc}=-\mathbf{F}_\infty^{\rm all}+\mathbf{F}_\infty^{\rm loc}$, where $\mathbf{F}_\infty^{\rm loc}=\mathbf{F}_\perp|_{\tau_0\to\infty}$. The term $-\mathbf{F}_\infty^{\rm all}=\pi N[\mathbf{u}\times\hat{\mathbf{z}}]$ then cancels the sum of Magnus and Iordanskii forces in $\mathbf{F}_\perp$ and we find that the transverse force indeed arises from the localized states alone, $\mathbf{F}_\perp=\mathbf{F}_{\rm sf}^{\rm loc}+\mathbf{F}_\infty^{\rm loc}$ (note that $\mathbf{F}_{\rm sf}^{\rm loc}|_{\tau_0\to\infty}\to 0$).

The characteristic feature of the transverse force is that it vanishes in the limit $\omega_0\tau\ll 1$: the Iordanskii force together with the spectral flow force cancels the part of the Magnus force which contains the vortex velocity, in agreement with usual experimental findings. The results of the microscopic analysis of the vortex dynamics thus show that the claim in Ref. 25 of the Magnus force being the only transverse force acting on a moving vortex is incorrect.

VI. TORQUE ON A NONAXISYMMETRIC VORTEX

Assume that the vortex core has no axial symmetry. Such nonaxisymmetric vortices exist in superfluid phases of ^{3}He and in d-wave superconductors. Let us further assume that the vortex core can rotate with respect to the heat bath—such rotating vortices have been experimentally observed in superfluid ^{3}He B (Ref. 26) (in a d-wave superconductor the asymmetry is coupled to the crystalline axes inhibiting the free rotation of the vortex core). If the vortex core rotates with respect to the heat bath, it experiences a torque from the ambient liquid which can be obtained from Eq. (32) with the choice $\delta\check{\mathcal{H}}=\delta\beta(\partial\check{\mathcal{H}}/\partial\beta)$, where $\delta\beta$ is a rotation angle around the z axis. The torque per unit length then follows from

$$T_z=-\int d^2r\frac{\partial\Omega_s}{\partial\beta}=$$

$$-\int d^2r\int\frac{d\epsilon}{4}\int\frac{dS_F}{(2\pi)^3 v_F}\mathrm{Tr}\left[\frac{\partial\check{\mathcal{H}}}{\partial\beta}\check{g}^{(\rm nst)}\right]. \quad (45)$$

In the coordinate frame (s,b), the rotation angle becomes the angle α along the momentum increment. Therefore,

$$T_z=-\int db\ ds\int\frac{d\epsilon}{2}\int\frac{dS_F}{(2\pi)^3 v_F}\mathrm{Tr}\left[\frac{\partial\check{\mathcal{H}}}{\partial\alpha}\check{g}_-\right]f_1 \quad (46)$$

and using the identity (25) we obtain

$$T_z=-\pi\sum_n\int db\int\frac{dS_F}{(2\pi)^3 v_F}v_\perp\frac{\partial E_n}{\partial\alpha}f_1.$$

Replacing the impact parameter b by the angular momentum μ and using the equation of motion

$$\frac{\partial L_n}{\partial t}=-\frac{\partial E_n}{\partial\alpha},$$

we find that the total torque follows from the transfer of the angular momentum L_n from the quasi-particle excitations to the vortex core,

$$T_z=\frac{1}{2}\sum_n\int\frac{dp_z}{2\pi}\int\frac{d\mu\ d\alpha}{2\pi}\frac{\partial L_n}{\partial t}f_1. \quad (47)$$

The above approach has been used to find the friction torque on a rotating nonaxisymmetric vortex core.[27]

VII. CONCLUSIONS

We have demonstrated that the description of the vortex dynamics in superfluid Fermi systems based on the kinetic equations for the quasiclassical Green functions of the microscopic non-stationary theory is equivalent to the concept of semiclassical particles localized in the vortex core, with a distribution function governed by the kinetic Boltzmann equation in its canonical form. We have derived expressions for the force and torque acting on a moving vortex and have shown that they can be represented as the momentum transfer to the vortex from the localized particles. Our analysis provides a microscopic justification of the phenomenological approach based on semiclassical dynamics.

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APPENDIX: DERIVATION OF THE IDENTITIES

Consider the Green function matrix $\mathcal{G}_\epsilon^{R(A)}$ and expand it in the eigenfunctions of a quasiclassical particle with the momentum $p\sim p_F\gg\xi^{-1}$ moving along the trajectory defined by its velocity $\mathbf{v}_F$ and the impact parameter b. The wave functions depend on the distance along the trajectory s. We find that

$$\check{\mathcal{G}}_\epsilon^{R(A)}(\mathbf{v},b;s_1,s_2)=\begin{pmatrix} G^{R(A)} & F^{R(A)}\\ -F^{\dagger R(A)} & \overline{G}^{R(A)}\end{pmatrix}$$

$$=-\sum_n\frac{\mathcal{U}_n(\mathbf{v},b;s_1)\mathcal{U}_n^\dagger(\mathbf{v},b;s_2)}{\epsilon-E_n(b)\pm i\delta}, \quad (A1)$$

where the eigenvectors are

$$\mathcal{U}=\begin{pmatrix} u\\ -v\end{pmatrix},\quad \mathcal{U}^\dagger=(u^*\quad v^*).$$

