Oscillator model for vacuum Rabi splitting in microcavities

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The vacuum-field Rabi splitting of an optical mode in a cavity interacting with a system of N excitations (atomic or electronic) represented by harmonic oscillators is obtained quantum mechanically both without and with coupling to dissipation. The optical lineshapes and the scaling of the Rabi splitting with the number of oscillators and its dependence on the damping parameters are given. Corresponding results are given for a classical treatment of the oscillators, and it is shown that to the leading order in the dissipation, the quantum mechanical results for absorption are the same as the classical ones. [S0163-1829(99)09515-6]

I. INTRODUCTION

The couplings between electronic excitations and optical modes have been of considerable interest in connection both with atoms in optical cavities and also with optically active semiconductors in semiconductor microcavities. The emission and reflection spectra of atoms in optical cavities are known to exhibit splittings due the coupling of their dipole transitions with the excitations of the vacuum-radiation field.[1,2] These coupled modes are called the vacuum-field Rabi oscillations, and the splittings between them are the vacuum-field Rabi splittings. In the case of semiconductor microcavities the optical modes are coupled to excitons whose modes are relatively sharp, and exciton-photon modes are often called cavity polaritons.[3,4] These couplings are important for understanding such effects as optical bistability and laser action.

Theoretical studies of these coupled modes often have been made using a two-level spin model to represent the electronic excitations of the atoms or of the solid. In this representation the higher-lying states of the excitation are neglected. The difference in energy between the two levels is taken to be the excitation energy between the lowest two states of the atom, $\hbar\omega = \Delta E_a$, or the excitation energy of an exciton in a semiconductor, $\hbar\omega = E_g - E_b$, where E_g is the band-gap energy and E_b is the exciton binding energy. A single two-level system coupled to an electromagnetic mode both without[5] and with dissipation[6,7] has been studied. Dissipation represents, for example, the finite Q of the cavity, which could be due to leakage of the electromagnetic modes from the cavity. Systems with N two-level systems have also been studied,[6] where it has been shown that the Rabi splitting is proportional to $\sqrt{N}$, and the correlation functions giving lineshapes have been studied.

An harmonic oscillator can also be used to represent the atomic excitations, and this representation offers considerably greater ease of mathematical treatment than does the spin representation. In this case the higher-lying states are represented only approximately by the evenly spaced states of the oscillator. It has been shown that the oscillator exhibits vacuum-field Rabi oscillations when coupled to an electromagnetic mode,[1,8] and results have been given for the Rabi splitting of N classical oscillators in the presence of dissipation.[2] A detailed quantum-mechanical study of a single-harmonic oscillator representing the electromagnetic mode coupled to dissipation has been given.[9]

To date, however, to our knowledge a quantum-mechanical treatment of the vacuum-field Rabi splitting of N harmonic oscillators with damping has not been given. We give such a treatment here. The dependence of the optical lineshape on the number of oscillators N and on the dissipation is presented. A derivation of the classical N harmonic-oscillator model is also given, and we find that to leading order in the friction the quantum mechanical and classical results are the same.

In Sec. II results are given for a model of N classical oscillators with dissipation. A quantum-mechanical treatment for N oscillators with dissipation is given in Sec. III. An outline of the quantum-mechanical treatment for one oscillator with dissipation[9] is given in the Appendix.

II. CLASSICAL OSCILLATOR WITH FRICTION

We begin by considering a system of N oscillators with frequencies ω_1 representing electronic excitations coupled to one oscillator of frequency ω_0 representing a single mode of the radiation field. In the absence of dissipation the system is described by the Lagrangian

$$L = T - V = \frac{1}{2}\dot{x}_0^2 + \frac{1}{2}\sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2}\omega_0^2 x_0^2 - \frac{1}{2}\sum_{i=1}^{N} \omega_i^2 x_i^2 + \alpha \sum_{i=1}^{N} x_0 x_i . \tag{1}$$

Here the dots represent time derivatives, and α represents the coupling between the oscillators and the radiation field, which is proportional to the oscillator strength of the electronic transition.[10]

In order to represent dissipation we introduce a general friction matrix

$$F=\frac{1}{2}\sum_{i,j=0}^{N}\dot{x}_iF_{ij}\dot{x}_j. \tag{2}$$

The equations of motion are given as usual[11] by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)-\frac{\partial L}{\partial x_i}+\frac{\partial F}{\partial \dot{x}_i}=0, \quad i=0,1,\ldots N. \tag{3}$$

For simplicity we choose the friction matrix to be diagonal $F_{ii}=\gamma_i\delta_{ii}$ with $\gamma_i=\gamma_0$ for $i=0$ and $\gamma_i=\gamma_1$ for $i=1,\ldots N$.

