Magnetic penetration length and irreversibility of a disordered granular superconductor with π junctions

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We calculate the magnetic penetration length $\lambda(T)$ of a disordered granular superconductor with π junctions in zero magnetic field, using a mean-field replica method. The superconductor is modeled by an array of Josephson junctions whose couplings are drawn randomly from a Gaussian distribution centered at $J_0>0$, with width J. For disorder strength $\delta\equiv J/J_0<1$ there are three thermodynamical phases of the array separated by continuous transitions: (i) the high-temperature normal phase, (ii) the reversible superconducting phase, and (iii) the low-temperature superconducting glass phase with broken ergodicity. For a range of disorder δ near 1 there is a further possibility of reentry into a low-temperature normal glass phase. For $\delta\geq 1$ there are only two phases: (i) the high-temperature normal phase and (ii) the low-temperature normal glass phase with broken ergodicity. In the superconducting glass phase we calculate both the Gibbs averaged and the single-state-averaged magnetic penetration lengths. [S0163-1829(99)04702-5]

I. INTRODUCTION

Irreversibility of magnetic properties of type-II superconductors in external magnetic fields is characteristic of a significant part of the *T-H* phase diagram. Because of it, it is meaningless to speak of a ''state'' of the superconductor in the thermodynamic sense, and quantities such as magnetization and magnetic penetration length can no longer be defined as equilibrium properties of the sample. The boundary between reversible and irreversible regions of the phase diagram came to be known as ''the irreversibility line,'' and much experimental and theoretical effort went into understanding its nature.[1–4]

On the other hand, irreversibility of type-II superconductors in *zero* applied magnetic field is more surprising as well as subtle. An astonishing manifestation of it is the so-called Wohlleben effect, or paramagnetic Meissner effect:[5–8] it is found that some otherwise diamagnetic samples become *paramagnetic* at low temperatures, depending on their history. This effect is observed primarily—although not exclusively—on annealed granular samples with large superconducting grains, with good contacts between the grains. Soon after the experimental discovery of the Wohlleben effect it was suggested[7] that a possible explanation might involve the presence of the so-called π junctions between some of the superconducting grains. Regardless of the microscopic mechanism, a π junction is phenomenologically described as a Josephson junction with an effectively *negative* critical current: its energy is minimum when the phase difference between the superconducting condensate wave functions at the opposite ends of the junction is π, in contrast with the regular Josephson junction, whose energy is minimum when the two wave functions are in phase.

When a fraction of all junctions in a network are π junctions, phases on grains are frustrated, and the ground state (or ground states) of the network contains phases which may not all be the same, so there are permanent microscopic currents flowing between the grains. These currents create microscopic (local) magnetic moments, which may be aligned by a weak external magnetic field as the sample is being cooled down from a high-temperature normal state to a low-temperature superconducting state. This alignment of magnetic moments is then thought to cause the global paramagnetic response of the sample. When the sample is being cooled in the absence of an external magnetic field, local magnetic moments will freeze with random orientation, and the global response of the sample will be diamagnetic. These are the essential ingredients of the theory of Sigrist and Rice,[8] who find a quantitative agreement of their theory with experiment. While their approach allows one to calculate the magnetic susceptibility, it does not give any information about the coherent superconducting properties of superconducting arrays, for they model the superconducting network as a collection of independent, noninteracting, microscopic current loops, while it is precisely the Josephson interaction among the grains, which is responsible for superconductivity of the whole sample in the first place.

In this work we calculate the magnetic penetration length $\lambda(T)$ of a model granular superconductor with π junctions, as the defining property of a superconductor. A finite $\lambda(T)$ indicates a superconductor with Meissner effect (the bulk of the sample is free of magnetic fields), and infinite $\lambda(T)$ indicates a sample in the normal state, unable to expel static magnetic fields.

Our model includes Josephson interaction between the grains, as well as randomness of the coupling strengths, with the possibility of some couplings being negative—the key ingredient leading to frustration and irreversibility of macroscopic properties. The concepts of randomness and frustration, as well as that of irreversibility, are familiar from the theory of spin glasses,[9–11] and we will apply the techniques developed there to the present problem of a disordered granular superconductor with π junctions. Electromagnetic interaction between microscopic currents in the array is neglected, and we assume zero applied magnetic field.

II. REPLICA FORMALISM

We consider a model granular superconductor described by the Hamiltonian

$$-\beta\mathcal{H}=\beta\sum_{\mathbf{n},\mu} J_\mu(\mathbf{n})\cos\Delta_\mu\phi(\mathbf{n}), \tag{1}$$

where $\Delta_\mu\phi(\mathbf{n})\equiv\phi(\mathbf{n}+\hat{\mu})-\phi(\mathbf{n}),\hat{\mu}$ are unit vectors of a hypercubic lattice, integer vectors $\mathbf{n}$ label the nodes of the lattice, $\phi(\mathbf{n})$ are phases of the superconducting condensates on individual grains, assumed to be placed on the lattice, and $J_\mu(\mathbf{n})$ are Josephson coupling strengths between grains at $\mathbf{n}$ and $\mathbf{n}+\hat{\mu}$. The coefficients $J_\mu(\mathbf{n})$ are drawn randomly from a distribution

$$P[J_\mu(\mathbf{n})]=\frac{1}{(2\pi J^2/z)^{1/2}}e^{-z[J_\mu(\mathbf{n})-J_0/z]^2/2J^2}. \tag{2}$$

Bonds with negative $J_\mu(\mathbf{n})$ represent the π junctions, and $J_0>0$ so that ''on average'' the superconducting phase coherence is supported. z is the number of nearest neighbors, which appears here so that it would not appear explicitly in most of the following developments.

Using the replica trick $\overline{\ln Z}=\lim_{n\to 0}(1/n)(\overline{Z^n}-1)$, where the overline denotes disorder averaging, we write

$$\overline{Z^n}=\mathrm{Tr}_\phi\int\prod_{\mathbf{n},\mu}P[J_\mu(\mathbf{n})]dJ_\mu(\mathbf{n})e^{-\beta\sum_{a=1}^{n}\mathcal{H}^a}\equiv\mathrm{Tr}_\phi e^{-\beta\mathcal{H}^{(n)}}, \tag{3}$$

the last equality being a definition of the replica Hamiltonian

$$-\beta\mathcal{H}^{(n)}=\beta J_0 z^{-1}\Bigg\{\sum_{\mathbf{n},\mu}\sum_a\cos\Delta_\mu\phi^a(\mathbf{n}) +\frac{J^2\beta}{2J_0}\sum_{\mathbf{n},\mu}\sum_{a,b}\cos\Delta_\mu\phi^a(\mathbf{n})\cos\Delta_\mu\phi^b(\mathbf{n})\Bigg\}. \tag{4}$$

III. HELICITY MODULUS IN THE MEAN-FIELD APPROXIMATION

At this point it is useful to introduce the concept of the helicity modulus—the straightforward connection between the helicity modulus and the magnetic penetration length will be given at the end of this section.

A definition best suited to our method is the original definition of Fisher, Barber, and Jasnow:[12]

$$\Upsilon(T)=\lim_{L\to\infty}\frac{2L^2}{\pi^2}[\mathcal{F}^-(T;L)-\mathcal{F}^+(T;L)], \tag{5}$$

where $\mathcal{F}^+(T;L)$ [or $\mathcal{F}^-(T;L)$] is the free energy per particle of an infinite system subject to periodic (or antiperiodic) boundary conditions of periodicity L along one direction. $\Upsilon(T)$ is a macroscopic variable measuring the stiffness of the system to an externally imposed twist in the phases by means of antiperiodic boundary conditions. At zero temperature, the difference in the right-hand side (rhs) of Eq. (5) is just the domain-wall energy. At finite temperature it is the free energy of a domain wall. Therefore a finite $\Upsilon(T)$ indicates a phase-coherent state, whereas a vanishing $\Upsilon(T)$ indicates a phase incoherent, or thermally disordered, state.

