Landau theory of bicriticality in a random quantum rotor system

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(Received 25 November 1998)

We consider here a generalization of the random quantum rotor model in which each rotor is characterized by an M-component vector spin. We focus entirely on the case not considered previously, namely when the distribution of exchange interactions has nonzero mean. Inclusion of nonzero mean permits ferromagnetic and superconducting phases for $M=1$ and $M=2$, respectively. We find that quite generally, the Landau theory for this system can be recast as a zero-mean problem in the presence of a magnetic field. Naturally then, we find that a Gabay-Toulouse line exists for $M>1$ when the distribution of exchange interactions has nonzero mean. The solution to the saddle point equations is presented in the vicinity of the bicritical point characterized by the intersection of the ferromagnetic ($M=1$) or superconducting ($M=2$) phase with the paramagnetic and spin glass phases. All transitions including the ferromagnet–spin-glass transition are observed to be second order. At zero temperature, we find that the ferromagnetic order parameter is nonanalytic in the parameter that controls the paramagnet-ferromagnet transition in the absence of disorder. Also for $M=1$, we find that replica symmetry breaking is present but vanishes at low temperatures. In addition, at finite temperature, we find that the qualitative features of the phase diagram, for $M=1$, are *identical* to what is observed experimentally in the random magnetic alloy $LiHo_xY_{1-x}F_4$. [S0163-1829(99)00418-X]

I. INTRODUCTION

Transport in granular metals is mediated by activated transport among the metallic grains. In granular superconductors composed of spatially separated metallic grains, such single particle charging events ultimately lead to a destruction of phase locking between the grains. This state of affairs obtains because the particle number n and phase θ associated with each grain are conjugate variables. If two grains phase lock, the resultant infinite uncertainty in the particle number leads necessarily to single particle charging. Should the single particle charging energy sufficiently exceed the Josephson coupling energy between grains, superconductivity is quenched.[1]

The simplicity of the physics underlying the quantum phase transition in an array of superconducting islands implies that the resultant Hamiltonian

$$H=-E_C\sum_i \left(\frac{\partial}{\partial\theta_i}\right)^2-\sum_{\langle i,j\rangle} J_{ij}\cos(\theta_i-\theta_j) \tag{1}$$

is characterized by only two parameters: (1) a charging energy E_C and (2) a Josephson coupling energy J_{ij}. In writing Eq. (1), we assumed that the islands occupy regular sites on a two-dimensional (2D) lattice and only nearest neighbor $\langle i,j\rangle$ Josephson coupling is relevant. For the ordered case in which $J_{\langle i,j\rangle}=J_0$, the superconductor-insulator transition is well studied[2] as the parameter E_C/J_0 increases. In sufficiently disordered systems, however, the Josephson energies are not all equal and in fact can be taken to be random.

We are concerned in this paper with the case in which the Josephson energies are random and characterized by a Gaussian distribution

$$P(J_{ij})=\frac{1}{\sqrt{2\pi J^2}}\exp\left[-\frac{(J_{ij}-J_0)^2}{2J^2}\right] \tag{2}$$

with nonzero mean J_0. While random Josephson systems have been treated previously,[3–5] such studies have focused predominantly on the zero mean case in which $J_0=0$. This limit differs fundamentally from the nonzero mean case, because the superconducting phase exists only when $J_0\neq 0$. Hence, for $J_0=0$, there is an absence of an ordering transition. Nonetheless, the zero-mean case is still of physical interest because as Read, Sachdev, and Ye (RSY)[3] as well as Miller and Huse[4] have shown, a zero temperature transition from a quantum spin glass to a paramagnet occurs as the strength of the quantum fluctuations increases. To put the random Josephson model in the context of the work of RSY, it is expedient to introduce the change of variables

$$\mathbf{S}_i=(\cos\theta_i,\sin\theta_i). \tag{3}$$

The resultant Josephson Hamiltonian

$$H=-E_C\sum_i \left(\frac{\partial}{\partial\theta_i}\right)^2+\sum_{\langle i,j\rangle} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j \tag{4}$$

is recast as a two-component ($M=2$) interacting spin problem with random magnetic interactions. This model is easily generalizable to describe interactions among any M-component spin operator (or quantum rotor) in the group $O(M)$. $M=1$ corresponds to Ising spins and is relevant to random magnetic[6] systems such as $LiHo_{0.167}Y_{0.833}F_4$ whereas the $M=3$ limit is applicable to spin fluctuations in the cuprates.[7] In this paper we focus primarily on the $M=1$ and $M=2$ cases in which the ordered phases correspond to a ferromagnet and a superconductor, respectively.

Two types of fluctuations control the transitions between the phases: (1) dynamic quantum fluctuations arising from the charging energy and (2) static fluctuations induced by the disorder. For nonzero mean, three phases [spin-glass, paramagnet, ferromagnet ($M=1$), or superconductor ($M=2$)] are expected to meet at a bicritical point. Experimentally, a

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bicritical point is anticipated whenever J_0, J, and E_C are on the same order of magnitude. It is the physics at this bicritical point that we focus on in this paper. To discriminate between these phases, we distinguish between thermal averages $\langle \cdots \rangle$ and averages over disorder $[\cdots]$. In the superconductor ($M=2$) or ferromagnetic phases ($M=1$), the disorder and thermal average of the local spin operator $[\langle S_{i\nu}\rangle]\neq 0$ is nonzero. In the spin glass phase, the thermal average $\langle S_{i\nu}\rangle\neq 0$, while $[\langle S_{i\nu}\rangle]=0$. At zero temperature, quantum fluctuations and static disorder conspire to lead to a vanishing of the static moment in the paramagnetic phase, that is, $\langle S_{i\nu}\rangle=0$. RSY have performed an extensive study of the spin-glass–paramagnetic boundary using the replica formalism[8] in the case of zero mean in which the purely ordered phase is absent. We adopt this formalism here in our analysis of the bicritical region. The only prior study on the nonzero mean case is that of Hartman and Weichman[9] who studied numerically the spherical limit $M\to\infty$ of the quantum rotor Hamiltonian. In their study, they found that the spin-glass phase is absent in the $M\to\infty$ limit for $d=2$. In the present work, we will not address the issue of dimensionality because we limit ourselves to a mean-field description in which all fields are homogeneous in space. Our results then are valid above some upper critical dimension that can only be determined by including fluctuations around the mean-field solution.

This paper is organized as follows. In Sec. II, we use the replica formalism to average over the disorder explicitly and obtain the effective Landau free-energy functional. We show that the leading terms in the Landau action resulting from the nonzero mean are analogous to those arising from an external magnetic field in the zero-mean problem. As a result, the appropriate saddle-point equations can be solved using a direct analogy to the zero-mean problem in the presence of a magnetic field. Explicit criteria are presented in Sec. III for the stability of each of the phases in the vicinity of bicriticality. We construct the phase diagram at finite temperature and find excellent agreement with the experimental results of Reich *et al.*[6] on the magnetic system $\mathrm{LiHo}_x\mathrm{Y}_{1-x}\mathrm{F}_4$. The explicit solution for $M\geq 2$ is presented in the last section with a special emphasis on the Gabay-Toulouse[10] line. In Sec. IV we analyze the possibility of replica symmetry breaking along the de Almeida–Thouless[11] line in the $M=1$ case.

