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Heisenberg spins on a cylinder in an axial magnetic field

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For classical Heisenberg spins in the continuum limit (i.e., the nonlinear σ model) on an elastic cylinder in an external axial magnetic field we find that the corresponding Euler-Lagrange equation is the double sine-Gordon (DSG) equation. The DSG soliton adopts a characteristic length ξ which is smaller than the radius of the cylinder. This mismatch of length scales results in a geometric frustration in the region of the soliton and is relieved by the deformation of the cylinder. We also find the DSG kink soliton lattice and pulse soliton lattice solutions and show that they cause a periodic deformation of the cylinder. [S0163-1829(98)51126-5]

Cylindrical structures, in particular microtubules,[1] abound in nature. They may either be made out of magnetic materials or may enclose magnetorheological fluids.[2] To study their magnetoelastic properties, one may treat their surfaces as a continuum of classical spins. Along this line of thinking, here we explore the elastic consequences of classical Heisenberg-coupled spins on a deformable cylinder in the presence of topological spin solitons and an external magnetic field. Recently, it has become possible to fabricate magnetic thin films in a cylindrical shape.[3]

The continuum limit of the Heisenberg Hamiltonian for classical ferromagnets or antiferromagnets for isotropic spin-spin coupling is the nonlinear σ model.[4–8] The total Hamiltonian for a deformable, magnetoelastically coupled manifold is given by $H=H_{magn}+H_{el}+H_{m-el}$, where H_{magn}, H_{el} and H_{m-el} represent the magnetic, elastic, and magnetoelastic energy, respectively. In this paper we will focus mainly on the magnetic part. The magnetic part (a variant of the nonlinear σ model) on a circular cylinder in an external axial magnetic field is given by

$$H_{magn}=J\int\int_{cyl}(\nabla n)^2dS-g\mu\int\int_{cyl}\mathbf{n}\cdot\mathbf{B}dS, \tag{1}$$

where J denotes the coupling energy between neighboring spins, $\hat{\mathbf{n}}$ is the magnetization unit vector, $\hat{\mathbf{n}}=(\cos\theta,\sin\theta\cos\phi,\sin\theta\sin\phi)$, and dS is the surface element of the deformable circular cylinder. B is the applied magnetic field along the axis of the cylinder (Fig. 1), μ is the magnetic moment, and g denotes the g factor of the electrons in the magnetic material.

Assuming cylindrical symmetry and homogeneous boundary conditions at both ends of the cylinder ($\theta=0$ as $z\to\pm\infty$) we get

$$H_{magn}=2J(2\pi\rho_0)\int_{-\infty}^{\infty}\left\{\frac{\theta_z^2}{2}+V(\theta)\right\}dz, \tag{2a}$$

with

$$V(\theta)=\frac{\sin^2\theta}{2\rho_0^2}+\frac{1}{\rho_B^2}(1-\cos\theta), \tag{2b}$$

where the magnetic length scale $\rho_B^2=2J/g\mu B$.

The Euler-Lagrange equation for the above Hamiltonian is the double sine-Gordon (DSG) equation

$$\theta_{zz}=\frac{1}{2\rho_0^2}\sin 2\theta+\frac{1}{\rho_B^2}\sin\theta. \tag{3}$$

The DSG equation arises in several physical contexts including in the study of spin waves in (the B phase of) superfluid ^{3}He,[9,10] in the problem of self-induced transparency,[11] poling process of certain polymers,[12,13] propagation of resonant optical pulses through degenerate media,[14] compressible XY chain of dipoles with piezoelectric coupling,[15] soliton-magnon bound states in organic antiferromagnets,[16] misfit dislocations on reconstructed metallic surfaces,[17] DNA excitations,[18] etc. The 2π soliton solution of the DSG equation consistent with the boundary conditions is

$$\theta(z)=2\tan^{-1}\left(\frac{\rho_B}{\xi\sinh\frac{z}{\xi}}\right),\quad \xi=\frac{\rho_0\rho_B}{(\rho_0^2+\rho_B^2)^{1/2}}, \tag{4}$$

where ξ is the characteristic length of the 2π soliton. Equivalently, $\xi=\sqrt{\rho_0\rho_B}\sqrt{\rho_0\rho_B/(\rho_0^2+\rho_B^2)}$. Thus the soliton chooses a characteristic length which is smaller than the geometric mean of the cylinder radius and the magnetic length. With increasing magnetic field the characteristic length of the soliton decreases. Note that Eq. (4) is equivalent to the following solution obtained in the context of anisotropic Heisenberg ferromagnetic chains[19]

$$\theta(z)=2\sin^{-1}\left\{\left(\cosh^2\frac{z}{\xi}-\frac{\xi^2}{\rho_0^2}\sinh^2\frac{z}{\xi}\right)^{-1/2}\right\}.$$