We adopt definition (5) even for systems with quenched disorder, implicitly assuming that the disorder ensemble averaging is performed within both $\mathcal{F}^+$ and $\mathcal{F}^-$. This can be done as in the previous section by going into the replica space. In order to evaluate $\mathcal{F}^-$, we apply antiperiodic boundary conditions to every replica along all spatial directions. This is merely for convenience. In any case, we are allowed to do so for isotropic systems (the replica Hamiltonian is isotropic, even though perhaps some disorder realizations are not).

In the ground state of the replica Hamiltonian the phases $\phi^a(\mathbf{n})$ absorb the twist imposed by the boundary conditions uniformly, i.e.,

$$\phi^a(\mathbf{n})=\phi_0+(n_x+n_y+n_z)\varepsilon, \tag{6}$$

where $\varepsilon=\pi/L$. It is convenient to choose the constant $\phi_0=0$. With this choice, $\cos[\phi^a(\mathbf{n})-(n_x+n_y+n_z)\varepsilon]=1$ and $\sin[\phi^a(\mathbf{n})-(n_x+n_y+n_z)\varepsilon]=0$. We expect therefore that at any *finite* temperature $\langle\cos\theta^a(\mathbf{n})\rangle=\gamma$ and $\langle\sin\theta^a(\mathbf{n})\rangle=0$, where we defined $\theta^a(\mathbf{n})\equiv\phi^a(\mathbf{n})-(n_x+n_y+n_z)\varepsilon$ to be the angle measured in a rotating reference frame. Here γ is a constant independent of the site index $\mathbf{n}$, which allows for a standard mean-field treatment. After substitution the replica Hamiltonian (4) becomes

$$-\beta\mathcal{H}_\varepsilon^{(n)}=\beta J_0 z^{-1}\Bigg\{\sum_{\mathbf{n},\mu}\sum_a\cos[\Delta_\mu\theta^a(\mathbf{n})+\varepsilon] +\frac{J^2\beta}{2J_0}\sum_{\mathbf{n},\mu}\sum_{a,b}\cos[\Delta_\mu\theta^a(\mathbf{n})+\varepsilon]\times\cos[\Delta_\mu\theta^b(\mathbf{n})+\varepsilon]\Bigg\}. \tag{7}$$

The usual mean-field ansatz $0\approx(A-\langle A\rangle)(B-\langle B\rangle)$, where A and B are arbitrary operators and $\langle\cdots\rangle$ denotes thermal averaging, now leads to

$$\cos[\theta^a(\mathbf{n}+\hat{\mu})-\theta^a(\mathbf{n})+\varepsilon]\approx[\gamma\cos\theta^a(\mathbf{n})+\gamma\cos\theta^a(\mathbf{n}+\hat{\mu})-\gamma^2]\cos\varepsilon, \tag{8}$$

after which the first term becomes

$$\sum_{\mathbf{n},\mu}\sum_a\cos[\theta^a(\mathbf{n}+\hat{\mu})-\theta^a(\mathbf{n})+\varepsilon]\approx z\cos\varepsilon\sum_{\mathbf{n}}\sum_a\gamma[\cos\theta^a(\mathbf{n})-\gamma/2]. \tag{9}$$

We further define $c_{ab}=\langle \cos\theta^a(\mathbf{n})\cos\theta^b(\mathbf{n})\rangle$ and $s_{ab}=\langle \sin\theta^a(\mathbf{n})\sin\theta^b(\mathbf{n})\rangle$. Finally, the mean-field replica Hamiltonian corresponding to the system with a twist becomes

$$-\beta\mathcal{H}_\varepsilon^{(n)}=\frac{1}{\mathcal{T}}\sum_{\mathbf{n}}\Bigg\{\cos\varepsilon\sum_a \gamma[\cos\theta^a(\mathbf{n})-\gamma/2] +\frac{\delta^2}{2\mathcal{T}}\sum_{ab}\Bigg(\left[\cos\theta^a(\mathbf{n})\cos\theta^b(\mathbf{n})-\frac{1}{2}c_{ab}\right] \times(c_{ab}\cos^2\varepsilon+s_{ab}\sin^2\varepsilon) +\left[\sin\theta^a(\mathbf{n})\sin\theta^b(\mathbf{n})-\frac{1}{2}s_{ab}\right] \times(c_{ab}\sin^2\varepsilon+s_{ab}\cos^2\varepsilon)\Bigg)\Bigg\}, \tag{10}$$

where $\mathcal{T}\equiv 1/(\beta J_0)$ is the dimensionless temperature and $\delta\equiv J/J_0$ is a dimensionless parameter, which we will refer to as the disorder strength.

Denoting L^D the volume of the cube of linear extent L, the resulting free energy per particle

$$\mathcal{F}_\varepsilon(\gamma,\{c_{ab}\},\{s_{ab}\})=-\frac{1}{\beta}\lim_{\substack{n\to 0\\ L\to\infty}}\frac{1}{nL^D}[\mathrm{Tr}_\theta\exp(-\beta\mathcal{H}_\varepsilon^{(n)})-1] \tag{11}$$

is a function of ε and variational parameters γ, $\{c_{ab}\}$, and $\{s_{ab}\}$, which satisfy saddle-point equations

$$0=\frac{\partial\mathcal{F}_\varepsilon}{\partial\gamma}=\frac{\partial\mathcal{F}_\varepsilon}{\partial c_{ab}}=\frac{\partial\mathcal{F}_\varepsilon}{\partial s_{ab}}, \tag{12}$$

which are equivalent to the mean-field self-consistency conditions. In the thermodynamic limit $\varepsilon\to 0$, and the helicity modulus can be written as a partial derivative

$$\Upsilon=\frac{2}{z}\left(\frac{\partial^2\mathcal{F}_\varepsilon}{\partial\varepsilon^2}\right)_{\varepsilon=0} =-\frac{2}{\beta z}\lim_{\substack{n\to 0\\ L\to\infty}}\frac{1}{nL^D}\mathrm{Tr}_\theta\left(\frac{\partial^2}{\partial\varepsilon^2}\exp[-\beta\mathcal{H}_\varepsilon^{(n)}]\right)_{\varepsilon=0}, \tag{13}$$

where ε is now regarded as a continuous variable, and the factor $2/z$ appears because we apply the twist in all spatial directions instead of just one.

Performing the suggested differentiation, while holding γ, $\{c_{ab}\}$, and $\{s_{ab}\}$ constant, we arrive at

$$\Upsilon/J_0=\gamma^2+\frac{\delta^2}{\mathcal{T}}\lim_{n\to 0}\frac{1}{n}\sum_{ab}(c_{ab}-s_{ab})^2. \tag{14}$$

Here the expectation values γ, $\{c_{ab}\}$, and $\{s_{ab}\}$ are to be calculated at $\varepsilon=0$, i.e., without any imposed twist.

We now make a connection between the helicity modulus expressed above as a function of mean-field order parameters of the model, and the magnetic penetration length, which we ultimately want to calculate. We note that the twist angle ε enters the Hamiltonian in the same way as a uniform external vector potential $\mathbf{A}$, if such were applied to the system. The correspondence is $\varepsilon\leftrightarrow(2\pi/\phi_0)\int_{\mathbf{n}}^{\mathbf{n}+\hat{\mu}}\mathbf{A}\cdot d\mathbf{l}$. Since in general

$$\frac{1}{4\pi\lambda^2(T)}=\frac{2}{z}\sum_\mu\left(\frac{\partial^2\mathcal{F}}{\partial A_\mu^2}\right)_{\mathbf{A}=0}, \tag{15}$$

we have

$$\frac{1}{4\pi\lambda^2(T)}=\left(\frac{\phi_0}{2\pi}\right)^2\Upsilon(T). \tag{16}$$

In the following sections we will analyze the saddle-point equations (12) in the limit $n\to 0$, first in the replica-symmetric (RS) ansatz, then in the replica symmetry-breaking (RSB) ansatz.