II. LANDAU ACTION

Central to the construction of a Landau theory of the bicritical region is the free-energy functional. For a quantum-mechanical system, this is obtained by explicitly including in the partition function

$$Z=\mathrm{Tr}(e^{-\beta H})=\mathrm{Tr}\left\{e^{-\beta H_0}\hat{T}\exp\left[-\int_0^\beta H_1(\tau)d\tau\right]\right\} \tag{5}$$

the time evolution according to a reference system. For this problem, the Hamiltonian for free quantum rotors

$$H_0=-E_C\sum_i\left(\frac{\partial}{\partial\theta_i}\right)^2 \tag{6}$$

describes our reference system. The perturbation

$$H_1(\tau)=e^{H_0\tau}\sum_{\langle i,j\rangle}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j e^{-H_0\tau} \tag{7}$$

corresponds to the random magnetic interactions. The trace in the partition function is evaluated over the complete set of quantum rotor states. A primary hurdle in evaluating the partition function is the average over the random spin interactions. It is now standard to perform this average[5] by replicating the spin system n times and using the identity

$$\ln[Z]=\lim_{n\to 0}\frac{[Z^n]-1}{n} \tag{8}$$

to obtain the partition function Z. We first must then evaluate $\langle Z^n\rangle$. Formally, the replicated partition function is defined through Eq. (5) by replacing H_0 and H_1 by their replicated equivalents

$$H_i^{\mathrm{eff}}=\sum_{a=1}^n H_i^a, \tag{9}$$

where the superscript indexes the individual replicas and $i=0,1$. Within the replica formalism, the average over the random interactions with the Gaussian distribution gives rise to a quartic spin interaction. This quartic spin interaction is easily reduced to a quadratic interaction by use of the Hubbard-Stratanovich transformation. Let us define the Fourier transforms

$$\mathbf{S}^a(k,\tau)=\frac{1}{\sqrt{N}}\sum_i \mathbf{S}_i^a(\tau)e^{i\mathbf{k}\cdot\mathbf{R}_i} \tag{10}$$

of the local spin operator for each site and the corresponding transform of the nearest-neighbor interaction

$$J(k)=J\sum_{\langle ij\rangle}e^{\mathbf{k}\cdot\mathbf{r}_{ij}}=2J\sum_{i=1}^d\cos k_i a \tag{11}$$

The replicated partition function now takes on the form

$$[Z^n]=Z_0^n\int D\Psi DQ e^{-F_{\mathrm{eff}}[\Psi,Q]} \tag{12}$$

where the effective free energy

$$\begin{aligned}F_{\mathrm{eff}}[\Psi,Q]=&\sum_{a,k}\int_0^\beta d\tau\Psi_\mu^a(k,\tau)[\Psi_\mu^a(k,\tau)]^*\\&+\sum_{a,b,k,k'}Q_{\mu\nu}^{ab}(k,k',\tau,\tau')[Q_{\mu\nu}^{ab}(k,k',\tau,\tau')]^*\\&-\ln\Bigg[\Bigg\langle\hat{T}\exp\Bigg(2\int_0^\beta d\tau\sum_{a,k}\sqrt{J_0(k)}\Psi_\mu^a(k,\tau)\\&\times S_\mu^a(-k,\tau)+\sum_{a,b,k,k'}\int_0^\beta\int_0^\beta d\tau d\tau'\\&\times\sqrt{2J(k)J(k')}Q_{\mu\nu}(k,k',\tau,\tau')\\&\times S_\mu^a(k,\tau)S_\nu^b(-k',\tau')\Bigg)\Bigg\rangle_0\Bigg]\end{aligned} \tag{13}$$

is now a functional of the auxilliary fields Q and Ψ which appear upon use of the Hubbard-Stratanovich transformation to decouple the quartic spin term proportional to $J^2(k)$ and the quadratic term scaling as $J_0(k)$, respectively. In writing this expression, we used the Einstein convention where repeated spin (but not replica) indices are summed over $Z_0 = \mathrm{Tr}[\exp(-\beta H_0)]$ and

$$\langle A\rangle_0 = \frac{1}{Z_0^n}\,\mathrm{Tr}(e^{-\beta H_0^{\mathrm{eff}}}A). \tag{14}$$

The fields Q and Ψ play fundamentally different roles. The proportionality of the Ψ field to the mean of the distribution implies that this field determines the ordering transition (superconducting for $M=2$ or ferromagnetic for $M=1$). This can be seen immediately upon differentiating the free energy with respect to Ψ. The self-consistent condition is that

$$\Psi_\mu^a(k,\tau) = \langle S_\mu^a(k,\tau)\rangle. \tag{15}$$

Hence, a nonzero value of $\Psi_\mu^a(k,\tau)$ implies ordering. It is for this reason that Ψ functions as the order parameter for the ferromagnetic or superconducting phase within Landau theory. Likewise, differentiation of the free energy with respect to Q reveals that

$$Q_{\mu\nu}^{ab}(k,k',\tau,\tau') = \langle S_\mu^a(k,\tau)S_\nu^b(k',\tau')\rangle \tag{16}$$

is the self-consistency condition for the Q matrices. For quantum spin glasses, it is the diagonal elements of the Q matrix

$$D(\tau-\tau') = \lim_{n\to 0}\frac{1}{Mn}\langle Q_{\mu\mu}^{aa}(k,k',\tau,\tau')\rangle \tag{17}$$

in the limit that $|\tau-\tau'|\to\infty$ that serves as the effective Edwards-Anderson spin-glass order parameter[3,5] within the Landau theory. A disclaimer is appropriate here as D is nonzero in the spin-glass as well as the paramagnet phases. However, its behavior is sufficiently different in the three phases: in the paramagnetic and ferromagnetic (superconducting) phases, as we will show, $D(\omega)$ still has the form $\sqrt{\omega^2+\Delta^2}$. The gap Δ vanishes at the transition to the spin-glass phase giving rise to the long-time behavior of $D(\tau)$.

Before we analyze the form of the gap, we must obtain the effective Landau action. The goal here is to obtain a polynomial functional of Q and Ψ from which a saddle point analysis can be performed. We proceed in the standard way and perform a cumulant expansion on the free energy. For the case of zero mean, this procedure is documented quite closely in RSY. In addition to the terms containing the Q matrices studied by RSY, the nonzero mean case will contain powers of Ψ_μ^a as well as cross terms. The resultant action must contain, of course, only even powers of Ψ_μ^a. For an analysis of the bicritical point, it is sufficient to retain quadratic and quartic terms in Ψ_μ^a. Terms of this kind are completely analogous to those derived previously by Doniach.[2] Of the cross terms, the simplest are of the form $\Psi_\mu^a\Psi_\nu^b Q_{\mu\nu}^{ab}$, $\Psi_\mu^a\Psi_\nu^a Q_{\mu\nu}^{aa}$, and $\Psi_\mu^a\Psi_\mu^a Q_{\nu\nu}^{aa}$. We have confirmed explicitly that retension of the latter two terms in which only one replica index occurs leads only to the renormalization of various coupling constants and minor modification of the phase boundaries near the bicritical region. Hence, we do not consider such terms. Likewise, we do not retain terms of the form $Q^{aa}Q^{bb}$ because this term vanishes in the limit $n\to 0$.

We find then that the effective action

$$\begin{aligned}
\mathcal{A} = {} & \frac{1}{t}\int d^dx\Bigg\{\frac{1}{\kappa}\int d\tau\sum_a\left(r+\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\right)Q_{\mu\mu}^{aa}(x,\tau_1,\tau_2)\big|_{\tau_1=\tau_2=\tau}+\frac{1}{2}\int d\tau_1 d\tau_2\sum_{a,b}[\nabla Q_{\mu\nu}^{ab}(x,\tau_1,\tau_2)]^2\\
& -\frac{\kappa}{3}\int d\tau_1 d\tau_2 d\tau_3\sum_{a,b,c}Q_{\mu\nu}^{ab}(x,\tau_1,\tau_2)Q_{\nu\rho}^{bc}(x,\tau_2,\tau_3)Q_{\rho\mu}^{ca}(x,\tau_3,\tau_1)+\frac{1}{2}\int d\tau\sum_a[uQ_{\mu\nu}^{aa}(x,\tau,\tau)Q_{\mu\nu}^{aa}(x,\tau,\tau)\\
& +vQ_{\mu\mu}^{aa}(x,\tau,\tau)Q_{\nu\nu}^{aa}(x,\tau,\tau)]\Bigg\}+\frac{1}{2g}\int d^dx\Bigg\{d\tau\sum_a\Psi_\mu^a(x,\tau)\left[-\gamma-\frac{\nabla^2}{2}\right]\Psi_\mu^a(x,\tau)\\
& +\int d\tau\sum_a\frac{\partial}{\partial\tau_1}\Psi_\mu^a(x,\tau_1)\frac{\partial}{\partial\tau_2}\Psi_\mu^a(x,\tau_2)\big|_{\tau_1=\tau_2=\tau}+\frac{\zeta}{2}\int d\tau\sum_a[\Psi_\mu^a(x,\tau)\Psi_\mu^a(x,\tau)]^2\Bigg\}\\
& -\frac{1}{\kappa t g}\int d^dx\int d\tau_1 d\tau_2\sum_{a,b}\Psi_\mu^a(x,\tau_1)\Psi_\nu^b(x,\tau_2)Q_{\mu\nu}^{ab}(x,\tau_1,\tau_2)+\cdots
\end{aligned} \tag{18}$$