We look for solutions of Eq. (3) in the form of linear combinations of harmonics $e^{i\Omega t}$ with frequencies that are given by the determinental equation

$$\det|V_{ij}-\Omega^2T_{ij}+i\Omega F_{ij}|=0. \tag{4}$$

This equation factorizes to give $N-1$ degenerate solutions with complex frequencies

$$\Omega_d=i\frac{\gamma_1}{2}+\sqrt{\omega_1^2-\frac{\gamma_1^2}{4}} \tag{5}$$

and two solutions satisfying

$$(\omega_0^2-\Omega^2+i\gamma_0\Omega)(\omega_1^2-\Omega^2+i\gamma_1\Omega)-\alpha^2N=0. \tag{6}$$

Near resonance ($\omega_1\approx\omega_0$) and taking $\gamma_i/\omega_i\ll 1$ and $\alpha\sqrt{N}/\omega_i\ll 1$ this equation has two solutions

$$\Omega_{0,1}^2\approx\omega_1^2+i\frac{\omega_1}{2}(\gamma_0+\gamma_1)\mp\frac{1}{2}\sqrt{4N\alpha^2-\omega_1^2(\gamma_0-\gamma_1)^2}. \tag{7}$$

For $|\gamma_1-\gamma_0|>2\alpha\sqrt{N}/\omega_1$ the solutions are overdamped. For $|\gamma_1-\gamma_0|<2\alpha\sqrt{N}/\omega_1$, on the other hand, the solutions are damped-harmonic oscillators whose frequencies are separated by

$$\Delta\Omega=\Omega_1-\Omega_0\cong\sqrt{\frac{\alpha^2N}{\omega_1^2}-\frac{1}{4}(\gamma_1-\gamma_0)^2}. \tag{8}$$

This is equivalent to the result obtained in Ref. 2. If the widths γ_0 and γ_1 are both zero, we obtain the same result as that obtained in the model using two-level systems,[1] $\Delta\Omega\approx\alpha\sqrt{N}/\omega_1$ where the Rabi splitting is proportional to $\sqrt{N}$.

It is straightforward to generalize the result in Eq. (8) to the case where the couplings α_i are all different from one another. The corresponding term in the Lagrangian Eq. (1) is $\Sigma_{i=1}^{N}\alpha_ix_ix_0$, and neglecting dissipation

$$\Delta\Omega\approx\frac{\sqrt{N}}{\omega_i}\sqrt{\langle\alpha_i^2\rangle} \tag{9}$$

where $\langle\alpha_i^2\rangle=1/N\Sigma_i^N\alpha_i^2$.

Within the classical model discussed in this section, the absorption lineshape can be obtained from the energy-dissipation rate of oscillators when the system is driven by an applied-harmonic force $f(t)=f_0\cos\omega t$, acting on oscillators representing atomic (or excitonic) dipoles $n\geqslant 1$. The resulting system of linear equations can be solved simply, and the rate of dissipation is obtained from the friction form[11] in Eq. (2) averaged over one period of $f(t)$ in the limit of large times. Assuming the resonance condition $\omega_0\approx\omega_1$, and omitting terms that are resonant at negative frequencies, we obtain the following dependence of the dissipation rate on the frequency of applied force:

$$2\overline{F}(\omega)=\frac{f_0^2N}{4\omega_1^2}\frac{\gamma_1(\omega-\omega_1)^2+\gamma_0\overline{\alpha}^2N/4+\gamma_1\gamma_0^2}{[(\omega-\omega_1)^2-\overline{\alpha}^2N/4-\gamma_0\gamma_1/4]^2+(\gamma_0+\gamma_1)^2(\omega-\omega_1)^2/4}, \tag{10}$$

where $\overline{\alpha}=\alpha/\omega$. This spectrum has two peaks whose separation is the Rabi splitting in the absorption as represented in the classical oscillator model. The splitting Δ_{RS} is different from the difference of the resonance frequencies given in Eq. (8). The maxima of the absorption can be found from the zeros of $\partial\overline{F}/\partial\omega$. For example, when γ_0 and γ_1 are small compared to α/ω_1, the distance between peaks can be approximated as

$$\Delta_{RS}\approx\sqrt{\overline{\alpha}^2N-\frac{1}{2}\gamma_0(\gamma_1-\gamma_0)}+O(\gamma_0^2\gamma_1^2), \tag{11}$$

and the full widths at half maximum of the coupled oscillators are given in terms of the widths of the uncoupled modes by $\gamma\approx 1/2(\gamma_0+\gamma_1)$.