IV. REPLICA-SYMMETRIC SOLUTION

In this case we assume perfect permutational symmetry of replicas. The thermal averages c_{ab} and s_{ab} are denoted as follows:

$$c_{aa}=1/2+\tilde{c}/2,\qquad s_{aa}=1/2-\tilde{c}/2,\qquad c_{ab}=c,\quad s_{ab}=s\quad a\neq b. \tag{17}$$

In the Hamiltonian (10) (with $\varepsilon=0$) we complete the squares as follows:

$$\sum_{ab}\left(\cos\theta^a\cos\theta^b-\frac{1}{2}c_{ab}\right)c_{ab} =c\left(\sum_a\cos\theta^a\right)^2+(1/2+\tilde{c}/2-c)\sum_a\cos^2\theta^a -\frac{1}{2}\frac{(1+\tilde{c})^2}{4}n-\frac{1}{2}c^2n(n-1), \tag{18}$$

and likewise for the sines. Integration over the angles is made possible by a repeated use of the identity

$$e^{ax^2}=\frac{1}{\sqrt{2\pi}}\int d\lambda\, e^{-\lambda^2/2+\sqrt{2a}\lambda x}. \tag{19}$$

After taking the $n\to 0$ limit we arrive at

$$\mathcal{F}/J_0=\frac{1}{2}\gamma^2-\frac{\delta^2}{2\mathcal{T}}\left[\frac{1}{2}c^2+\frac{1}{2}s^2-\frac{1}{4}(1+\tilde{c})^2-s+\frac{1}{2}\right] -\mathcal{T}\int d\lambda\, G_{\delta^2 c}(\lambda)\int d\sigma\, G_{\delta^2 s}(\sigma)f(\lambda,\sigma), \tag{20}$$

where $G_\xi(u)\equiv(2\pi\xi)^{-1/2}\exp(-u^2/2\xi)$, and

$$f(\lambda,\sigma)=\ln\int_{-\pi}^{\pi}d\theta\exp\left\{\frac{1}{\mathcal{T}}[(\gamma+\lambda)\cos\theta+\sigma\sin\theta] +\frac{\delta^2}{2\mathcal{T}^2}(\tilde{c}-c+s)\cos^2\theta\right\}. \tag{21}$$

In order to locate the critical point(s) and the mean-field critical behavior of this system, we first assume that the order parameters γ^2, c, s, and $\tilde{c}$, are all small in the neighborhood of a critical point. We next expand the free-energy functional (20) to third order in these variables. The critical temperature $\mathcal{T}_c$ is found as the temperature below which the free energy has a saddle point for nontrivial (i.e., nonzero) values of some or all of the order parameters. It turns out that we have to treat separately the cases $\delta<1$ (weak disorder) and $\delta\geqslant 1$ (strong disorder). Our results are summarized in the following table (expansions are in the variable $\tau=1-\mathcal{T}/\mathcal{T}_c$):

	$\delta<1$	$\delta\geqslant 1$
$\mathcal{T}_c$	1/2	$\delta/2$
γ^2	$\tau(2-3\delta^2+\delta^4)/(1+2\delta^2-\delta^4)+\mathcal{O}(\tau^2)$	0
c	$\tau(2-\delta^2)/(1+2\delta^2-\delta^4)+\mathcal{O}(\tau^2)$	$\tau/2+\mathcal{O}(\tau^2)$
s	$\mathcal{O}(\tau^2)$	$\tau/2+\mathcal{O}(\tau^2)$
$\tilde{c}$	$\tau(1-\delta^2)/(1+2\delta^2-\delta^4)+\mathcal{O}(\tau^2)$	0.

This set of data reveals a peculiarity: if one were to imagine a process in which the disorder strength δ is being continuously tuned from a value below 1 to a value above 1, the behavior of order parameters should continuously change from that in the left column to that in the right column (unless there is a discontinuous phase transition associated with a discontinuous change of symmetry). The s-order parameter behaves differently at first sight. Its power-law expansion in the neighborhood of the critical line changes from at least quadratic in τ for $\delta<1$ to linear in τ for $\delta\geqslant 1$. The obvious escape is that the relation $s=\mathcal{O}(\tau^2)$ is invalid exactly at $\delta=1$, but since it holds for any $\delta<1$, the region of its validity must be continuously shrinking to zero as δ approaches 1 from below. This behavior follows naturally if, for $\delta<1$, there is another line of criticality $\mathcal{T}_s$, lying below $\mathcal{T}_c$, and which runs into $\mathcal{T}_c$ exactly at $\delta=1$ (see Fig. 1). Continuing this line of thought even further, if $s=\mathcal{O}(\tau_s)$, $\tau_s\equiv 1-\mathcal{T}/\mathcal{T}_s$, just below $\mathcal{T}_s$, we must have $s\equiv 0$ everywhere between $\mathcal{T}_s$ and $\mathcal{T}_c$. This is of course consistent with $s=\mathcal{O}(\tau^2)$ just below $\mathcal{T}_c$.

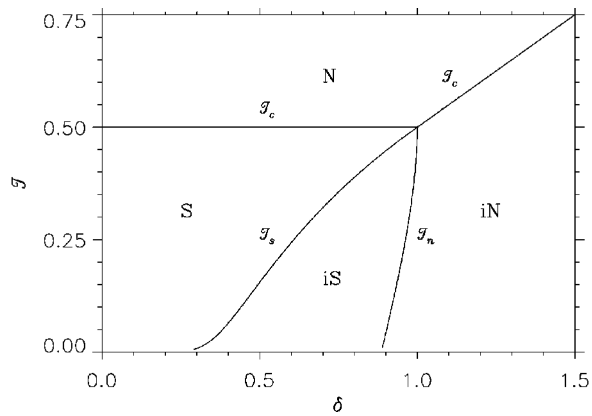

FIG. 1. Mean-field phase diagram of a model granular superconductor with π junctions. N is the normal phase, S is the reversible superconducting phase, iS is the irreversible superconducting glass phase with broken ergodicity, and iN is the normal phase with broken ergodicity. The $\mathcal{T}_n$ phase boundary has been calculated in the RS ansatz, and is therefore approximate.

The reason why this transition line did not appear in our previous analysis is that near $\mathcal{T}_s$ only the s order parameter can be considered small, while the others, γ^2, c, and $\tilde{c}$, are "large." Therefore, in order to obtain the functional form of $\mathcal{T}_s(\delta)$, we have to treat them exactly, and expand only with respect to s.