contains three types of terms. (1) The terms that depend purely on the Q matrices, as in the terms in the first curly bracket in Eq. (18), have been studied previously by RSY in the context of quantum spin-glass–paramagnet transition. (2) The Ψ-dependent terms in the second curly bracket give rise to the ferromagnet or superconducting phases. They are controlled by the parameter γ. (3) The competition between these two transitions is mediated by the last term in Eq. (18). To reiterate, other cross terms exist—for example, the term noted previously. However, such terms have no bearing on the critical region. It suffices then to truncate the action at the level of Eq. (18). The last term in Eq. (18) bears a strong resemblance to the term that appears in the Landau action for the zero-mean problem in the presence of a magnetic field h.

In fact, we can obtain the corresponding term from Eq. (18) by the transformation $\Psi^a_\mu \rightarrow h\delta_{\mu,1}\sqrt{\kappa g/2t}$. This mapping is not unexpected. In the nonzero mean problem, ordered spins create an effective ''magnetic field'' that acts as a source term for the Q matrices. This analogy is particularly powerful and serves as a useful check as to the validity of our saddle-point equations. Further, this mapping is true even in the classical case.

A word on the coupling constants is in order. The parametrization of the action in terms of the coupling constants κ, t, and g was obtained by appropriately rescaling the fields Q and Ψ as well as the space and time coordinates. Fundamentally, g is a function of E_C and J_0, while κ and t are functions of E_C and J only. u, v, and ζ are related to a four-point spin correlation function evaluated in the long-time limit as discussed by RSY. In the zero-mean case, the phase diagram demarcating spin glass and paramagnetic stability is determined by the parameter r which determines the strength of the quantum fluctuations. When these fluctuations exceed a critical value, that is $r > r_c$, a transition to a paramagnetic phase occurs.[3] In the problem at hand, two parameters, r and γ, determine the phase diagram. The coupling constant γ is directly related to the parameter E_C/J_0 and E_C/J as well. The latter dependence arises as a result of the removal of the quadratic term

$$\int d^dx d\tau_1 d\tau_2 \sum_{a,b} [Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)]^2, \tag{19}$$

by the transformation[3] $Q \rightarrow Q - C\delta^{ab}\delta_\mu(\tau_1-\tau_2)$. From microscopic considerations, it follows that $\kappa^2 t/2 = 1$. This is an important simplification because we will show that the ferromagnet-paramagnet boundary is determined by the line $\gamma = 2\Delta/\kappa^2 t = \Delta$.

III. SADDLE POINT ANALYSIS: PHASE DIAGRAM

In terms of the frequency-dependent order parameters

$$\Psi_\mu(k,\omega) = \frac{1}{\beta}\int_0^\beta d\tau \Psi^a_\mu(k,\tau) e^{-i\omega\tau} \tag{20}$$

and

$$Q^{ab}_{\mu\nu}(k_1,k_2,\omega_1,\omega_2) = \frac{1}{\beta^2}\int_0^\beta \int_0^\beta d\tau_1 d\tau_2 Q^{ab}_{\mu\nu}(k_1,k_2,\tau_1,\tau_2) e^{-i\omega_1\tau_1 - i\omega_2\tau_2}, \tag{21}$$

the three phases in our problem form under the following conditions. For the paramagnetic phase near the bicritical point, the saddle-point equations are satisfied when $\Psi^a_\mu = 0$ but as RSY have shown

$$Q^{ab}_{\mu\nu}(k,\omega_1,\omega_2) = (2\pi)^d \beta \delta_{\mu\nu}\delta^{ab}\delta^d(k)\delta_{\omega_1+\omega_2,0} D(\omega_1). \tag{22}$$

We have retained the ansatz for the Q matrix used by RSY for the paramagnetic phase. Similarly, the ordering parameter also vanishes in the spin-glass phase $\Psi_\mu = 0$ and for the replica-symmetric solution

$$Q^{ab}_{\mu\nu}(k,\omega_1,\omega_2) = (2\pi)^d \delta^d(k)\delta_{\mu\nu}[\beta D(\omega_1)\delta_{\omega_1+\omega_2,0}\delta^{ab} + \beta^2 \delta_{\omega_1,0}\delta_{\omega_2,0} q^{ab}]. \tag{23}$$

The possibility of a replica broken-symmetry solution will also be explored by including terms of $O(Q^4)$ in the Landau action. In the ordered phase (that is ferromagnetic or superconductor) both Ψ and Q are nonzero. As ferromagnetism has generally been studied with a frequency-independent order parameter, we explore a static ansatz of the form

$$\Psi^a_\mu(k,\omega) = (2\pi)^d \delta^d(k) \beta \delta_{\omega,0}\delta_{\mu,1}\psi \tag{24}$$

for Ψ^a_μ and

$$Q^{ab}_{\mu\nu}(k,\omega_1,\omega_2) = (2\pi)^d \delta^d(k)\delta_{\mu\nu}[\beta \overline{D}(\omega_1)\delta_{\omega_1+\omega_2,0}\delta^{ab} + \beta^2 \tilde{q}\,\delta_{\omega_1,0}\delta_{\omega_2,0}\delta^{ab} + \beta^2 \delta_{\omega_1,0}\delta_{\omega_2,0} q^{ab}] \tag{25}$$

for the Q matrices. Our explicit inclusion of $\tilde{q}$ as the $\omega = 0$ diagonal element of the Q matrices implies that we can redefine $D(\omega) = \overline{D}(\omega) + \beta\tilde{q}\delta_{\omega,0}$. Consequently, we can set $\overline{D}(\omega) = 0$ and assume that q^{ab} is purely off-diagonal. The replica-symmetric solution corresponds to $q^{ab} = q$ for all $a \neq b$. Initially, we will explore only this case.

A. $M = 1$: Ferromagnetic order

We now specialize to the $M = 1$ ferromagnetic case as crucial differences can occur with the analysis for $M > 1$. If we substitute Eqs. (24) and (25) into the Landau action, we obtain a free-energy density of the form

$$\begin{aligned}\frac{\mathcal{F}}{n} = {} & \frac{1}{\beta\kappa t}\sum_{\omega\neq 0}(\omega^2 + r)\overline{D}(\omega) + \frac{r\tilde{q}}{\kappa t} - \frac{\kappa}{3\beta t}\sum_{\omega\neq 0}\overline{D}^3(\omega) \\ & + \frac{u+v}{2t}\left(\tilde{q} + \frac{1}{\beta}\sum_{\omega\neq 0}\overline{D}^2(\omega)\right)^2 + \frac{1}{2g}\left(-\gamma\psi^2 + \frac{\zeta\psi^4}{2}\right) \\ & - \frac{\kappa\beta^2}{3t}\left(\tilde{q}^3 + 3\tilde{q}\frac{\mathrm{Tr}\, q^2}{n} + \frac{\mathrm{Tr}\, q^3}{n}\right) - \frac{\beta\psi^2}{\kappa g t}\left(\tilde{q} + \frac{\sum_{ab} q^{ab}}{n}\right).\end{aligned} \tag{26}$$