III. QUANTUM OSCILLATORS WITH DISSIPATION

Here we derive the modes and also the correlation functions of a system of N quantum-mechanical harmonic oscillators representing the electronic excitations coupled to an optical mode in the case where both the N oscillators and the optical mode are coupled to a dissipative-loss mechanism. The present paper generalizes that of Ref. 9, which treated the case of a single oscillator coupled to an optical mode with dissipation. For clarity we outline the results for the case of one oscillator in the Appendix. In this paper the source of dissipation is represented in a general way by a mechanism with closely spaced energy levels like that of a heat bath. Physically the dissipation could represent, for example, photon leakage from the cavity and exciton scattering from phonons.

The Hamiltonian for a set of N harmonic oscillators each coupled linearly to one oscillator representing the optical mode is

$$H_{osc}=\frac{p_0^2}{2}+\frac{\omega_0^2}{2}x_0^2+\frac{1}{2}\sum_{i=1}^{N}(p_i^2+\omega_1^2x_i^2)-\alpha\sum_{i=1}^{N}x_0x_i. \tag{12}$$

The coupling to the ''0'' oscillator can be included either through the coordinate or the momentum operators. In the resonant approximation,[10] which is sufficient for ω_0 close to ω_1, the Hamiltonian is given in terms of creation and annihilation operators as

$$H_{\rm osc}=\hbar\omega_0 a_0^+ a_0+\sum_{i=1}^{N}\hbar\omega_1 a_i^+ a_i-\frac{\hbar\bar{\alpha}}{2}\sum_{i=1}^{N}(a_0^+ a_i+a_i^+ a_0)+C_0, \tag{13}$$

where $\bar{\alpha}=\alpha/\sqrt{\omega_0\omega_1}$, $C_0=\hbar/2(\omega_0+N\omega_1)$, and $a_j=\sqrt{\omega_0/2\hbar}x_j+i/\sqrt{2\hbar\omega_j}p_j$.

In the absence of dissipation the spectrum is found easily by a linear transformation which diagonalizes $H_{\rm osc}$:

$$b_i=\sum_j A_{ij}a_j, \tag{14a}$$

$$H_{\rm osc}=\sum_{i=1}^{N}\hbar\Omega_i b_i^+ b_i, \tag{14b}$$

where energy is measured relative to C_0. Then using the relation $[b,H_{\rm osc}]=\hbar\Omega_n b_n$, we obtain from this quantum-mechanical treatment the same results that we found in the classical case, viz. that there are $N-1$ degenerate frequencies $\Omega=\omega_1$ and two nondegenerate frequencies given by

$$\Omega_{0,1}=\frac{1}{2}(\omega_0+\omega_1)\mp\frac{1}{2}\sqrt{(\omega_0-\omega_1)^2+\bar{\alpha}^2N}, \tag{15}$$

so that at resonance the Rabi frequency is $\Delta\Omega=\bar{\alpha}\sqrt{N}=\alpha\sqrt{N}/\omega_1$ and it is again proportional to $\sqrt{N}$.

In order to describe the effects of dissipation that give rise to nonzero linewidths of the coupled transitions, we couple each of the oscillators linearly to a loss mechanism

$$H=H_{\rm osc}+H_l+g_0p_0\Gamma_0+\sum_{i=1}^{N}g_1p_n\Gamma_n. \tag{16}$$

Here we have taken the oscillators $i=0,1,\ldots N$ to be coupled to some operator coordinates Γ_j of the loss mechanism, and H_l is the unspecified Hamiltonian of the loss mechanism. It does not matter whether the coupling is via the coordinates or the momenta. The loss mechanism is described by a diagonal density matrix in Eq. (A3).

At resonance the matrix A in Eq. (13) can be approximated by $A_r=A(\omega_0=\omega_1)$ giving

$$A_r=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2N}} & \frac{1}{\sqrt{2N}} & \cdots & \frac{1}{\sqrt{2N}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2N}} & \frac{1}{\sqrt{2N}} & \cdots & \frac{1}{\sqrt{2N}} \\ 0 & A_{21} & A_{22} & \ldots & A_{2N} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & A_{N1} & A_{N2} & \ldots & A_{NN} \end{pmatrix}. \tag{17}$$