It is somewhat more convenient to expand the saddle-point equation $\partial\mathcal{F}/\partial s=0$ to second order in s, than to expand $\mathcal{F}$ to third order in s and then differentiate. Either procedure yields the saddle-point equation for s,

$$0=s\left\{1-2\left(\frac{\delta^2}{2\mathcal{T}^2}\right)\int d\lambda\, G_{\delta^2 c}(\lambda)h_2^2(\mathcal{T},\lambda)+8s\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^2\int d\lambda\, G_{\delta^2 c}(\lambda)h_2^3(\mathcal{T},\lambda)+\mathcal{O}(s^2)\right\}. \tag{22}$$

Here we have introduced a new function,

$$h_2(\mathcal{T},\lambda)\equiv\frac{\int_{-\pi}^{\pi}d\theta\sin^2\theta\exp\{1/\mathcal{T}(\gamma+\lambda)\cos\theta+\delta^2/(2\mathcal{T}^2)(\tilde{c}-c)\cos^2\theta\}}{\int_{-\pi}^{\pi}d\theta\exp\{1/\mathcal{T}(\gamma+\lambda)\cos\theta+\delta^2/(2\mathcal{T}^2)(\tilde{c}-c)\cos^2\theta\}}, \tag{23}$$

for brevity. In this expression γ, c, and $\tilde{c}$ satisfy the saddle-point equations

$$\partial\mathcal{F}/\partial\gamma=\partial\mathcal{F}/\partial c=\partial\mathcal{F}/\partial\tilde{c}=0. \tag{24}$$

When the zeroth-order term in the brackets of Eq. (22) is positive, the solution is trivial, $s=0$. The critical temperature $\mathcal{T}_s$ is found as the point where this term vanishes, which yields implicitly for $\mathcal{T}_s$:

$$1=\frac{\delta^2}{\mathcal{T}_s^2}\int d\lambda\, G_{\delta^2 c}(\lambda)h_2^2(\mathcal{T}_s,\lambda)\quad(\delta<1). \tag{25}$$

Equation (25) for $\mathcal{T}_s$, along with Eqs. (24) and $s=0$, can be solved numerically; the result is presented in Fig. 1.

We can now determine the behavior of all order parameters just below the $\mathcal{T}_s$ line, to first order in τ_s. To this accuracy the order parameters γ, c, and $\tilde{c}$ will be determined by Eqs. (24), with $s=0$. This is seen from the expansion of the free-energy functional $\mathcal{F}$ in powers of s: this expansion does not contain a term linear in s, since $\partial\mathcal{F}/\partial s=0$ at $\mathcal{T}_s$. Therefore, expressions (24), with $s=0$, will provide values of γ, c, and $\tilde{c}$ correct to $\mathcal{O}(\tau_s^2)$ near $\mathcal{T}_s$.

Combining Eq. (22) with Eq. (25), we find

$$s=\left(\frac{\mathcal{T}_s^2}{2\delta^2}\right)\frac{\int d\lambda\, G_{\delta^2 c}(\lambda)[h_2^2(\mathcal{T},\lambda)-h_2^2(\mathcal{T}_s,\lambda)]}{\int d\lambda\, G_{\delta^2 c}(\lambda)h_2^3(\mathcal{T}_s,\lambda)}+\mathcal{O}(\tau_s^2). \tag{26}$$

We have expressed s in terms of known quantities [recall that γ, c, and $\tilde{c}$ are determined *independently* of s to $\mathcal{O}(\tau_s^2)$ accuracy].

Our analysis of the replica-symmetric solution is not complete, however. At least in the RS ansatz there is another, reentrant, phase transition *below* the $\mathcal{T}_s$ line. This line, which we denote $\mathcal{T}_n$, is the line of a reentrant phase transition back into the normal state, where the primary order parameter γ vanishes. In order to locate $\mathcal{T}_n$, we impose a stronger condition

$$\lim_{\gamma\to 0^+}\frac{1}{\gamma}\frac{\partial\mathcal{F}}{\partial\gamma}=0, \tag{27}$$

which explicitly excludes the trivial solution $\gamma\equiv 0$ from our consideration. This condition, along with the usual

$$\partial\mathcal{F}/\partial c=\partial\mathcal{F}/\partial s=\partial\mathcal{F}/\partial\tilde{c}=0, \tag{28}$$

leads to a closed-form expression for $\mathcal{T}_n$:

$$\frac{1}{2}-\mathcal{T}_n=\frac{1}{2}\int_0^\infty d\lambda\,\lambda\, e^{-\lambda^2/2}\left[\frac{I_1(\delta/\mathcal{T}_n\sqrt{1/2-\mathcal{T}_n}\lambda)}{I_0(\delta/\mathcal{T}_n\sqrt{1/2-\mathcal{T}_n}\lambda)}\right]^2, \tag{29}$$

where $I_0(x)$ and $I_1(x)$ are modified Bessel functions. For completness, $s=c=1/2-\mathcal{T}_n$, and $\tilde{c}=\gamma=0$ at $\mathcal{T}_n$. Equation (29) can be solved numerically for $\mathcal{T}_n$, with the result presented in Fig. 1.

V. REPLICA SYMMETRY-BROKEN SOLUTION

It is a rule rather than an exception that in solving problems with quenched disorder with replicas, one ends up breaking the replica symmetry in some region of the phase diagram. The reason is the instability of the free-energy functional to small deviations away from the RS solution.[13] The stability of the RS solution is indicated by the eigenvalues of the reduced Hessian,

$$S_{ij}=\frac{\partial^2}{\partial X_i\partial X_j}\mathcal{F}_\varepsilon(\gamma,\{c_{ab}\},\{s_{ab}\}), \tag{30}$$

where $X_i=c_{ab}$ or s_{ab}, $a\neq b$, and $\varepsilon=0$. Differentiation and eigenvalue calculation in Eq. (30) must be done before the limit $n\to 0$ is taken. All eigenvalues of the Hessian must be positive in order that the RS solution be stable.

Here we adopt a more pragmatic (and not particularly rigorous) approach, in which replica symmetry breaking is allowed from the outset, by working with a larger set of order parameters. In addition to the ubiquitous RS solution, at low temperatures there will also appear a nontrivial RSB solution. In such a case we simply assume that the RSB solution is the stable one.

At first we allow one-step replica symmetry breaking (1-RSB), known to produce quantitatively correct results for a broad range of temperatures (except near zero) in the theory of the Ising spin glass.[14] In this approach the n replicas are divided into n/m blocks of m replicas in each block. The thermal averages c_{ab} and s_{ab} then assume a small number of values:

$$c_{aa}=1/2+\tilde{c}/2,$$

$$s_{aa}=1/2-\tilde{c}/2,$$

$$c_{ab}=c_1,\quad s_{ab}=s_1\quad a\neq b,\quad \{a,b\}\in\text{block},$$

$$c_{ab}=c_0,\quad s_{ab}=s_0\quad a\neq b,\quad \{a,b\}\notin\text{block}. \tag{31}$$

In the Hamiltonian (10) (with $\varepsilon=0$) we now complete the squares as follows:

$$\begin{aligned}\sum_{ab}&\left(\cos\theta^a\cos\theta^b-\frac{1}{2}c_{ab}\right)c_{ab}\\ &=c_0\left(\sum_a\cos\theta^a\right)^2+(c_1-c_0)\sum_{k=1}^{n/m}\left(\sum_{a\in\text{block}(k)}\cos\theta^a\right)^2\\ &\quad+\left(\frac{1+\tilde{c}}{2}-c_1\right)\sum_a\cos^2\theta^a-\frac{1}{2}\frac{(1+\tilde{c})^2}{4}n\\ &\quad-\frac{1}{2}c_1^2 n(m-1)-\frac{1}{2}c_0^2 n(n-m),\end{aligned} \tag{32}$$

and likewise for the sines. Integration over the angles is again facilitated by the identity (19). In the end, after taking the $n\to 0$ limit,

$$\mathcal{F}/J_0=\frac{1}{2}\gamma^2-\frac{\delta^2}{2\mathcal{T}}\left[\frac{1}{2}c_1^2(1-m)+\frac{1}{2}c_0^2m+\frac{1}{2}s_1^2(1-m)+\frac{1}{2}s_0^2m-\frac{(1+\tilde{c})^2}{4}-s_1+\frac{1}{2}\right]$$
$$-\mathcal{T}\int d\lambda_0G_{\delta^2c_0}(\lambda_0)\int d\sigma_0G_{\delta^2s_0}(\sigma_0)\frac{1}{m}\ln\int d\lambda_1G_{\delta^2(c_1-c_0)}(\lambda_1-\lambda_0)\int d\sigma_1G_{\delta^2(s_1-s_0)}(\sigma_1-\sigma_0)\exp[mf_1(\lambda_1,\sigma_1)], \tag{33}$$

where

$$f_1(\lambda,\sigma)=\ln\int_{-\pi}^{\pi}d\theta\exp\left\{\frac{1}{\mathcal{T}}[(\gamma+\lambda)\cos\theta+\sigma\sin\theta]+\frac{\delta^2}{2\mathcal{T}^2}(\tilde{c}-c_1+s_1)\cos^2\theta\right\}. \tag{34}$$