The 2π soliton is depicted in Fig. 1. Also note that the limit $B\to 0$ ($\rho_B\to\infty$) cannot be taken in Eq. (4) to obtain a sine-

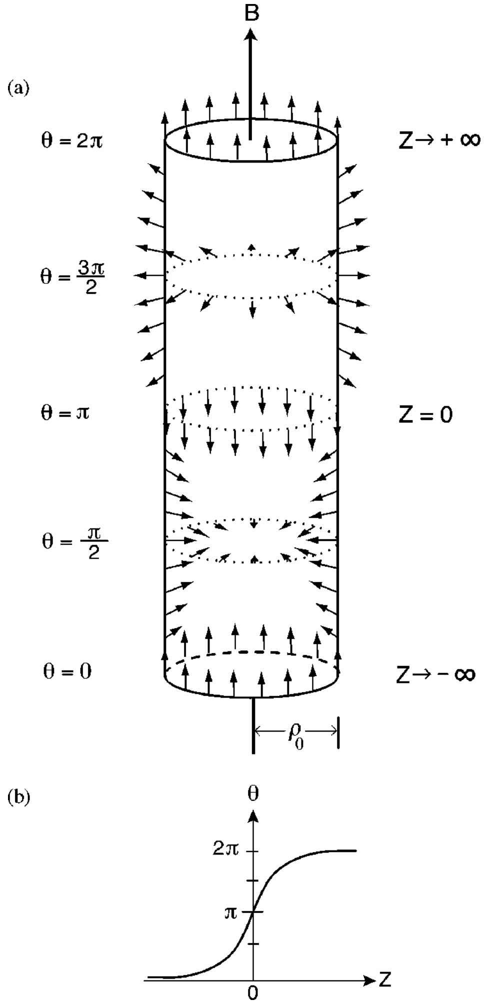

FIG. 1. Cylindrically symmetric $0\rightarrow 2\pi$ twist soliton on an infinite cylinder in the presence of an axial magnetic field.

Gordon kink solution since the $\theta(z)$ boundary conditions for a DSG kink are $(0,2\pi)$ at $(-\infty,\infty)$ as opposed to $(0,\pi)$ for a sine-Gordon kink.

The energy of the soliton

$$E_s=16\pi J\left\{\left(1+\frac{\rho_0^2}{\rho_B^2}\right)^{1/2}+\frac{\rho_0^2}{\rho_B^2}\sinh^{-1}\frac{\rho_B}{\rho_0}\right\} \tag{5}$$

is larger than $16\pi J$ (the mimimum energy for the homotopy class[4,20] with winding number $Q=2$). Only in the limit $\rho_B\rightarrow\infty$ $(B\rightarrow 0)$ we get $E_s\rightarrow 16\pi J$ and $\xi\rightarrow\rho_0$. Now let us consider that part of the Hamiltonian H_1 which does not involve the magnetic field B (or ρ_B),

$$H_1=2\pi J\int_{-\infty}^{\infty}\left\{\rho_0\theta_z^2+\frac{\sin^2\theta}{\rho_0}\right\}dz. \tag{6}$$

This part of the energy is minimized by a soliton solution which satisfies the self-dual equations[21] or in other words by a soliton solution which has ρ_0, the radius of the cylinder, as its characteristic length.

Our soliton solution has a characteristic length $\xi<\rho_0$ (as in the case of a soliton lattice when considering a single π soliton), which leads to a geometric frustration.[21] The origin of this geometric frustration lies in the applied magnetic field B which introduces a new length scale into the system. Now, in order to release this frustration the cylinder (if it is elastic) will deform, i.e., to decrease its radius in the region where the soliton is centered.[21] This will miminize the energy in H_1.