Here $A_{21},\ldots A_{NN}$ can be chosen to be real. They satisfy the conditions:

$$\sum_{i=1}^{N}A_{ji}=0, \tag{18}$$

which follows from $[b_i^+,b_j]=0$, $j=0$ and 1, $i=2,\ldots,N$, and orthonormality

$$\sum_{i=1}^{N}|A_{ji}|^2=1 \quad\text{and}\quad \sum_{i=1}^{N}A_{ji}A_{j'i}=0 \quad\text{for}\quad j\neq j'.$$

The transformed position and momentum operators are denoted by Q and P, and they are related to operators b,b^+ in the Heisenberg representation in the usual way

$$b_n(t)=\sqrt{\frac{\Omega_n}{2\hbar}}Q_n(t)+\frac{i}{\sqrt{2\hbar\Omega_n}}P_n(t). \tag{19}$$

Then the Hamiltonian is

$$H=\frac{1}{2}\sum_{n=0}^{N}P_n^2+\frac{1}{2}\Omega_0^2Q_0^2+\frac{1}{2}\Omega_1Q_1^2+\frac{1}{2}\sum_{i=2}^{N}\omega_1^2Q_i^2+H_l+\sum_{n=0}^{N}P_n\tilde{\Gamma}_n, \tag{20}$$

where

$$\tilde{\Gamma}_0=\left(\frac{\omega_0}{2\Omega_0}\right)^{1/2}g_0\Gamma_0+\left(\frac{\omega_1}{2N\Omega_0}\right)^{1/2}g_1\sum_{n=1}^{N}\Gamma_n, \tag{21a}$$

$$\tilde{\Gamma}_1=-\left(\frac{\omega_0}{2\Omega_1}\right)^{1/2}g_0\Gamma_0+\left(\frac{\omega_1}{2N\Omega_1}\right)^{1/2}g_1\sum_{n=1}^{N}\Gamma_n, \tag{21b}$$

and for

$$m\geqslant 2,\ \tilde{\Gamma}_m=g_1\sum_{n=1}^{N}A_{\rm mn}\Gamma_n. \tag{21c}$$

Equation (20) was obtained by inverting Eq. (14a) and using $(A^{-1})_{\rm ij}=A_{\rm ji}$ where we have chosen all $A_{\rm ij}$ to be real.

We now derive equations of motion for operators P_n, Q_n, $n=0,\ldots N$, using approximations similar to those used for the single oscillator in the Appendix. The couplings are thought to be turned on at $t=0$. The uncoupled operators $\Gamma_n^{(0)}(t)$ are taken to have zero diagonal elements and to have the following form for $t>0$:

$$[\Gamma_n^0(t)]_{\rm ij}=\gamma_{\rm ij}^{(n)}e^{i(E_i-E_j)/\hbar},\quad i\neq j, \tag{22}$$

where $\gamma^{(n)}$ are real and $\{E_i\}$ is the energy spectrum of the uncoupled loss mechanism H_l.

We assume that Γ_n,Γ_m for $n\neq m$ do not have nonzero matrix elements in the same subspace of the Hilbert space of $H_l^{(0)}$. Then

$$\gamma_{\rm ij}^{(n)}\gamma_{\rm ji}^{(m)}=\delta_{\rm nm}|\gamma_{\rm ij}^{(n)}|^2. \tag{23}$$

This property permits a separation of the coupled equations for $P_0,P_1,\ldots P_N$ into two coupled equations for P_0,P_1 and uncoupled equations for $P_2,\ldots P_N$.

Let us first consider the Hamiltonian in Eq. (16) in the absence of coupling between the oscillators and the optical mode, $\alpha=0$. H then separates into $N+1$ commuting parts each representing a damped-harmonic oscillator. We assume that the eigenvalues of dissipation source H_l are closely spaced, and we replace the summations by energy integrations and define[9]

$$B_n(\omega)=\int_0^\infty dE\rho(E+\hbar\omega)\rho(E)e^{E/k_BT}\gamma^{(n)}(E+\hbar\omega,E)\times\gamma^{(n)}(E+\hbar\omega,E), \tag{24}$$

where $\rho(E)$ is the density of states of the spectrum $\{E_i\}$, and the damping coefficients are

$$\beta_n=g_n^2\omega_n Z^{-1}B_n(\omega_n)[1-e^{\hbar\omega_n/k_BT}], \tag{25}$$

where Z is the partition function in Eq. (A3), with $\omega_1=\omega_2=\cdots\omega_N$. In the present model it follows that $\beta_1=\beta_2=\cdots=\beta_N$, and then there are only two different damping coefficients β_0 and β_1.