Formally, the free-energy density as defined here is the disorder-averaged free energy per replica per spin component. The last term in Eq. (26) arises from the cross term and generates off-diagonal components of the Q matrix. For $\psi = 0$, this term is absent and we generate the solution of RSY. We obtain the saddle point equations by differentiating Eq. (26) with respect to q, $\tilde{q}$, $\overline{D}$, and ψ in the $n \rightarrow 0$ limit. All of these quantities can be simplified once the appropriate gap parameter

$$\Delta^2 = r + \kappa(u+v)\left(\tilde{q} + \frac{1}{\beta}\sum_{\omega\neq 0}\overline{D}(\omega)\right) \tag{27}$$

is identified. The difference with the corresponding quantity in the work of RSY is the presence of the $\tilde{q}$ term in the gap which includes the contribution arising from the cross term in the Landau action. From the derivative equation with respect to $\bar{D}(\omega)$, we find that

$$\bar{D}(\omega)=-\frac{1}{\kappa}\sqrt{\omega^2+\Delta^2}. \tag{28}$$

The constraint $\Delta\geqslant 0$ is crucial for the stability of the free energy density and the minus sign in the above expression ensures that $D(\tau)>0$. The saddle-point equations for q and $\tilde{q}$ yield that

$$q=\frac{\psi^2}{2\kappa g\Delta} \tag{29}$$

and

$$\tilde{q}=\frac{\psi^2}{2\kappa g\Delta}-\frac{\Delta}{\kappa\beta}. \tag{30}$$

As expected, these equations form the basis for the bulk of our analysis and are identical to the RSY saddle-point equations with h replaced by ψ. Despite this mapping, h and ψ do serve different roles in the zero and nonzero mean theories. In the former, h is an external adjustable scalar quantity without any critical properties, whereas ψ plays the role of an order parameter in the nonzero mean case and must be treated on equal footing with Q.

The saddle-point equation for ψ

$$\psi[-\gamma+\Delta+\zeta\psi^2]=0 \tag{31}$$

implies that aside from the trivial solution $\psi=0$, the nontrivial solution corresponds to

$$\psi^2=\frac{1}{\zeta}(\gamma-\Delta). \tag{32}$$

In obtaining this equation, we used the fact that $\kappa^2 t/2=1$. If we specialize to low temperatures, that is low relative to the gap $(T<\Delta)$, the sum in the gap equation can be evaluated

$$\frac{1}{\beta}\sum_{\omega}\sqrt{\omega^2+\Delta^2}=\frac{\Lambda_\omega^2}{2\pi}+\frac{\Delta^2}{4\pi}\ln\frac{\Lambda_\omega^2}{\Delta^2} \tag{33}$$

and the resultant self-consistent equation takes the form

$$\Delta^2=r-r_c-\frac{u+v}{4\pi}\Delta^2\ln\frac{\Lambda_\omega^2}{\Delta^2}+\frac{u+v}{2g\zeta}\left(\frac{\gamma}{\Delta}-1\right). \tag{34}$$

We have introduced

$$r_c=(u+v)\frac{\Lambda_\omega^2}{2\pi} \tag{35}$$

and the cutoff Λ_ω is determined by the energy scale at which zero-point quantum fluctuations become important. The natural cutoff for such fluctuations in this problem is E_C, the charging energy.

The essential physics of the bicritical point is contained in Eqs. (32) and (34). From Eq. (32), it is clear that the ferromagnetic phase exists only when $\gamma\geqslant\Delta$. This ensures that $\psi^2>0$ in the ferromagnetic phase. The vanishing of the order parameter ψ signifies a termination of the ferromagnetic phase. For $\gamma\neq 0$ and $\Delta\neq 0$, the only line along which $\psi^2=0$ corresponds to $\gamma=\Delta$. If we substitute this condition into the gap equation, we find that the critical line separating the ferromagnet from the paramagnet at low temperatures ($T\ll\Delta$) is given by

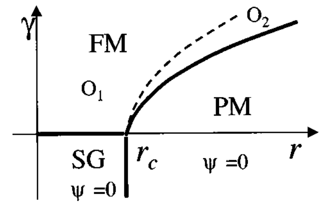

FIG. 1. Zero-temperature phase diagram demarcating the regions of ferromagnetism, paramagnetism, and spin-glass behavior for $M=1$. The parameters γ and r are determined by the dynamical quantum fluctuations and the static disorder. The curve separating the ferromagnet from the paramagnet scales roughly as $\gamma\approx\sqrt{r-r_c}$ up to logarithmic factors. Regions $O2$ and $O1$ are distinguished by the magnitude of the spin-wave gap Δ as well as the magnitude of the order parameter ψ, which is nonzero only in the ferromagnetic phase. The transition between these two regions occurs when $\gamma-\gamma_c\approx(r-r_c)^{3/2}$, where γ_c and r_c are determined by Eqs. (36) and (35), respectively.

$$\gamma=\Delta=\left(\frac{4\pi(r-r_c)}{(u+v)\ln[\Lambda_\omega^2/(r-r_c)]}\right)^{1/2} \tag{36}$$

and is depicted in Fig. 1. In obtaining this equation, we assumed that $|r-r_c|\ll 1$ and $\gamma\ll 1$. The bicritical point corresponds to $r=r_c$ and $\gamma=0$. At this point, the gap vanishes as does ψ. The essential nontrivial nature of this result is the nonanalytic dependence of γ on the quantum fluctuation parameter $r-r_c$.

In the vicinity of the bicritical point, distinct regimes partition the ferromagnetic phase that are determined by the magnitude of Δ and ψ. This behavior is shown in Fig. 1. The regions $O1$ and $O2$ are distinguished by their distance from the line $\gamma=\Delta$. The transition between $O1$ and $O2$ occurs when

$$\gamma-\left(\frac{4\pi(r-r_c)}{(u+v)\ln[\Lambda_\omega^2/(r-r_c)]}\right)^{1/2}\approx\frac{(r-r_c)^{3/2}}{\ln[\Lambda_\omega^2/(r-r_c)]}. \tag{37}$$

Because $r-r_c$ is much less than unity, algebraic dependence in Eq. (59) of the form $(r-r_c)^{3/2}$ implies that the $O2$ region is quite narrow. Within this narrow region, the gap is well approximated by Eq. (36) and the value of ψ is given by

$$\psi^2=\frac{1}{\zeta}\left[\gamma-\left(\frac{4\pi(r-r_c)}{(u+v)\ln[\Lambda_\omega^2/(r-r_c)]}\right)^{1/2}\right]\quad\text{region}\quad O2. \tag{38}$$

In the bulk of the ferromagnetic phase, region $O1$, the γ-dependent terms must be retained in Eq. (34) to accurately describe the gap $\Delta=\gamma-\zeta\psi^2$ with the ordering parameter given by

$$\psi^2=\frac{2g}{u+v}\left[\frac{u+v}{2\pi}\gamma^3\ln\frac{\Lambda_\omega}{\gamma}-\gamma(r-r_c)\right]\quad\text{region}\quad O1. \tag{39}$$