Next, we focus on the second part of the Hamiltonian H_2 which does involve the external magnetic field B:

$$H_2=4\pi J\int_{-\infty}^{\infty}\frac{\rho_0}{\rho_B^2}(1-\cos\,\theta)dz. \tag{7}$$

For our soliton solution [Eq. (4)] H_2 will take the form:

$$H_2=4\pi J\int_{-\infty}^{\infty}\frac{2}{\rho_B^2}\frac{\rho_0(z)}{\left[1+\left(1-\frac{\xi^2}{\rho_0^2(z)}\right)\sinh^2\frac{z}{\xi}\right]}dz$$

$$=8\pi J\int_{-\infty}^{\infty}f(\rho_0(z),z)dz$$

if we allow $\rho_0=\rho_0(z)$.

This part of the Hamiltonian will decrease in energy if we locally decrease the radius ρ_0 but keep ρ_0 constant in Eq. (4) for our soliton solution.

Therefore, for constant magnetic field if we keep the cylinder cross section constant at $z=\pm\infty$, the cylinder will deform in the region of the soliton. For small magnetic field (i.e., $\rho_B\gg\rho_0$) we get

$$E_s=16\pi J\left\{1+\frac{\rho_0^2}{2\rho_B^2}\left(1+2\,\ln\frac{2\rho_B}{\rho_0}\right)\right\}. \tag{8}$$

For large magnetic field (i.e., $\rho_B\ll\rho_0$) we get $E_s=16\pi J(2\rho_0/\rho_B)$.

The deformation of the cylinder can be understood physically as follows. At cylinder boundaries at $z=\pm\infty$ the spins are aligned with the direction of the magnetic field. However, since this is a 2π soliton, in the middle of the cylinder the spins are antiparallel to the direction of the magnetic field (Fig. 1). To reduce this magnetic energy cost the cylinder will try to deform (pinch) in the middle. The shape, $\rho_0(z)$ of the cylinder which minimizes the energy (to $16\pi J$) and at the same time respects the uniform boundary conditions for the vector field $\hat{\mathbf{n}}$ would come from finding a suitable solution of (the self-duality equations and) the DSG equation. We do not attempt to find a solution of this highly nonlinear system of differential equations but note that an ansatz for the radius of the cylinder $\rho_0(z)=\rho_0-\epsilon\rho_0\mathrm{sech}^2(z/\xi)$ (which in inself represents an elastic soliton) would lead to a lower energy than on a rigid cylinder with $\rho_0(z)=\rho_0$. Here $\epsilon<1$ is a small parameter. It can be shown explicitly by choosing an appropriate value of ϵ (Ref. 21) that the energy is lowered by deforming the cylinder.

In the absence of a magnetic field the self-dual equation for a cylinder with cylindrically symmetric field is $\theta_z=\pm\sin\,\theta/\rho_0$. For the DSG case, using Eq. (3) we can write

$$\theta_z=\pm\frac{\sin\,\theta}{\rho_0}\left[1+\frac{4\rho_0^2}{\rho_B^2}\frac{\sin^2\theta/2}{\sin^2\theta}\right]^{1/2}.$$

The function in square brackets is greater than one for all values of $\theta(z)$ and the self-dual equation is not satisfied. Therefore, the cylinder would deform in order to satisfy the self-dual equation and to lower the energy. Note that all solutions of this equation satisfy the Euler-Lagrange equation [Eq. (3)] but not vice versa.

Next, we consider the 2π soliton lattice solution of the DSG equation in two regimes: (i) $\rho_0 \geqslant \rho_B$ (large magnetic field) and (ii) $\rho_0 < \rho_B$ (small magnetic field).

(i) $\rho_0 \geqslant \rho_B$. In this case there is only one local maximum $V_{max} = 2/\rho_B^2$ in the zero to 2π range and occurs at $\theta = \pi$. We find the following (topological) kink lattice exact solution

$$\tan \frac{\theta_L(z)}{2} = \pm \frac{\rho_B}{\xi_L} \frac{\mathrm{cn}\left(\frac{z-z_0}{\xi_L}, k\right)}{\mathrm{sn}\left(\frac{x-x_0}{\xi_L}, k\right)}, \tag{9a}$$

with periodicity $d = 4K\xi_L$, where $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$ are Jacobi elliptic functions and $K(k)$ is the complete elliptic integral of the first kind.[22] Here

$$\xi_L^2 = \frac{\rho_B^2}{\tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_2}{2}}, \quad k^2 = \frac{\tan^2 \frac{\theta_2}{2}}{\tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_2}{2}}, \tag{9b}$$

and $\tan^2\theta_1/2$ and $\tan^2\theta_2/2$ are the two nonzero roots of

$$\left(1 - \frac{\rho_B^2 V_0}{2}\right) \tan^4 \frac{\theta}{2} + \rho_B^2 \left(\frac{1}{\xi^2} - V_0\right) \tan^2 \frac{\theta}{2} - \frac{\rho_B^2 V_0}{2} = 0, \tag{9c}$$

with $0 \leqslant V_0 < 2/\rho_B^2$. Here $V_0 = 0$ and $V_0 = 2/\rho_B^2$ correspond to $k = 1$ (single soliton case) and $k = 0$, respectively.