Returning to the $\alpha\neq 0$ case, we note that the equations of motion for the transformed operators $P_0,P_1,\ldots P_N$ in general are coupled because in the generalization of Eq. (A8) we obtain diagonal-matrix elements of

$$([\Gamma_n^{(0)}(t_1),[\Gamma_m^{(0)}(t_2),H_l^{(0)}(t_2)]])_{ii}=2Z\sum_k \hbar\omega_k\gamma_{ik}^{(n)}\gamma_{ki}^{(m)}\cos\omega_{ik}(t_1-t_2), \tag{26}$$

where $\omega_{ik}=(E_i-E_k)/\hbar$. In view of Eq. (23) for the Γ's, these matrix elements are nonzero only if $n=m$. Then using this property with $\tilde{\Gamma}$'s from Eqs. (21) and the orthonormality properties of A we obtain

$$\ddot{P}_0+\Omega_0^2P_0+\beta_{00}\dot{P}_1+\beta_{01}\dot{P}_2\mp-\Omega_0^2\tilde{\Gamma}_0^{(0)}(t), \tag{27}$$

$$\ddot{P}_1+\Omega_1^2P_1+\beta_{10}\dot{P}_1+\beta_{11}\dot{P}_2=-\Omega_1^2\tilde{\Gamma}_1^{(0)}(t), \tag{28}$$

$$\ddot{P}_n+\omega_1^2P_n+\beta_{nn}\dot{P}_n=-\omega_1^2\tilde{\Gamma}_n^{(0)}(t),\quad n\geqslant 2, \tag{29}$$

where

$$\beta_{nn}=\beta_1,\quad n\geqslant 2,$$

$$\beta_{00}=\frac{\omega_0\beta_0+\omega_1\beta_1}{2\Omega_0},$$

$$\beta_{11}=\frac{\omega_0\beta_0+\omega_1\beta_1}{2\Omega_1},$$

$$\beta_{10}=\beta_{01}=\frac{\omega_1\beta_1-\omega_0\beta_0}{2\sqrt{\Omega_1\Omega_0}}.$$

Thus, at resonance $\beta_{00}\approx\beta_{11}\approx 1/2(\beta_0+\beta_1)$ and $\beta_{10}=\beta_{01}\approx 1/2(\beta_1-\beta_0)$ to $O(\alpha\sqrt{N}/\omega_1)$. In the equation for $Q_0(t)$ the right-hand side in Eq. (27) is replaced by $\dot{\tilde{\Gamma}}_0^{(0)}(t)+\beta_{00}\tilde{\Gamma}_0^{(0)}(t)+\beta_{01}\tilde{\Gamma}_1^{(0)}(t)$ and similarly for $Q_1(t)$.

The solutions of Eq. (29) for $n\geqslant 2$ are damped oscillators with frequency $\varpi=\sqrt{\omega^2-\beta_1^2/4}$, and they have a damping rate $\beta/2$ which is the same as in the case of a single oscillator in the Appendix [Eqs. (A13)]. Equation (27) and (28) are coupled linear-differential equations with initial conditions for the Heisenberg operators given by the corresponding Schrödinger operators. They describe coupled driven damped oscillators and can be solved by Laplace transformation. The resonant frequencies are given by the solutions of

$$(\Omega^2-\Omega_0^2+i\beta_{00}\Omega)(\Omega^2-\Omega_1^2+i\beta_{11}\Omega)+\beta_{01}\beta_{10}\Omega^2=0, \tag{30}$$

where Ω_0 and Ω_1 are given in Eq. (15). Near resonance $\omega_0\approx\omega_1$, and to orders $O(\beta^2,\alpha^2)$ the solutions of Eq. (30) can be written

$$\Omega=\Omega_0^\pm\approx\pm\omega_0-\frac{i\beta_{00}}{2}\mp\frac{\beta_{00}^2}{4\omega_0}\mp\frac{1}{2\omega_0}\sqrt{N\alpha^2-\omega_0^2\beta_{01}\beta_{10}}, \tag{31a}$$

$$\Omega=\Omega_1^\pm\approx\pm\omega_1-\frac{i\beta_{11}}{2}\mp\frac{\beta_{11}^2}{4\omega_1}\pm\frac{1}{2\omega_0}\sqrt{N\alpha^2-\omega_1^2\beta_{01}\beta_{10}}. \tag{31b}$$