Hence, at the point $r=r_c$, we find a strong nonanalytic dependence $\psi\propto\gamma^{3/2}\sqrt{\ln\Lambda_\omega/\gamma}$ on the coupling constant γ. This is particularly important because it signifies that even at the mean-field level, deviations from the standard square-root dependence are present in the current theory. This result is fundamentally tied to the logarithmic dependence induced by the frequency summations and is caused by the zero-point quantum fluctuations. However, if we extrapolate our results to the regime $r-r_c\approx O(1)$, the second term in Eq. (39) dominates and we do recover that $\psi\propto\gamma^{1/2}$ in agreement with the expectation from standard mean-field Landau theories. Paramagnetic (PM) and spin-glass (SG) behavior obtain whenever $\gamma<\Delta$. In this regime, the nontrivial solution for ψ no longer holds and $\psi=0$ is the only valid solution. In this limit, our solution for Δ is identical to that of RSY and all of their results are recovered. For example, consider the free energy density (in units of $\kappa^2 t/2=1$)

$$\begin{aligned}\frac{\mathcal{F}}{n}&=\frac{\Lambda_\omega^4}{2\pi}\left(\frac{1}{6}+\frac{u+v}{8\pi}\right)-\frac{\Lambda_\omega^2(r-r_c)}{4\pi}-\frac{(r-r_c)^2}{4(u+v)}+\frac{\gamma^2(r-r_c)}{2(u+v)}\\&\quad-\frac{\gamma^4}{8\pi}\ln\frac{\Lambda_\omega}{\gamma}+\cdots\\&=\mathcal{F}_0-\frac{\gamma^4}{8\pi}\ln\frac{\Lambda_\omega}{\gamma}+\frac{\gamma^2(r-r_c)}{2(u+v)}+\cdots\end{aligned} \tag{40}$$

in region $O1$. This quantity is obtainable from Eq. (26) once the saddle-point solutions for region $O1$ are used and only the leading terms in γ and the cross term are retained. When $\gamma=0$, this expression is identical to that of RSY in the spin-glass phase. As anticipated, the leading γ dependence in the free-energy density is nonanalytic. This behavior originates from the nonanalytic behavior of the order parameter ψ on γ. We will see that this nonanalyticity does not survive for $M>1$ above and below the Gabay-Toulouse line.

Another result which we can obtain immediately is the nature of the ferromagnetic (FM)-SG transition. As indicated in Fig. 1, a transition from the FM to the SG occurs when $r<r_c$ and $\gamma\to 0$. In this limit, both ψ^2 and Δ tend to zero. However, their ratio is finite. Consequently, the order parameter q is given by

$$q=\frac{r_c-r}{\kappa(u+v)}. \tag{41}$$

This expression is identical to that of RSY and hence coupled with our earliear result for the PM-FM transition, we find that all order parameters associated with the bicritical point are continuous at the phase boundaries, thereby indicating that all transtions are second order.

Extending these results to finite temperatures $T\gg\Delta$ simply requires the evaluation of the frequency summation over frequencies in the gap equation for $T\gg\Delta$. Using the result in Eq. (2.l3) of RSY, we obtain a self-consistent condition for the gap

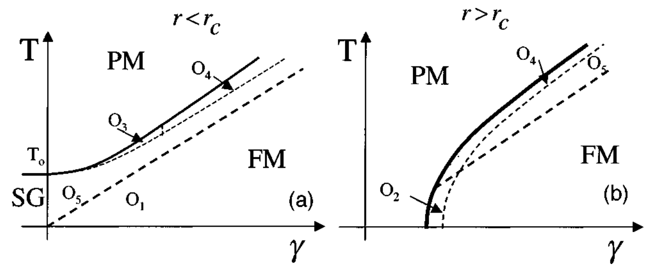

FIG. 2. $M=1$ finite-temperature phase diagrams for (a) $r_c-r>0$ and (b) $r-r_c>0$. Regions $O3$ and $O4$ lie close to the paramagnet boundary and hence have $\gamma\approx\Delta(T)$. Thermal fluctuations dominate in region $O5$ as well as in $O3$ and $O4$. In region $O1$, thermal fluctuations are negligible. The key difference between noted when r_c-r changes sign from positive to negative is the absence of the spin-glass phase for $r>r_c$.

$$\begin{aligned}\Delta^2&=r-r_c(T)-(u+v)T\Delta-\frac{u+v}{2\pi}\Delta^2\ln\frac{\Lambda_\omega}{T}\\&\quad+\frac{u+v}{2g\zeta\Delta}(\gamma-\Delta)\end{aligned} \tag{42}$$

in the regime $T\gg\Delta$ where we have defined $r_c(T)=r_c-(u+v)\pi T^2/3$. Recall that the transition between the ferromagnetic and paramagnetic phases occurs when $\gamma=\Delta$. Hence, in the units chosen here, this condition simplifies to $\gamma=\Delta(T)$. For $T\ll\sqrt{r-r_c(T)}$, the boundary for the paramagnetic-ferromagnetic state remains unchanged from the $T=0$ results discussed above. However, for $T\gg\Delta$, two distinct regimes

$$\gamma=\Delta(T)=\begin{cases}\dfrac{r-r_c(T)}{\sqrt{(u+v)}T} & \sqrt{r-r_c(T)}\ll T\quad\text{region}\quad O3,\\[2ex] \sqrt{\dfrac{2\pi^2T^2}{3\ln\dfrac{\Lambda_\omega}{T}}} & \sqrt{r-r_c}\ll T\quad\text{region}\quad O4\end{cases} \tag{43}$$

emerge depending on the magnitude of the thermal fluctuations. These regimes are depicted in Fig. 2(a). The crossover between these two regions occurs when $\gamma\propto\sqrt{r_c-r}$. In Fig. 2(a), $r_c-r>0$. The temperature

$$T_0=\sqrt{\frac{3(r_c-r)}{\pi(u+v)}} \tag{44}$$

is denoted explicitly in Fig. 2(a) as this is the lowest temperature at which region $O3$ obtains. Immediately below region $O3$ where $\gamma-\Delta_0(T)\propto[r-r_c(T)]^2/T$ and to the right of $O4$ where $\gamma-\Delta_0(T)\propto T^3/\sqrt{\ln\Lambda_\omega/T}$, a transition to a new region occurs in which the gap takes the form

$$\begin{aligned}\Delta&=\gamma-\frac{2g\zeta}{u+v}\left[\frac{(u+v)\gamma^3}{2\pi}\ln\frac{\Lambda_\omega}{T}+(u+v)\gamma^2T-[r\right.\\&\quad\left.-r_c(T)]\gamma\right]=\gamma-\zeta\psi^2(T).\end{aligned} \tag{45}$$

In this region, denoted as $O5$ in Fig. 2(a), as well as in regions $O3$ and $O4$, classical thermal fluctuations dominate. These regions can be construed as being quantum critical.

Further away in region $O1$, the ferromagnetic phase is impervious to thermal fluctuations. The crossover to this regime occurs when $\gamma \approx O(T)$. This partition is the dashed line separating region $O5$ from $O1$. In Fig. 2(b), the corresponding phases are shown for $r-r_c>0$. The key difference with the $r-r_c<0$ regime is the absence of the spin-glass phase.

Experimentally, the phase diagram has been measured for the random Ising spin system $LiHo_xY_{1-x}F_4$ at finite temperature. This system possesses all three phases discussed here. It then serves as a bench mark test of the phenomenological theory we have developed. While the overall features of the experimentally-determined phase diagram are similar to that shown in Fig. 2(a), it is worth looking closely at the form of the boundaries between the three phases. Particularly striking in the experimentally determined phase diagram[6] is the close to linear dependence of the PM-FM phase boundary away from the bicritical region but a nonlinear dependence on the doping level in the vicinity of the bicritical region. This dependence mirrors closely the behavior of the PM-FM finite temperature phase boundary shown in Fig. 2(a). While a quantitative comparison cannot be made because of the phenomenological nature of the coupling constants used in this model, the agreement with experiment is sufficiently striking and serves to justify the applicability of the model used here.

B. $M>1$

We consider now explicitly $M>1$. For the problem at hand, the ordered phase for $M=2$ corresponds to a superconductor. Analogous isotropic solutions can be obtained for $M>1$ with the transformations $u+v\rightarrow u+Mv$. However, because nonzero mean generates spontaneously an effective magnetization, there exists a possibility that the different spin components of the replica Q matrices might acquire fundamentally different values as first proposed by Gabay and Toulouse.[10] In the zero-mean case, this happens only when a magnetic field is present. However, in this case, the Gabay-Toulouse (GT) line exists for all $M>1$ as a result of the spontaneously-generated magnetization.