The method just described can be used repeatedly to include multiple-step replica symmetry breaking, and generalizes to a case of continuous (or infinite-step) replica symmetry breaking.[15,16] In this latter case the c and s order parameters become functions $c(x)$ and $s(x)$ on the interval [0,1], and provided these functions are continuous, the free energy acquires the form

$$\mathcal{F}/J_0=\frac{1}{2}\gamma^2-\frac{\delta^2}{2\mathcal{T}}\times\left[\frac{1}{2}\int_0^1dx[c^2(x)+s^2(x)]-\frac{(1+\tilde{c})^2}{4}-s(1)+\frac{1}{2}\right]-\mathcal{T}\int d\lambda G_{\delta^2c(0)}(\lambda)\int d\sigma G_{\delta^2s(0)}(\sigma)f(0|\lambda,\sigma), \tag{35}$$

where function $f(x|\lambda,\sigma)$ satisfies a nonlinear partial differential equation

$$\frac{\partial f}{\partial x}=-\frac{\delta^2}{2}\frac{dc(x)}{dx}\left[\frac{\partial^2f}{\partial\lambda^2}+x\left(\frac{\partial f}{\partial\lambda}\right)^2\right]-\frac{\delta^2}{2}\frac{ds(x)}{dx}\left[\frac{\partial^2f}{\partial\sigma^2}+x\left(\frac{\partial f}{\partial\sigma}\right)^2\right], \tag{36}$$

and a boundary condition

$$f(1|\lambda,\sigma)=f_1(\lambda,\sigma). \tag{37}$$

The free energy of the system at any temperature can be found from Eq. (35), where the parameters satisfy the saddle-point equations

$$0=\frac{\partial\mathcal{F}}{\partial\gamma}=\frac{\partial\mathcal{F}}{\partial\tilde{c}}=\frac{\delta\mathcal{F}}{\delta c(x)}=\frac{\delta\mathcal{F}}{\delta s(x)}. \tag{38}$$

The set of equations (35)–(38) defies all attempts to a closed-form solution, and approximations have to be made. As a rule, analytical results for $c(x)$ and $s(x)$ can be obtained only in the neighborhood of the critical point.[17] On the other hand, if one were to obtain an ‘‘exact’’ numerical solution valid in the entire temperature range, an ingenious scheme has been designed and used by Sommers and Dupont.[18]

Here we content ourselves with the solution of a 1-RSB model (33) and (34), by solving

$$0=\frac{\partial\mathcal{F}}{\partial\gamma}=\frac{\partial\mathcal{F}}{\partial\tilde{c}}=\frac{\partial\mathcal{F}}{\partial c_1}=\frac{\partial\mathcal{F}}{\partial c_0}=\frac{\partial\mathcal{F}}{\partial s_1}=\frac{\partial\mathcal{F}}{\partial s_0}=\frac{\partial\mathcal{F}}{\partial m}. \tag{39}$$

Again, we can solve these equations only in the neighborhood of the critical point, but this is to no detriment as this is precisely where they are expected to be valid.

Leaving computational details for the Appendix, we summarize that a nontrivial 1-RSB solution exists whenever the replica-symmetric order parameter s is nonzero, i.e., below the $\mathcal{T}_s$ line for $\delta<1$, and below the $\mathcal{T}_c$ line for $\delta\geqslant1$. This is in qualitative agreement with the complete analysis of the Hessian for a similar, infinite-ranged m-vector spin glass in a field,[19,20] according to which RSB is indicated simultaneously with the onset of the transverse order parameter. Analytical results for the 1-RSB essential for the present discussion are summarized below:

	c_1-c_0	s_1	s_0	m
$\delta<1$	$\mathcal{O}(\tau_s^2)$	$\mathcal{O}(\tau_s)$	$\mathcal{O}(\tau_s)$	$\mathcal{O}(\tau_s)$
$\delta\geqslant1$	$\mathcal{O}(\tau_c)$	$\mathcal{O}(\tau_c)$	$\mathcal{O}(\tau_c)$	$\mathcal{O}(\tau_c)$.

This table shows that the mathematical expression of the 1-RSB changes qualitatively at $\delta=1$. This alone, however, does not mean there must be another line of criticality below $\mathcal{T}_s$, as was the case in the previous section. Results for $\delta<1$ in this table were obtained assuming $\mathcal{O}(c_1)=1$, which breaks down at $\delta=1$ [the denominator in Eq. (A5) vanishes]. Nonetheless, the $\mathcal{T}_n$ line obtained in the RS ansatz does have this bias, see Fig. 1. It is clear that such a line must be present in the exact solution as well, separating the superconducting and the normal glass phases, although it may be vertical.

VI. HELICITY MODULUS AND IRREVERSIBILITY

The helicity modulus (14) becomes, in the RS ansatz (17),

$$\Upsilon^{\mathrm{RS}}/J_0=\gamma^2+\frac{\delta^2}{\mathcal{T}}[\tilde{c}^2-(c-s)^2]. \tag{40}$$

A numerical solution of Υ^{RS} as function of temperature and disorder is presented in Fig. 2. Immediately apparent are the nonanalytical points in Υ^{RS}, which mark the transition between the reversible superconductor and the superconducting

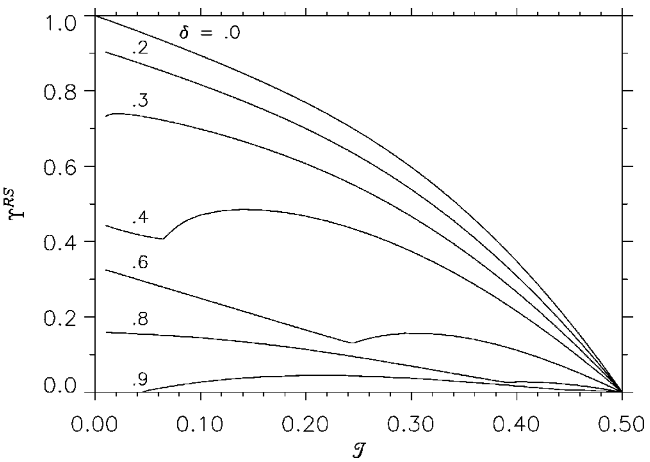

FIG. 2. Helicity modulus $\Upsilon^{\rm RS}(\mathcal{T})$, parametrized by disorder strength δ.

glass. This behavior resembles the nonanalyticity of magnetic susceptibility of spin glasses.[10,11] Mathematically it follows from the fact that $s=0$ above $\mathcal{T}_s$, and $s=\mathcal{O}(\tau_s)$ below $\mathcal{T}_s$. Another remarkable property is the positive slope of $\Upsilon^{\rm RS}(\mathcal{T})$ within the reversible phase, in the neighborhood of the glass transition temperature $\mathcal{T}_s$.