Expressions for $P(t)$ and $Q(t)$ are obtained straightforwardly by inverse Laplace transforms in the complex plane. The resulting expressions are rather lengthy and will not be given here. P and Q are found to oscillate with frequencies

$$\varpi_{0,1}=\omega_1\mp\frac{1}{2\omega_1}\sqrt{N\alpha^2-\omega_1^2\beta_{01}^2}, \tag{32}$$

where we have set $\omega_0=\omega_1$. The terms in P and Q have the form $A_1e^{-A_2t}\cos(\varpi_it+\phi)$ and involve time integrals of $\Gamma_n^{(0)}(t)$. The Rabi frequency, defined as the difference between two resonant frequencies of the interacting system, is given by

$$\Delta\varpi\approx\sqrt{\frac{\alpha^2N}{\omega_1^2}-\frac{1}{4}(\beta_0-\beta_1)^2}. \tag{33}$$

The Rabi splitting, on the other hand, is obtained from the position of the peaks in the optical spectrum and in general will be different from $\Delta\omega$ in Eq. (33). We define a dipole operator for our model as $d(t)=x_1(t)+\cdots+x_N(t)$. The linear response is given by the dipole-dipole correlation function

$$G(\tau)=\lim(t\to\infty)(i/\hbar)\langle[d(t+\tau),d(t)]\rangle, \tag{34}$$

and the absorption spectrum is obtained as an imaginary part of the susceptibility[6] $\chi''(\omega)$ where $\chi(\omega)=\int_0^\infty d\tau G(\tau)e^{i\omega\tau}$. In the evaluation of the commutator in Eq. (34) we use the following property that can be derived[9] for operators $\tilde{\Gamma}_i^{(0)}(t)$:

$$\langle[\tilde{\Gamma}_i^{(0)}(t),\tilde{\Gamma}_j^{(0)}(t)]\rangle \approx \frac{\hbar}{2\pi\omega_1}\beta_{ij}\int_0^\infty d\omega[e^{i\omega(t-t')}-e^{-i\omega(t-t')}], \quad i=0,1, \quad j=0,1, \tag{35}$$

where the right-hand side is understood in the operator sense in the integrals over the time. Using Laplace transformation for $\langle[d(t+\tau),d(t)]\rangle$ to the complex frequency domain and Eqs. (34) and (35), we find

$$\chi''(\omega)=\frac{N}{8}\frac{\beta_1(\omega-\omega_1)^2+\beta_0\overline{\alpha}^2N/4}{[(\omega-\omega_1)^2-\overline{\alpha}^2N/4-\beta_0\beta_1/4]^2+(\beta_0+\beta_1)^2(\omega-\omega_1)^2/4}. \tag{36}$$

The absorption spectrum has a doublet line shape, and the Rabi splitting in it is given by the separation between the peaks. If the dissipation coefficients β_i are much smaller than $\alpha\sqrt{N}/\omega_1$ the widths of each of the two peaks are approximately $(\beta_0+\beta_1)/2$. By comparing the results for the Rabi frequency and the absorption spectrum with the corresponding results in classical model in Eqs. (8) and (10), we see that the results obtained here for a quantum-mechanical treatment of the harmonic oscillator model with N oscillators is the same as that obtained using a classical approach to leading order in dissipation.

In the present paper we have considered the linewidths of interacting modes representing photons and excitons, each of which is coupled to a source of dissipation. We would like to point out that there is an alternative model with which to represent the linewidth, particularly in the case of the photon.[12] In it, the photon is taken to be coupled to a continuum of excitations,[13] which could represent the continuum of photon states into which the chosen photon state can ''leak.'' We have also considered a model in which the photon is coupled to a continuum in this way and the exciton is coupled to a source of dissipation. In this case we find that the absorption spectrum lineshape has the form of Eq. (36) where photon linewidth β_0 is replaced by a term proportional to the coupling strength to the continuum.[14]

Using the model considered in the present paper we then can obtain the emission spectrum by assuming that the photon oscillator is weakly coupled to the states outside the cavity and considering the decay of the initial cavity photon (or polariton) state. For this we evaluate correlation function $\langle a_0(\tau)a_0^+(0)\rangle$. It is found to be a sum of terms oscillating with frequencies given in Eq. (32) and damped by β_{00}. The spectrum is given by an expression similar to Eq. (36), except the numerator does not have the term depending on $\omega-\omega_1$. Therefore, the position of the peaks is determined by the minima of the denominator, and the Rabi splitting in the emission spectra is then given by

$$\Delta_{RS}=\sqrt{\overline{\alpha}^2N-\frac{1}{2}(\beta_0^2+\beta_1^2)}. \tag{37}$$

We notice that this expression can also be obtained heuristically from the classical oscillator model of Sec. II if we assume that the energy lost by the photon oscillator is transferred into the radiation field instead of dissipating into heat.