To explore the possibility of a GT line, we must generalize the ansatz for the Q matrices to explicitly break the symmetry between the spin components of Q. The simplest way of doing this is to divide the spin components of the Q matrix into longitudinal, $\mu=\nu=1$ and transverse, $\mu=\nu\neq 1$ sectors. Hence, in Eq. (25), we introduce the parameters q_L^{ab}, $\tilde{q}_L$, and $\overline{D}_L(\omega)$ for the longitudinal $\mu=\nu=1$ component and q_T^{ab}, $\tilde{q}_T$, and $\overline{D}_T(\omega)$ for the transverse components $\mu>1$. At the replica-symmetric level, both q_L^{ab} and q_T^{ab} are constants independent of the matrix label ab. We will call these constants q_L and q_T, respectively. The resultant expression for the free energy

$$\begin{aligned}\frac{\mathcal{F}}{n}=&\frac{1}{\beta\kappa t}\sum_{\omega\neq 0}(\omega^2+r)\overline{D}_L(\omega)+\frac{M-1}{\beta\kappa t}\sum_{\omega\neq 0}(\omega^2+r)\overline{D}_T(\omega)+\frac{r}{\kappa t}\tilde{q}_L+(M-1)\frac{r\tilde{q}_T}{\kappa t}-\frac{\kappa}{3\beta t}\sum_{\omega}[\overline{D}_L^3(\omega)+(M-1)\overline{D}_T^3(\omega)]\\&+\frac{u}{2t}\left\{\left[\tilde{q}_L+\frac{1}{\beta}\sum_{\omega\neq 0}\overline{D}_L(\omega)\right]^2+(M-1)\left[\tilde{q}_L+\frac{1}{\beta}\sum_{\omega\neq 0}\overline{D}_T(\omega)\right]^2\right\}+\frac{v}{2t}\left\{\tilde{q}_L+(M-1)\tilde{q}_T\right.\\&\left.+\frac{1}{\beta}\sum_{\omega\neq 0}[\overline{D}_L(\omega)+(M-1)\overline{D}_T(\omega)]\right\}^2-\frac{\kappa\beta^2}{3t}(\tilde{q}_L^3-3\tilde{q}_Lq_L^2+2q_L^3)-(M-1)\frac{\kappa\beta^2}{3t}(q_T^3-3\tilde{q}_Tq_T^2+2q_T^3)\\&-\frac{\beta\psi^2}{\kappa g t}(\tilde{q}_L-q_L)+\frac{1}{2g}\left(-\gamma\psi^2+\frac{\zeta\psi^4}{2}\right)\end{aligned}\tag{46}$$

is a generalization of Eq. (26) to an anisotropic system. The explicit factor of $M-1$ arises from the separation into transverse and longitudinal components.

If we approach the GT line from below, we find that the relevant saddle-point equations are

$$\begin{aligned}\Delta_L^2=&r+(u+v)\kappa\left[\tilde{q}_L+\frac{1}{\beta}\sum_{\omega\neq 0}\overline{D}_L(\omega)\right]\\&+(M-1)v\kappa\left[\tilde{q}_T+\frac{1}{\beta}\sum_{\omega\neq 0}\overline{D}_T(\omega)\right],\end{aligned}\tag{47}$$

$$\overline{D}_L(\omega)=-\frac{1}{\kappa}(\omega^2+\Delta_L^2)^{1/2},$$

$$\overline{D}_T(\omega)=-\frac{1}{\kappa}|\omega|,$$

$$q_L=\frac{\psi^2}{2\kappa g\Delta_L},$$

$$\tilde{q}_L=\frac{\psi^2}{2\kappa g\Delta_L}-\frac{\Delta_L}{\kappa\beta},$$

$$0=\psi(-\gamma+\zeta\psi^2+\Delta_L),$$

$$q_T=\tilde{q}_T=\frac{1}{\kappa}\left\{\frac{1}{\beta}\sum_{\omega}|\omega|+\frac{1}{u+(M-1)v}\times\left(\frac{v}{\beta}\sum_{\omega}\sqrt{\omega^2+\Delta_L^2}-r-\frac{v\psi^2}{2g\Delta_L}\right)\right\}$$

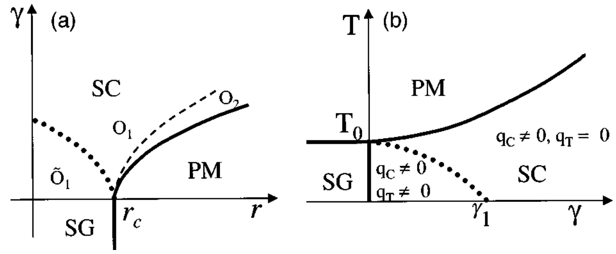

FIG. 3. (a) Zero and (b) finite-temperature phase diagrams for $M>1$. The criteria for distinguishing regions $O1$ and $O2$ are identical to the $M=1$ case but except $u+v\rightarrow u+Mv$. The GT line separates regions $\tilde{O}1$ from $O1$. This line terminates at $\gamma_1=\sqrt{u(r_c-r)/v}$. Below this line both the transverse and longitudinal components of the replica off-diagonal components of the Q matrix are nonzero. The location of the GT line is given by Eq. (49). (b) Finite temperature GT line as determined by Eq. (50).

which are a direct generalization of the $M=1$ equations to the anisotropic system. We are particularly interested in the solution in the γ-r plane where $q_T=0$. This demarcates the GT line. Below the GT line, $\tilde{q}_T=q_T\neq 0$, while above, the transverse replica off-diagonal component of Q vanishes. Note this state of affairs does not occur unless $\psi\neq 0$. If we substitute the nontrivial solution for ψ into the expression for Δ_L^2, we find that within logarithmic accuracy at zero temperature, we recover the result obtained previously for $M=1$ but with $u+v\rightarrow u+Mv$. The phase diagram hence is identical to that shown in Fig. 1. However, a new region, $\tilde{O}1$, appears. This is illustrated in Fig. 3(a). To find the line demarcating this region we must solve for the transverse replica off-diagonal component of Q. After several manipulations of the set of equations in Eq. (47), we find that

$$q_T=\frac{1}{\kappa(u+Mv)}\left[r_c-r-\frac{v}{u}\Delta_L^2\right]. \tag{48}$$

If we use the fact that $\Delta_L\approx\gamma$, we find that the GT line occurs when

$$\gamma=\sqrt{\frac{u}{v}(r_c-r)}. \tag{49}$$

The phase diagram depicting this line at zero temperature is shown in Fig. 3(a). In the region labeled $\tilde{O}1$, $q_T\neq 0$ and $q_L\neq 0$, whereas in $O1$ only $q_L\neq 0$. Hence, we have identified the zero-temperature GT line. At finite temperature, the generalization of the Eq. (49) is simply

$$\gamma=\sqrt{\frac{u}{v}[r_c(T)-r]}=\sqrt{\frac{v}{u}\left[r_c-r-(u+Mv)\frac{\pi T^2}{3}\right]}. \tag{50}$$

Hence, the GT line is now a surface in the γ, r, and T space, a slice of which is shown in Fig. 3(b). At the point $\gamma=0$, we recover the isotropic result that

$$q_T=q_L=\frac{r_c(T)-r}{\kappa(u+Mv)} \tag{51}$$

in the spin-glass phase. This expression is the generalization of Eq. (41) to $M>1$ and hence reflects the continuous nature of the order parameter at the multiple phase boundaries.