In the continuous RSB scenario

$$\Upsilon/J_0=\gamma^2+\frac{\delta^2}{\mathcal{T}}\left\{\tilde{c}^2-\int_0^1 dx[c(x)-s(x)]^2\right\}. \qquad (41)$$

This is the Gibbs average over the entire phase space, which is often designated as the field-cooled average in the context of spin glasses. There is a difference, however, between the ''field'' in the spin-glass context, and the ''field'' in the context of superconductivity. While in the former case the magnetic field **H** couples directly to the order parameter (magnetization), in the latter case the coupling to the superconducting order parameter (complex scalar wave function) is minimal through the vector potential **A**. One therefore hesitates to use the term ''field-cooled'' for the Gibbs average in the present context. In fact, cooling a disordered superconductor in a magnetic field and subsequently turning off the field would lead to trapped magnetic flux and excess vortices inside the sample, which is out of equilibrium. By contrast, a spin glass is a nonergodic system *in equilibrium*. In the next section we will discuss possible ways to measure the Gibbs averaged helicity modulus of a granular superconductor.

In the phase with broken ergodicity, manifested mathematically by the breaking of replica symmetry, the system explores only one thermodynamic state—one ''valley'' of the complex free-energy landscape. Therefore, we also need to calculate the single-state average as another experimentally relevant quantity. Following Parisi's interpretation,[21] the single-state average Υ_1 is obtained when, instead of integrating over the entire interval [0,1] as in Eq. (41), one uses only values at $x=1$, i.e., $c(1)$ and $s(1)$:

$$\Upsilon_1/J_0=\gamma^2+\frac{\delta^2}{\mathcal{T}}\{\tilde{c}^2-[c(1)-s(1)]^2\}. \qquad (42)$$

In the 1-RSB ansatz (31), the analog of Eq. (41) reads

$$\Upsilon/J_0=\gamma^2+\frac{\delta^2}{\mathcal{T}}[\tilde{c}^2-m(c_0-s_0)^2-(1-m)(c_1-s_1)^2]. \qquad (43)$$

This is the Gibbs-averaged value of Υ. In the same ansatz the values entering the single-state average, i.e., $c(1)$ and $s(1)$, are approximated by c_1 and s_1, respectively, and the single-state-averaged helicity modulus is

$$\Upsilon_1/J_0=\gamma^2+\frac{\delta^2}{\mathcal{T}}[\tilde{c}^2-(c_1-s_1)^2]. \qquad (44)$$

One can see by inspection that both Υ and Υ_1 vanish when $\delta\geq 1$. Therefore we have to classify both phases, N and iN, see Fig. 1, as *normal* (nonsuperconducting). Phase iN has broken ergodicity, and must be irreversible in some sense, although the helicity modulus is insensitive to it. This is apparently a consequence of the fact that the $U(1)$ symmetry is not broken at $\mathcal{T}_c$ (since $\gamma\equiv 0$).

For $\delta<1$ the U(1) symmetry is broken at $\mathcal{T}_c$. The sample is in a superconducting state S with no trace of glassiness for $\mathcal{T}_s<\mathcal{T}<\mathcal{T}_c$ (replica symmetry is unbroken), and the helicity modulus is given by Eq. (40). Below $\mathcal{T}_s$, however, *both* U(1) and replica symmetries are broken, and we have a nonergodic, irreversible, glassy superconductor iS. When measuring the helicity modulus of the iS phase, one can, in principle, distinguish between the single-state averaged and the Gibbs-averaged Υ. Near the $\mathcal{T}_s$ line the difference $\Upsilon_1-\Upsilon=\mathcal{O}(\tau_s^2)$. More specifically,

$$(\Upsilon_1-\Upsilon)/J_0=2\frac{\delta^2}{\mathcal{T}_s}c_1 m(s_1-s_0)+\mathcal{O}(\tau_s^3). \qquad (45)$$

The significance of this result is that *the existence of the glassy superconducting state in granular superconductors can be probed by measuring the penetration length* $\lambda(T)$: the single-state average Υ_1 and the Gibbs average Υ will appear as two branches of the $1/\lambda^2(T)$ vs T curve. The single-state average branch is the ''stiffer'' of the two, corresponding to a shorter penetration length. At $\mathcal{T}_s$ the two branches merge smoothly with the RS solution: by inspection of the 1-RSB solution for s_1, Eq. (A2), and the RS solution for s, Eq. (26), one can see that $s_1=s+\mathcal{O}(\tau_s^2)$. Hence $\Upsilon^{\rm RS}$, Υ_1, and Υ are all equal to $\mathcal{O}(\tau_s^2)$. The nonanalytical point in the temperature dependence of the helicity modulus remains the characteristic of the S-iS transition in the 1-RSB solution, and we believe the same is true of the exact solution.

At the reentrant transition at $\mathcal{T}_n$ the helicity modulus vanishes, and the sample becomes normal (and remains glassy). The RS analysis suggests that this is a continuous transition.

VII. DISCUSSION AND CONCLUSIONS

We have analyzed a model disordered granular superconductor with π junctions in a mean-field replica approximation. Quenched disorder as well as randomness in the sign of the couplings between superconducting grains are realized by working with an ensemble, in which each coupling strength is drawn randomly from a Gaussian distribution. The width-to-average ratio δ of this distribution serves as the characteristic of disorder.

There is a close correspondence between the present model and the previously mentioned m-vector spin glass in a field,[22–24] defined by the Hamiltonian

$$\mathcal{H}=-\sum_{\mathbf{n},\mathbf{n}'} J_{\mathbf{n},\mathbf{n}'}\mathbf{S}(\mathbf{n})\cdot\mathbf{S}(\mathbf{n}')-\mathbf{H}\cdot\sum_{\mathbf{n}}\mathbf{S}(\mathbf{n}). \tag{46}$$

In this Hamiltonian the first summation is over all pairs $\mathbf{n},\mathbf{n}'$, and the distribution $P(J_{\mathbf{n},\mathbf{n}'})$ is a zero-centered Gaussian. Spins $\mathbf{S}(\mathbf{n})$ have m components and a fixed length, $\mathbf{S}^2(\mathbf{n})=m$. Case $m=2$ is the random-bond XY model in a "field," and is relevant to our discussion.

When $\mathbf{S}(\mathbf{n})$ represent magnetic moments, as understood in the spin-glass context, $\mathbf{H}$ is the usual external magnetic field. If $\mathbf{S}(\mathbf{n})$ were to represent phases of a superconductor, as in this work, $\mathbf{H}$ in Eq. (46) would represent a fictitious field, for which there is no experimental realization. This is why the second term in Eq. (46) does not appear in our Hamiltonian (1). In fact, we assumed zero applied magnetic field throughout this paper. This simply reflects our inability to solve any XY-type models with minimal coupling to the field, since infinitesimal changes of the field bring about qualitative changes to the Hamiltonian.[25,26]

The most interesting common feature of the present model and the m-vector model is the separate U(1) and ergodicity breaking for weak disorder. In both models the U(1) symmetry breaking occurs first: in the m-vector model (45) it is the external field, which is responsible for the U(1) symmetry breaking, while in model (1) this symmetry is broken spontaneously when the "on average" ferromagnetic coupling overcomes thermal effects. At a lower temperature $\mathcal{T}_s$, from within the ordered phase there emerges a phase with broken ergodicity, a superconducting glass (the iS phase) in the present model, and the "canted ferromagnet" in the m-vector model. The analog of the $\mathcal{T}_s$ line of the present model is the Gabay-Toulouse line[22] of the m-vector model.