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APPENDIX

Here we derive the correlation functions that determine optical line shapes for a single oscillator representing the electromagnetic mode of a cavity, in the presence of dissipation. We follow the development given in Ref. 9 for this case. The Hamiltonian is written

$$H=H_{osc}+H_l+gP\Gamma, \tag{A1}$$

where

$$H_{osc}=\frac{P^2}{2}+\frac{1}{2}\omega^2X^2. \tag{A2}$$

P and X are the momentum and position operators of the oscillator of frequency ω, and Γ is an operator representing the coordinate of the loss mechanism. H_l is the (unspecified) Hamiltonian of the source of the dissipation, which is pictured as having closely spaced eigenstates. It is represented by a diagonal density matrix

$$\rho_{nm}=\delta_{nm}Z^{-1}e^{-E_n/k_BT}, \tag{A3a}$$

where

$$Z=\sum_n e^{-E_n/k_BT}. \tag{A3b}$$

We assume that when the coupling $g=0$ the diagonal elements of Γ are zero.

The equations of motion of the Heisenberg operators for P and X are

$$\dot{P}=-\omega^2X, \tag{A4a}$$

$$\dot{X}=P+g\Gamma. \tag{A4b}$$

They are equivalent to the integral equations

$$P(t)=P^{(0)}(t)-\omega g\int_0^t dt'\,\Gamma(t')\sin[\omega(t-t')], \tag{A5a}$$

$$X(t)=X^{(0)}(t)+g\int_0^t dt'\,\Gamma(t')\cos\omega(t-t'), \tag{A5b}$$

where in the interaction picture operators are labeled with the superscript ''(0).'' Then

$$P^{(0)}(t)=P(0)\cos\omega t-\omega X(0)\sin\omega t \tag{A6a}$$

$$Q^{(0)}(t)=Q(0)\cos\omega t+\frac{1}{\omega}P(0)\sin\omega t. \quad \text{(A6b)}$$

$\Gamma^{(0)}_{\rm ij}(t)$ gives the dissipative coupling taken to have only off-diagonal terms and is given by the matrix

$$\Gamma^{(0)}_{\rm ij}(t)=\Gamma^{(0)}_{\rm ij}(0)e^{i(E_t-E_j)t/\hbar}=\gamma_{\rm ij}e^{i\omega_i t},$$

with $\gamma_{\rm ii}=0$, which gives $\langle\Gamma^{(0)}(t)\rangle=0$.

$\Gamma(t)$ can be found from the equation $i\hbar\dot{\Gamma}(t)=[\Gamma(t),H_l(t)]$, where $H_l(t)$ satisfies the equation

$$H_l(t)=H_l(0)+\frac{g}{i\hbar}\int_0^t dt'[H_l(t'),\Gamma(t')]P(t'). \quad \text{(A7)}$$

$\Gamma(t)$ satisfies the integral equation

$$\Gamma(t)=\Gamma^0(t)+\frac{g}{\hbar^2}\int_0^t dt'\int_0^{t'}dt''e^{iH_l^0(t-t')/\hbar}\times[\Gamma(t'),[\Gamma(t''),H_l(t'')]P(t'')e^{-iH_t^0(t-t')/\hbar}, \quad \text{(A8)}$$

which is then substituted into Eq. (A5a) to give an equation for $P(t)$.

In order to obtain a tractable result, Γ and H_l are replaced by $\Gamma^{(0)}$ and $H_l^{(0)}$, the expression $[\Gamma^0(t'),[\Gamma^0(t''),H_l^0(t'')]]P(t'')$ is replaced by its expectation value averaged over ρ in Eq. (A3), and sums are replaced by integrals.

The following quantities are defined

$$B(\omega)=\int_0^\infty dE\rho(E+\hbar\omega)\rho(E)\gamma^2(E+\hbar\omega,E)e^{-E/k_BT}, \quad \text{(A9)}$$

$$\beta(\omega)=2\omega g^2Z^{-1}\pi B(\omega)(1-e^{-\hbar\omega/k_BT}). \quad \text{(A10)}$$

Then for times $t\gg\omega^{-1}$, as shown in Ref. 9,

$$\ddot{P}+\beta\dot{P}+\omega^2P=-\omega^2g\Gamma^{(0)}(t), \quad \text{(A11)}$$

$$\ddot{X}+\beta\dot{X}+\omega^2X=g\dot{\Gamma}^0(t)+g\beta\Gamma^{(0)}(t). \quad \text{(A12)}$$