We now consider the region above the GT line (see Fig. 4). This region was not analyzed by RSY. However, this region is of considerable interest because although q_T vanishes in this region, the nonzero transverse component of the order parameter becomes gapped, significantly different in character from the longitudinal one. This can be seen immediately by setting $q_T=\tilde{q}_T=0$ in Eq. (46) and differentiating with respect to $\bar{D}_T(\omega)$. From this operation, we find that contrary to the ungapped transverse component of Q below the GT line,

$$\bar{D}_T(\omega)=-\frac{1}{\kappa}(\omega^2+\Delta_T^2)^{1/2} \tag{52}$$

with

$$\Delta_T^2=r+\frac{u+(M-1)v}{\beta}\sum_\omega\sqrt{\omega^2+\Delta_L^2}+v\left(\frac{\psi^2}{2g\Delta_L}-\frac{1}{\beta}\sum_\omega\sqrt{\omega^2+\Delta_L^2}\right) \tag{53}$$

above the GT line. The corresponding expression for Δ_L is easily obtained from the first equation in Eq. (47) by setting $q_T=0$. The ferromagnetic phase above the GT line can be divided into two regions $O1$ and $O2$ which are now different with respect to the relative magnitudes of Δ_L and Δ_T. In the region $O2$ which is completely analogous to the corresponding region in the $M=1$ case, Δ_L and Δ_T are almost equal and are given by Eq. (36) with $u+v\rightarrow u+Mv$. The condition for crossover to $O1$ in which Δ_L and Δ_T are somewhat different in magnitude is given by Eq. (37). This conclusion is reached by manipulating the system of equations in Eq. (47) with $q_T=\tilde{q}_T=0$ and $\bar{D}_T(\omega)$ given by Eq. (52). The resultant expression

$$v\Delta_L^2-(u+v)\Delta_T^2=-ur+\frac{u(u+Mv)}{\beta}\sum_\omega\sqrt{\omega^2+\Delta_T^2}, \tag{54}$$

which is valid at any temperature, contains both the transverse and longitudinal gaps. Within the approximation that $\Delta_L\approx\gamma$, we obtain that at zero temperature in $O1$,

$$\Delta_T=\left(\frac{4\pi[v\gamma^2+u(r-r_c)]}{u(u+Mv)\ln\{\Lambda_\omega^2/[v\gamma^2+u(r-r_c)]\}}\right)^{1/2}. \tag{55}$$

From this equation it is immediately clear that Δ_T is logarithmically smaller than Δ_L. Also, we easily recover the GT line $\gamma=\sqrt{u(r_c-r)/v}$, simply by solving $\Delta_T=0$.

At finite temperature, we can formally distinguish three limiting cases: (1) $\Delta_L\gg T$ and $\Delta_T\gg T$, (2) $\Delta_L\gg T$ and $\Delta_T\ll T$, and (3) $\Delta_L\ll T$ and $\Delta_T\ll T$. The regime $\Delta_L\ll T$ and $\Delta_T\gg T$ does not exist as Δ_T is always less than Δ_L. Case (1) is identical to the $T=0$ limit whereas cases (2) and (3) are the high-temperature limit with respect to Δ_T. To describe cases (2) and (3), we calculate the sum in Eq. (54) in the high-temperature limit with the approximation that $\Delta_L \approx \gamma$. Within logarithmic accuracy, the resultant equation

$$v\gamma^2 - u[r_c(T)-r] = u(u+Mv)\left(\frac{\Delta_T^2}{2\pi}\ln\frac{\Lambda_\omega}{T} + T\Delta_T\right) \tag{56}$$

is similar in structure to Eq. (42). Consequently, within cases (2) and (3), two distinct regimes denoted by $O5$ and $O5'$ and $O6$ and $O6'$, respectively, arise. These regions are illustrated in Fig. 4. In the regions superscripted with a prime, two conditions hold

$$\sqrt{v\gamma^2-[r_c(T)-r]} \ll T,$$

$$\Delta_T = \frac{v\gamma^2 - u[r_c(T)-r]}{T} \quad O5 \quad \text{and} \quad O6. \tag{57}$$

Contrastly, in the unprimed regions

$$\sqrt{v\gamma^2-(r_c-r)} \ll T,$$

$$\Delta_T = \sqrt{\frac{2\pi T^2}{3\ln(\Lambda_\omega/T)}} \quad O5' \quad \text{and} \quad O6'. \tag{58}$$

The transition between the primed and unprimed regions occurs when $\sqrt{v\gamma^2 - u[r_c(T)-r]} \approx T/\ln\Lambda_\omega/T$, whereas the transition between $O6'$ and $O1$ occurs when $\sqrt{v\gamma^2 - u(r_c - r)} \approx T$. The difference between regions $O5$ and $O5'$ and regions $O6$ and $O6'$ is that $T \ll \Delta_L$ in the latter whereas the opposite is true in the former. We have taken particular care in distinguishing the primed from the unprimed regions because they imply that the GT transition is identical in structure to the ordinary paramagnet–spin-glass transition described by RSY. Except, only the transverse component of Q is affected at the GT transition. In fact, regions $O5'$ and $O6'$ are quantum critical with respect to the GT transition while the temperature dependence of the transverse component is ''classical'' in regions $O5$ and $O6$. In $O1$, thermal fluctuations are subservient to quantum fluctuations for both transverse and longitudinal components of Q.

In each of these regions, a key question that can be addressed is how does the transverse gap renormalize ψ and Δ_L. Consider first the regime $\gamma \gg T$. In this regime, we find that

$$\Delta_L = \gamma - \zeta\psi^2(T) = \gamma - \frac{2g\zeta}{u+Mv}\left[\frac{\gamma^3}{2\pi}\ln\frac{\Lambda_\omega}{\gamma} + (r_c - r)\gamma - \frac{(M-1)v\gamma}{u(u+Mv)}\Delta_T^2\right]. \tag{59}$$

It follows immediately from Eq. (55) that the transverse contribution to the longitudinal gap is logarithmically small. As a result, the order parameter ψ is well described by its value given by Eq. (39) with $u+v \to u+Mv$. Consider now the high-temperature regime. In this case, Δ_L is exactly given by Eq. (45) plus the term proportional to Δ_T^2 in Eq. (59). However, we checked explicitly that in every subregime, the correction due to Δ_T^2 is subdominant to the leading terms. Consequently, the order parameter ψ and Δ_L are both unrenormalized by the transverse gap to the leading order.

Using the solutions delineated in Eq. (47), we can calculate the free-energy density below and above the GT line. Above and below the GT line, we find that the free-energy density at zero-temperature up to the leading γ-dependent terms

$$\frac{\mathcal{F}}{n} = \mathcal{F}_0(u+v \to u+Mv, \zeta) - \frac{(M-1)v\Lambda_\omega^2\gamma^2}{4\pi u} + \frac{\gamma^2(r-r_c)}{2u(u+Mv)}\left(1 - \frac{(M-1)v}{u}\right) + \cdots, \tag{60}$$

contains the standard analogous contribution from the $M=1$ analysis as well as q_T- and Δ_T-dependent terms arising from Eq. (46). It is the contribution from the latter terms that results in a suppression of the $\gamma^4 \ln\Lambda_\omega/\gamma$ as the leading γ-dependent terms when $r = r_c$. At the bicritical point, we find that in contrast to the $M=1$ case, the leading term in the free energy density is analytic in the coupling constant γ. This term is of the form $\Lambda_\omega^2\gamma^2$.

IV. REPLICA-SYMMETRY BREAKING

The requisite[10] for a replica asymmetric solution within Landau theory of spin glasses is the presence of a $Q^{4\text{th}}$ term in the Landau action. We specialize to $M=1$ for simplicity. To facilitate such an analysis, we must extend the cumulant expansion of Eq. (13) to the next order in perturbation theory. As RSY have performed such an analysis for the spin-glass phase, we focus on the ferromagnetic case. While several types of fourth-order terms occur, the most relevant is of the form

$$-\frac{y_1}{6t}\int d^dx \int d\tau_1 d\tau_2 \sum_{ab} [Q^{ab}(x,\tau_1,\tau_2)]^4. \tag{61}$$

This term will give rise to a $(q^{ab})^4$ contribution to the free energy. Our focus is the resultant change in the free energy

$$\frac{\Delta\mathcal{F}}{n} = -\frac{\kappa\beta^2}{3t}(\mathrm{Tr}\,q^3 + 3\tilde{q}\,\mathrm{Tr}\,q^2) - \frac{\beta\psi^2}{\kappa g t}\sum_{a,b} q^{a,b} - \frac{y_1\beta}{6t}\sum_{ab}(q^{ab})^4. \tag{62}$$