There is one, we believe important, difference in how these two models are treated mathematically. In the standard solution of the m-vector model one makes an, in general, nonequivalent substitution in the partition sum,

$$\mathrm{Tr}_{\mathbf{S}^a(\mathbf{n})}\equiv\int\left[\prod_{\mu=1}^{m} dS_\mu^a(\mathbf{n})\right]\delta(\mathbf{S}^a(\mathbf{n})\cdot\mathbf{S}^a(\mathbf{n})-m)$$
$$\rightarrow\int\prod_{\mu=1}^{m} dS_\mu^a(\mathbf{n}), \tag{47}$$

plus a much weaker constraint $\langle\mathbf{S}^a(\mathbf{n})\cdot\mathbf{S}^a(\mathbf{n})\rangle=m$. In other words, the fixed length constraint is enforced only on average, which may be justified for large m, where there are $m-1$ Euler angles per spin and the analysis becomes difficult, but it is suspect for $m=2$ or 3. In this work $(m=2)$ we work with the polar angles $\theta^a(\mathbf{n})$. The fixed length constraint is thus always in place, not just on average. We believe that this property is important for the calculation of the helicity modulus, which probes the response to a long-wavelength twist of the phase.

We found that the iS phase of a granular superconductor is accessible for detection using a penetration length probe, Υ. At $\mathcal{T}_s$ there is a discontinuity of the first derivative of $\Upsilon(\mathcal{T})$. Just above $\mathcal{T}_s$ the helicity modulus *increases* with temperature. This is made possible by the interplay between thermal fluctuations, randomness of the effective coupling in the direction transverse to the direction of the primary order parameter γ, and the fixed length constraint, which in the language of phase variables reads simply $\cos^2\theta(\mathbf{n})+\sin^2\theta(\mathbf{n})=1$. Below $\mathcal{T}_s$ one, in principle, measures two values of Υ; the Gibbs averaged and the single-state averaged. A simple slow cooling of a sample in zero magnetic field from the normal state to the superconducting state will prepare a sample equilibrated *within its ergodicity space*. Below $\mathcal{T}_s$, a measurement on such a sample will produce the single-thermodynamic-state average Υ_1. The Gibbs average is less straightforward to obtain. In the case of a spin glass this is done by first aligning spins by an applied field, which is turned off when the sample is cooled. The system then breaks up into spatial domains, each domain adjusting to the local disorder individually. Hopefully the domains will cover all the thermodynamic states, even the thermodynamicly rare ones. Measurement on this heterogeneous sample will yield the Gibbs averaged quantity. In analogy with this procedure, we are looking for a way to constrain the superconducting sample in a state "most dissimilar" to the single thermodynamic state that would have been obtained by slow annealing, then to release the constraint and perform the measurement after equilibration. Since this constraint cannot be imposed by the magnetic field, we have to turn to other means of experimental preparation of the sample for Gibbs averaging. In the case of a granular superconductor we may think of preparing the sample at temperature $\mathcal{T}_2$ by slow annealing, then rapidly quenching to $\mathcal{T}_1<\mathcal{T}_2$, and doing the measurement at $\mathcal{T}_1$. The slow annealing will prepare the sample in a single thermodynamic state *at* $\mathcal{T}_2$. Recall that it is the nature of a spin glass that the free-energy landscape evolves chaotically as function of temperature. Therefore, the thermodynamic state in which the system is at $\mathcal{T}_2$ is very dissimilar to the thermodynamic state that the system would have been in if the annealing had been continued down to $\mathcal{T}_1$. The rapid quench to $\mathcal{T}_1$ (the experimental challenge is in the word "rapid") will put the system into a new free-energy landscape, to which it will have to adjust. The hope is that the sample will break up into spatial domains, each in a different thermodynamic state (all individually equilibrated at $\mathcal{T}_1$). The subsequent measurement on this sample will yield an approximation to the Gibbs averaged helicity modulus Υ.

There appears to be a critical value $\delta_s\sim 0.26$ below which the superconducting glass does not exist, not even at $\mathcal{T}=0$ (see Fig. 1). We have been unable to further substantiate this claim, as one has to either characterize the ground state of the system with random π bonds, which is not trivial, or to solve the mean-field equations in the singular limit $\mathcal{T}\rightarrow 0$. Vannimenus *et al.* investigated ground-state properties of a related model.[27] They find that when a single π junction of strength $-K$ is immersed in an infinite lattice of regular junctions, it takes a supercritical strength $K>K_c>0$ to perturb the spins around the π junction from perfect alignment. Since the distortion field of such a junction is that of a dipole, i.e., short ranged, one can expect that a dilute system of π junctions of small but finite strength will have a pefectly collinear ground state. Indeed, this is true as long as the π junctions do not percolate.[27] The π bonds will be invisible to the order parameter γ, which will be 1. However, the elastic properties of this ground state *will* be affected. This is why the zero-temperature helicity modulus—or phase stiffness—will be less than 1 for $0<\delta<\delta_c$, even though the system is clearly not a glass (see Fig. 2, curve $\delta=0.2$). This is also supported by our numerical results, which suggest that at zero temperature $\gamma=1$ for $0<\delta<\delta_c$. Only for $\delta>\delta_c$ we see γ detach from 1. Simultaneously, s becomes nonzero, the replica symmetry is broken, and the system becomes a glass. The apparent existence of a critical $\delta_c>0$ in the present model is nonetheless surprising, since the Gaussian distribution (2) always allows for π junctions of arbitrary strength.

In our RS analysis there is a window $0.886<\delta<1$, for which there is a reentrant transition into the normal glass phase at $\mathcal{T}_n$ (see Fig. 1). This result is unreliable, as the exact solution with broken replica symmetry is needed in that region. The actual $\mathcal{T}_n$ line may turn out to be vertical, as in the Sherrington-Kirkpatrick spin glass.[11]

Our main result is the behavior of the experimentally relevant helicity modulus in the superconducting phases S and iS (Fig. 2). Although the original problem was formulated as a short-ranged model, the mean-field analysis makes the range of interaction effectively infinite. The results therefore have to be verified using more reliable methods, applicable to realistic short-ranged systems in two and three dimensions. The most interesting question of course is whether or not the irreversible iS phase exists in a realistic system with short-range interactions. When trying to answer this question one immediately runs into problems of interpretation. The concept of replica symmetry breaking, so essential for infinite-ranged models, becomes ill-defined for short-ranged models. For instance, in systems amenable to a field-theoretical treatment in $4-\epsilon$ dimensions, replicas serve merely as a mathematical device to organize the perturbation series[28] (in ϵ disorder cannot be treated perturbatively). In this case there is no obvious mechanism that would break replica symmetry. Instead, one looks for disorder-generated renormalization group (RG) fixed points in order to identify finite-temperature phase transitions (which give rise to weak glasses),[29] or for RG flows running away towards strongly coupled, disorder-dominated phases (strong glasses),[30] while replica symmetry remains unbroken. Although quite successful, even this is not a universal tool, and one has to use functional RG methods[31,32] whenever there happens to be a zero-temperature disordered fixed point.

All of the above phases—weak glass, strong glass, and zero-temperature glass—are possible alternatives for the iS phase of an actual disordered superconductor. Further work along these lines is in progress and will be presented in a future publication.