These are equivalent to the equations of a damped driven oscillator and can be solved for X and P by the Laplace transformation. The solutions are

$$P(t)=e^{-\beta t/2}\left\{\frac{1}{\varpi}\left[-\omega^2X(0)+\frac{1}{2}\beta P(0)\right]\sin\varpi t+P(0)\cos\varpi t\right\}-\frac{\omega^2}{\varpi}g\int_0^t dt'\Gamma^{(0)}(t')\times e^{-\beta/2(t-t')}\sin\varpi(t-t'), \quad \text{(A13a)}$$

$$X(t)=e^{-\beta t/2}\left\{\frac{1}{\varpi}\left[P(0)+\frac{1}{2}\beta X(0)\right]\sin\varpi t+X(0)\cos\varpi t\right\}-g\int_0^t dt'\Gamma^{(0)}(t')e^{\beta/2(t-t')}\left[\cos\varpi(t-t')+\frac{\beta}{2\varpi}\sin\varpi(t-t')\right], \quad \text{(A13b)}$$

where $\varpi\equiv\sqrt{\omega^2-\beta^2/4}$. The resulting average of the commutator of P and X over the density matrix in Eq. (3) is equal to $i\hbar$, and thus the corresponding creation and annihilation operators

$$a(t)=\left(\frac{\omega}{2\hbar}\right)^{1/2}X(t)+\frac{i}{(2\hbar\omega)^{1/2}}P(t) \quad \text{(A14)}$$

can be evaluated. Finally, the optical spectrum is obtained from the correlation functions[1]

$$G_1(t,\tau)=\langle a^+(t+\tau)a(t)\rangle, \quad \text{(A15a)}$$

$$G_2(t,\tau)=\langle a(t)a^+(t+\tau)\rangle, \quad \text{(A15b)}$$

We evaluate them for times t, $\tau\gg\omega^{-1}$, which is reasonable for ω in the range of optical frequencies. To leading order in β/ω we obtain

$$G_1(t,\tau)=e^{-\beta\tau/2+i\varpi\tau}\left[\frac{1}{e^{\hbar\omega/\mathrm{kT}}-1}+e^{-\beta t}\left(\langle a^+(0)a(0)\rangle-\frac{1}{e^{\hbar\omega/\mathrm{kT}}-1}\right)\right], \quad \text{(A16a)}$$

$$G_2(t,\tau)=e^{-\beta\tau/2+i\varpi\tau}\left[\frac{1}{1-e^{-\hbar\omega/\mathrm{kT}}}+e^{-\beta t}\left(\langle a(0)a^+(0)\rangle-\frac{1}{1-e^{-\hbar\omega/\mathrm{kT}}}\right)\right]. \quad \text{(A16b)}$$

The linear response to an applied weak perturbation is given by a commutator correlation function

$$G(\tau)=\langle[a(t),a^+(t+\tau)]\rangle=e^{-\beta\tau/2+i\varpi\tau}+O(\beta/\omega), \quad \text{(A17)}$$

which represents a damped oscillator with width β.

[1]G. S. Agarwal, J. Opt. Soc. Am. B **2**, 480 (1985).
[2]Y. Zhu, D. J. Ganthier, S. E. Moriu, Q. Wu., H. J. Carmichael, and T. W. Mossber, Phys. Rev. Lett. **64**, 2499 (1990).
[3]C. Weisbuch, M. Nishioka, A. Ishikawa, and Y. Arakawa, Phys. Rev. Lett. **69**, 3314 (1992).
[4]T. Gutbrot, M. Bayer, A. Forchel, J. P. Reithmaier, T. L. Reinecke, S. Rudin, and P. A. Knipp, Phys. Rev. B **57**, 9950 (1998).
[5]E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[6]G. S. Agarwal, Phys. Rev. Lett. **53**, 1732 (1984).
[7]G. S. Agarwal and R. R. Puri, Phys. Rev. A **33**, 1757 (1986).
[8]The harmonic-oscillator representation does not exhibit driven Rabi oscillations in external fields, but this phenomenon is not the subject of the present study.
[9]I. R. Senitsky, Phys. Rev. **119**, 670 (1960).
[10]H. Haug and S. Koch, *Quantum Theory of the Optical and Electronic Properties of Semiconductors*, 3rd ed. (World Scientific, Singapore, 1994).
[11]H. Goldstein, *Classical Mechanics* (Addision Wesley, Reading, MA, 1950).
[12]L. C. Andreani, V. Savona, P. Schwendimann, and A. Quattropani, Superlattices Microstruct. **15**, 453 (1984).
[13]U. Fano, Phys. Rev. **124**, 1866 (1961).
[14]S. Rudin and T. L. Reinecke (unpublished).