The presence of the ψ^2 term suggests that within the space of ultrametric functions[12] $q(x)$ on the interval $0 \leq x \leq 1$, we should choose an ansatz for q,

$$q(s) = \begin{cases} q_0 & 0<s<s_0, \\ \dfrac{\kappa\beta s}{2y_1} & s_0<s<s_1, \\ q_1 & s_1<s<1 \end{cases} \tag{63}$$

that has two distinct plateaus. This insight is based on an analogy with the replica broken-symmetry solution in the presence of a magnetic field. From continuity, we must have that $q_0 = \kappa\beta s_0/2y_1$ and $q_1 = \kappa\beta s_1/2y_1$. The constants s_0 and s_1 can be determined from the saddle-point equations for

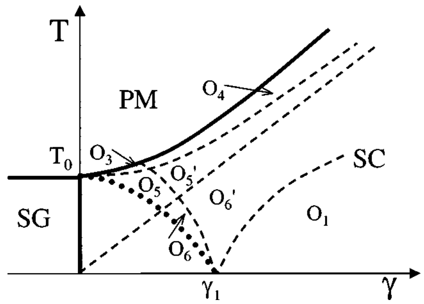

FIG. 4. Phase diagram illustrating the distinct regimes that occur at finite temperature above the GT line denoted with solid circles. Regions $O3$ and $O4$ are as described in Fig. 2(a) previously. Regions $O5'$ and $O6'$ quantum critical with respect to the GT transition. The difference between regions $O5$ and $O5'$ and regions $O6$ and $O6'$ is that $T \ll \Delta_L$ in the latter whereas the opposite is true in the former.

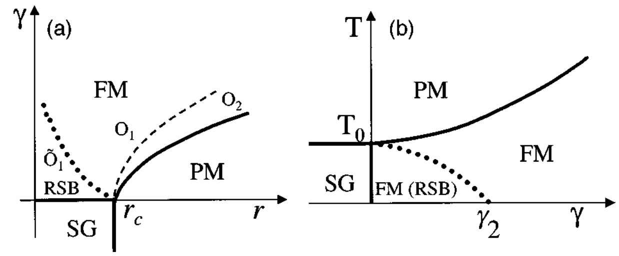

FIG. 5. (a) Phase diagram illustrating replica-symmetry breaking for $M=1$ in the $\gamma - r$ plane. The broken symmetry region is denoted RSB. Regions $O1$ and $O2$ are as before. (b) Replica-symmetry breaking region (shaded region) at finite temperature. The finite temperature criterion is given by Eq. (68). The point $\gamma_2 = 2y_1(r_c-r)^2/[3\kappa^2 g(u+v)]$ is determined by the replica-symmetry breaking condition specified in Eq. (68).

$\Delta\mathcal{F}$. Upon differentiating with respect to q_1, we find that $\tilde{q} = q_1 - y_1 q_1^2/\kappa\beta$. The corresponding equation for q_0 provides a relationship

$$q_0 = \left(\frac{3\psi^2}{4y_1\kappa g} \right)^{1/3} \tag{64}$$

between q_0 and ψ. Replica symmetry breaking occurs when $q_0 < q_1$. To leading order in temperature, we can approximate $q_1 \approx \tilde{q}$, where $\tilde{q}$ is given by

$$\tilde{q} = \frac{\psi^2}{2\kappa g \Delta}. \tag{65}$$

Hence, the boundary demarcating the replica-symmetric solution is determined by

$$\psi^{4/3} = \left(\frac{6\kappa^2 g^2}{y_1} \right)^{1/3} \Delta \tag{66}$$

which we obtain upon equating q_0 and q_1. If we use Eq. (39) for ψ which is valid in region $O1$ and use the fact that in this region $\Delta \approx \gamma$, we obtain

$$\gamma = \frac{2y_1(r_c - r)^2}{3\kappa^2 g(u+v)^2} \quad r < r_c \tag{67}$$

as the condition for replica symmetry breaking at low temperatures $T \ll \gamma$. This condition for replica-symmetry breaking extends continuously to the spin-glass phase agreeing with the work of RSY. The phase diagram illustrating the replica-broken symmetry region is depicted in Fig. 5. At finite temperature, we use Eq. (45) for ψ and obtain a generalization

$$\gamma = \frac{2y_1}{3\kappa^2 g} \left[\gamma T + \frac{r_c - r}{u+v} \right]^2 \quad r < r_c \tag{68}$$

of the replica-breaking condition which extends smoothly over to the zero-temperature condition.

To estimate the strength of the replica-symmetry breaking, we define the effective broken ergodicity order parameter

$$\Delta_q = q_1 - \int_0^1 q(s)\,ds. \tag{69}$$

If we substitute the expression for $q(s)$ and integrate, we obtain

$$\Delta_q = \frac{y_1 T}{\kappa} [q_1^2(T) - q_0^2(T)]. \tag{70}$$

Because both q_0 and q_1 are finite at low temperatures, $\Delta_q \to 0$ as $T \to 0$. The weakness of the replica-symmetry breaking in the ferromagnetic phase is in accord with the weak-symmetry breaking found by RSY in the spin-glass phase. Implicit in the replica-broken solution in Eq. (63) is the presence of many degenerate energy minima in the energy landscape. The weakness of replica symmetry-breaking at low temperatures in this model within the ferromagnetic phase suggests that the ferromagnetic phase is energetically homogeneous.

V. SUMMARY

We have constructed here a Landau theory near the bicritical point for a ferromagnet, spin glass, and paramagnet. The analogous analysis was also performed for $M>1$ in which the ordered phase for $M=2$ is a superconductor. All transitions were found to be second order in contrast to the work of Hartman and Weichman[9] who claimed that the FM-SG transition was first order in the spherical limit. The key difference between our treatments is that fluctuations are absent in the current analysis. As a result our work indicates that fluctuations might drive the FM-SG transition to a first order transition. A key result of this work is the formal equivalence between the role of a nonzero mean and the presence of a magnetic field in the zero mean problem. This observation is equally valid for classical systems. Resilience of the ferromagnet against thermal fluctuations occurs when $T < \gamma$, where γ is the coupling constant that ultimately determines the rigidity of the ferromagnet phase. Additional features of our analysis that are particularly striking are (1) the nonanalytic dependence of ψ on γ, namely, $\psi \propto \gamma^{3/2}/\ln \Lambda_\omega/\gamma$, near the bicritical region, in the vicinity of the bicritical point and (2) the subsequent leading nonanalytic dependence of the free-energy density in region $O1$ on γ for $M=1$. We showed that this behavior ceases for M

>1 below and above the GT line. The reason underlying this difference with the $M=1$ case is the presence of $M-1$ components of Q that yield leading analytic contributions of order $O(\gamma^2)$ to the free energy density. An additional feature which our analysis brings out for $M>1$ is the similarity between the Gabay-Toulouse transition with the spin-glass–paramagnet transition with zero mean. This similarity has been noticed previously in the context of classical spin glasses.[13] A key surprise found in the analysis of the GT transition is the subleading depdence of ψ and Δ_L on the transverse gap and q_T. This suggests that the GT line should have only weak experimentally detectable features in the superconducting phase, for example. The excellent agreement observed with the experimental results on $LiHo_xY_{1-x}F_4$ for the case of $M=1$ is encouraging that similar agreement will be found with experiments on the analogous superconducting systems.

While we referred to the point of intersection between all the phases as being bicritical, it is in fact multicritical. This state of affairs obtains as a result of the presence of replica symmetry breaking and the Gabay-Toulouse instability. As in the $M=1$ spin-glass case, we also showed that the nonergodicity parameter is linear in temperature illustrating the weakness of replica symmetry breaking in the ferromagnetic phase. Though we did not treat explicitly replica-symmetry breaking for $M>1$, such symmetry breaking is expected in this case as well when quartic terms are included in the action. In a future study, we will extend this analysis beyond mean field and report on the renormalization group analysis of the ''bicritical'' region.

ACKNOWLEDGMENTS

We thank S. Sachdev for useful comments during the early stages of this project. This work was funded by the DMR of the NSF and the ACS petroleum research fund.

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