APPENDIX A: CASE $\delta<1$ (WEAK DISORDER)

In order to obtain the leading behavior of s_1, s_0, $t_c\equiv c_1-c_0$, and m for temperatures $\mathcal{T}$ just below $\mathcal{T}_s$, it is necessary to expand the free-energy functional to $\mathcal{O}(\tau_s^6)$, where $\tau_s\equiv 1-\mathcal{T}/\mathcal{T}_s$. We recall that $\mathcal{T}_s$ is given implicitly by Eq. (25). Since $\mathcal{T}_s<\mathcal{T}_c$, parameters γ^2, c_1, and $\tilde{c}$ are $\mathcal{O}(1)$, and cannot be considered small. With a bit of hindsight we expect $s_1=\mathcal{O}(\tau_s)$, $s_0=\mathcal{O}(\tau_s)$, $t_c=\mathcal{O}(\tau_s^2)$, and $m=\mathcal{O}(\tau_s)$. The following expansion is based heavily on the identity $\int dz G_\xi(z)f(z)=\exp[(\xi/2)d^2/dz^2]f(z)|_{z=0}$, which holds when the Taylor series of the function $f(z)$ has an infinite radius of convergence (otherwise the series on the rhs is only asymptotic). After a fair amount of work we obtain

$$\begin{aligned}
\frac{\mathcal{F}}{J_0\mathcal{T}}=&\frac{1}{2\mathcal{T}}\gamma^2-\left(\frac{\delta^2}{2\mathcal{T}^2}\right)\left[\frac{1}{2}c_1^2-mc_1t_c+\frac{1}{2}mt_c^2+\frac{1}{2}s_1^2(1-m)+\frac{1}{2}s_0^2m-\frac{1}{4}(1+\tilde{c})^2+\frac{1}{2}\right]\\
&-\int d\lambda\, G_{\delta^2c_1}(\lambda)\left[f-\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^2s_1^2h_2^2+\frac{8}{3}\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^3s_1^3h_2^3-15\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^4s_1^4h_2^4+\frac{544}{5}\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^5s_1^5h_2^5\right]\\
&-m\int d\lambda\, G_{\delta^2c_1}(\lambda)\left[\left(\frac{\delta^2}{2\mathcal{T}^2}\right)(\mathcal{T}\partial_\lambda f)^2t_c-\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^2(\mathcal{T}^2\partial_\lambda\partial_\lambda f)^2t_c^2+\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^2h_2^2(s_1^2-s_0^2)\right.\\
&-2\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^3(\mathcal{T}\partial_\lambda f)(\mathcal{T}\partial_\lambda h_2^2)t_cs_1^2+2\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^3(\mathcal{T}\partial_\lambda h_2)^2t_cs_0^2-4\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^3h_2^3(s_1^3-s_1s_0^2)\\
&\left.+\frac{1}{3}\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^4(h_4-9h_2^2)^2s_1^4+4\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^4h_2^2(h_4-6h_2^2)s_1^2s_0^2-\frac{1}{3}\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^4(h_4-3h_2^2)^2s_0^4\right]\\
&-\frac{m^2}{3}\int d\lambda\, G_{\delta^2c_1}(\lambda)4\left(\frac{\delta^2}{2\mathcal{T}^2}\right)^3h_2^3(s_1^3-3s_1s_0^2+2s_0^3)+\mathcal{O}(\tau_s^6).
\end{aligned}$$

Here

$$f\equiv f(\mathcal{T},\lambda)=\ln\int_{-\pi}^{\pi}d\theta\exp\left\{\frac{1}{\mathcal{T}}(\gamma+\lambda)\cos\theta+\frac{\delta^2}{2\mathcal{T}^2}(\tilde{c}-c_1)\cos^2\theta\right\} \tag{A1}$$

and

$$h_p \equiv h_p(\mathcal{T},\lambda) = \frac{\int_{-\pi}^{\pi} d\theta \sin^p\theta \exp\{1/\mathcal{T}(\gamma+\lambda)\cos\theta + \delta^2/(2\mathcal{T}^2)(\tilde{c}-c_1)\cos^2\theta\}}{\int_{-\pi}^{\pi} d\theta \exp\{1/\mathcal{T}(\gamma+\lambda)\cos\theta + \delta^2/(2\mathcal{T}^2)(\tilde{c}-c_1)\cos^2\theta\}}, \quad p=2,4. \tag{A2}$$

This expansion can be most readily obtained when one first expands formally in Eq. (33) with respect to m to $\mathcal{O}(m^3)$, then expands the coefficients of all powers of m in terms of t_c, s_1, and s_0 to appropriate order so that the entire expression for $\mathcal{F}$ is accurate to $\mathcal{O}(\tau_s^6)$. This is the lowest-order expansion necessary for determination of s_1, s_0, t_c, and m to leading order in τ_s.

Algebraic saddle-point equations for s_1, s_0, t_c, and m are not difficult to solve. The 1-RSB solution is

$$s_1 = \left(\frac{\mathcal{T}_s^2}{2\delta^2}\right) \frac{\int d\lambda\, G_{\delta^2 c_1}(\lambda)[h_2^2(\mathcal{T},\lambda) - h_2^2(\mathcal{T}_s,\lambda)]}{\int d\lambda\, G_{\delta^2 c_1}(\lambda) h_2^3(\mathcal{T}_s,\lambda)} + \mathcal{O}(\tau_s^2), \tag{A3}$$

$$s_0 = \mathcal{O}(\tau_s), \tag{A4}$$

$$t_c = 2\left(\frac{\delta^2}{2\mathcal{T}_s^2}\right)^2 (s_1^2 - s_0^2) \frac{\int d\lambda\, G_{\delta^2 c_1}(\lambda)[\mathcal{T}_s \partial_\lambda h_2(\mathcal{T}_s,\lambda)]^2}{1 - (\delta^2/\mathcal{T}_s^2)\int d\lambda\, G_{\delta^2 c_1}(\lambda)[\mathcal{T}_s^2 \partial_\lambda \partial_\lambda f(\mathcal{T}_s,\lambda)]^2} + \mathcal{O}(\tau_s^3), \tag{A5}$$

and

$$m = \left(\frac{\delta^2}{2\mathcal{T}_s^2}\right) \frac{s_1^3}{(s_1 - s_0)^2} \frac{\int d\lambda\, G_{\delta^2 c_1}(\lambda) h_4(\mathcal{T}_s,\lambda)[h_4(\mathcal{T}_s,\lambda) - 12 h_2^2(\mathcal{T}_s,\lambda)]}{\int d\lambda\, G_{\delta^2 c_1}(\lambda) h_2^3(\mathcal{T}_s,\lambda)} + \left(\frac{\delta^2}{2\mathcal{T}_s^2}\right) \frac{s_0(s_1+s_0)}{s_1 - s_0} \frac{\int d\lambda\, G_{\delta^2 c_1}(\lambda)[h_4(\mathcal{T}_s,\lambda) - 3h_2^2(\mathcal{T}_s,\lambda)]^2}{\int d\lambda\, G_{\delta^2 c_1}(\lambda) h_2^3(\mathcal{T}_s,\lambda)} + \mathcal{O}(\tau_s^2). \tag{A6}$$

The equation for s_0 is cubic and we do not include its explicit solution here, apart from the fact that $s_0 = \mathcal{O}(\tau_s)$.

APPENDIX B: CASE $\delta \geqslant 1$ (STRONG DISORDER)

Again, in order to obtain the leading behavior of all variables, γ^2, $\tilde{c}$, c_1, c_0, s_1, s_0, and m, we have to expand $\mathcal{F}$ to $\mathcal{O}(\tau_c^6)$, where $\tau_c \equiv 1 - \mathcal{T}/\mathcal{T}_c$. We recall that $\mathcal{T}_c = \delta/2$. The expansion of the free-energy functional is lengthy and we will not present it here. The saddle-point equations have a replica symmetry-broken solution

$$\gamma = 0, \quad \tilde{c} = 0, \quad c_1 = s_1 = \frac{1}{2}\tau_c + \frac{25}{72}\tau_c^2 + \mathcal{O}(\tau_c^3), \quad c_0 = s_0 = \frac{1}{6}\tau_c + \mathcal{O}(\tau_c^2), \quad \text{and } m = \tau_c + \mathcal{O}(\tau_c^2).$$

We note that the equation $\partial\mathcal{F}/\partial\gamma = 0$ actually does not have a solution. Nonetheless, $\mathcal{F}$ has a global minimum with respect to γ at $\gamma = 0$, which is an endpoint of the definition interval.

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