The Green function matrix obeys the equation

$$[\epsilon_n(-i\nabla)-E_F-\epsilon\check{\tau}_3+\check{\mathcal{H}}]\mathcal{G}=\check{1}\delta(s_1-s_2), \quad (A2)$$

where $\epsilon_n(\mathbf{p})$ is the normal-state spectrum. The Bogoliubov wave functions $\mathcal{U}$ are solutions of the equation

$$[\epsilon_n(-i\nabla)-E_F+\check{\mathcal{H}}]\mathcal{U}=E\check{\tau}_3\mathcal{U} \quad (A3)$$

and satisfy the orthogonality conditions

$$\sum_n\check{\tau}_3\mathcal{U}_n(s_1)\mathcal{U}_n^\dagger(s_2)=\check{1}\delta(s_1-s_2), \quad (A4)$$

$$\int ds\mathcal{U}_n^\dagger(s)\check{\tau}_3\mathcal{U}_{n'}(s)=\delta_{n,n'}. \tag{A5}$$

Next, we determine the relation between the Green function $\check{\mathcal{G}}^{R(A)}$ and its quasiclassical counterpart $\check{g}^{R(A)}$. We represent the Green function as

$$\check{\mathcal{G}}=e^{ip_\perp(s_1-s_2)}[\check{a}_+\Theta(s_1-s_2)-\check{a}_-\Theta(s_2-s_1)], \tag{A6}$$

where $\check{a}_\pm(s)$ are functions of $s=(s_1+s_2)/2$; they vary over distances of the order of ξ. From Eq. (A2) we find $\check{a}_++\check{a}_-=i/v_\perp$. The functions $\check{a}_\pm$ satisfy the same Eilenberger equations as the quasiclassical Green functions $\check{g}^{R(A)}$. Using the boundary conditions at large distances, we obtain (compare with Ref. 17) $\check{a}_\pm=i(\check{1}\pm\check{g})/2v_\perp$. Thus,

$$\check{\mathcal{G}}^{R(A)}(s_1\to s_2)=\frac{i}{2v_\perp}[\check{g}^{R(A)}+\check{1}\,\mathrm{sgn}(s_1-s_2)]. \tag{A7}$$

We now are ready to derive Eqs. (22), (25), and (27). Consider the energies $|\epsilon|<\Delta_0$. We find

$$\begin{aligned}\frac{1}{2}\int ds\,\mathrm{Tr}[\check{\tau}_3(\check{g}^R-\check{g}^A)]\\ &=-iv_\perp\int ds\,\mathrm{Tr}\{\check{\tau}_3[\check{\mathcal{G}}^R(s,s)-\check{\mathcal{G}}^A(s,s)]\}\\ &=2\pi v_\perp\sum_n\delta(\epsilon-E_n)\int ds\,\mathrm{Tr}[\mathcal{U}_n^\dagger(s)\check{\tau}_3\mathcal{U}_n(s)]\\ &=2\pi v_\perp\sum_n\delta(\epsilon-E_n),\end{aligned}$$

which is Eq. (27). Note that the second term in Eq. (A7) vanishes under the trace.

We turn to derivation of Eq. (22). Integrating the first part of the relation by parts and assuming that $\check{g}_-$ vanishes at large distances from the vortex we obtain

$$\begin{aligned}\frac{1}{2}\int ds\,\mathrm{Tr}[(\mathbf{d}\cdot\hat{\nabla}\check{\mathcal{H}})(\check{g}^R-\check{g}^A)]\\ &=\frac{1}{2}\int ds\,\mathrm{Tr}[(\mathbf{d}\cdot\nabla\check{\mathcal{H}})(\check{g}^R-\check{g}^A)].\end{aligned} \tag{A8}$$

Second, we use again Eq. (A7) and find that the RHS of Eq. (A8) is

$$\begin{aligned}-iv_\perp\int ds\,\mathrm{Tr}\{(\mathbf{d}\cdot\nabla\check{\mathcal{H}})[\check{\mathcal{G}}^R(s,s)-\check{\mathcal{G}}^A(s,s)]\}\\ &=2\pi v_\perp\sum_n\delta(\epsilon-E_n)\int ds\,\mathrm{Tr}\left[\mathcal{U}_n^\dagger\left(d_s\frac{\partial\check{\mathcal{H}}}{\partial s}+d_b\frac{\partial\check{\mathcal{H}}}{\partial b}\right)\mathcal{U}_n\right]\\ &=2\pi v_\perp d_b\sum_n\delta(\epsilon-E_n)\int ds\,\mathrm{Tr}\left[\mathcal{U}_n^\dagger\frac{\partial\check{\mathcal{H}}}{\partial b}\mathcal{U}_n\right]\\ &=2\pi([\mathbf{v}_\perp\times\mathbf{d}]\cdot\hat{\mathbf{z}})\sum_n\frac{\partial E_n}{\partial b}\delta(\epsilon-E_n).\end{aligned}$$

This proves the identity (22). In the same way, we obtain for localized states

$$\begin{aligned}\frac{1}{2}\int ds\,\mathrm{Tr}\left[\frac{\partial\check{\mathcal{H}}}{\partial\alpha}(\check{g}^R-\check{g}^A)\right]\\ &=-iv_\perp\int ds\,\mathrm{Tr}\left[\frac{\partial\check{\mathcal{H}}}{\partial\alpha}[\check{\mathcal{G}}^R(s,s)-\check{\mathcal{G}}^A(s,s)]\right]\\ &=2\pi v_\perp\sum_n\delta(\epsilon-E_n)\int ds\,\mathrm{Tr}\left[\mathcal{U}_n^\dagger\frac{\partial\check{\mathcal{H}}}{\partial\alpha}\mathcal{U}_n\right]\\ &=2\pi v_\perp\sum_n\frac{\partial E_n}{\partial\alpha}\delta(\epsilon-E_n),\end{aligned} \tag{A9}$$

which proves the identity (25).

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