The energy of the kink lattice is obtained by substituting Eqs. (9a) and (9b) in Eq. (2) and evaluating various integrals,[22]

$$\begin{aligned} E_L = 16\pi J \Bigg[& \left\{ \frac{2\rho_0 \rho_B^2}{\xi} \frac{(\xi^2 - \xi_L^2)}{(\rho_B^2 - \xi^2)} - f(k) \right\} K \\ & - \left\{ \frac{\rho_B^2 - \xi_L^2}{\xi_L^2 - k'^2 \rho_B^2} f(k) \right\} E \\ & - \Bigg\{ \frac{2\rho_0 [\xi_L^2 - (2+k^2)\xi^2 + 2\rho_B^2(\xi^2 - \xi_L^2)]}{\xi(\rho_B^2 - \xi_L^2)} \\ & + \frac{2\rho_B^2 + \xi_L^2}{\rho_B^2} f(k) + \frac{k'^2(\rho_B^2 - \xi_L^2)}{\xi_L^2 - k'^2 \rho_B^2} f(k) \Bigg\} \Pi \Bigg], \end{aligned} \tag{10}$$

where $E(k)$ and $\Pi(1 - \xi_L^2/\rho_B^2, k)$ are the complete elliptic integrals of the second and third kind,[22] respectively, and

$$f(k) = \frac{2\rho_0 \xi}{\xi_L^2} + \frac{\rho_0 \rho_B^2 [\xi_L^2 - (2+k^2)\xi^2]}{\xi \xi_L^2 (\rho_B^2 - \xi_L^2)} + \frac{\rho_0 \rho_B^4 (\xi^2 - \xi_L^2)}{\xi \xi_L^2 (\rho_B^2 - \xi_L^2)^2}.$$

Note that in the limit $k \to 1$ the kink lattice solution and energy reduce precisely to the single soliton results.

Again, following the analysis after Eq. (7), we state that if we allow the cylinder to be elastic, it will have a periodic deformation in the presence of the kink lattice in order to lower the total energy of the system. This argument also follows from an analogy with the periodic pinch of the cylinder in the absence of the magnetic field (sine-Gordon soliton lattice solution[21]). Specifically, the ansatz for the radius $\rho_0(z) = \rho_0 - \epsilon \rho_0 \mathrm{cn}^2(z/\xi_L, k)$ would lead to a lower energy than the rigid case ($\rho_0(z) = \rho_0$). Here $\epsilon < 1$ is a small parameter. In addition, it can be shown explicitly by choosing an appropriate value of ϵ (Ref. 21) that the energy is lowered by periodically deforming the cylinder.

The interaction between two kinks can be obtained by evaluating E_L in the limit $k \to 1$. However, this is an algebraically tedious task. A more elegant way is to use the asymptotic form ($z \to \infty$) of Eq. (4) and then apply Manton's formula.[23] We get the potential between a kink-antikink pair separated by a distance d as

$$U_{DSG}(d) = -128\pi J \frac{\rho_0 \rho_B^2}{\xi^3} \exp\left(-\frac{d}{\xi}\right).$$

Note that, even though there is a magnetic field-dependent prefactor in the interaction, it decreases monotonically with increasing magnetic field. We contrast this with the interaction in the absence of the magnetic field (for a sine-Gordon kink-antikink pair[21])

$$U_{SG}(d) = -32\pi J \exp\left(-\frac{d}{\rho_0}\right).$$

The factor of 4 in U_{DSG}, as compared to U_{SG}, can be understood approximately as follows: For very small values of the magnetic field each 2π soliton can be considered as consisting of two π solitons (see Fig. 5 of Ref. 19). Thus, there are four kink-antikink interactions between the π solitons.

(ii) $\rho_0 < \rho_B$. In this case there is a local minimum $V_{min} = 2/\rho_B^2$ in the zero to 2π range and occurs at $\theta = \pi$. However, now there are two local maxima $V_{max} = \rho_0^2/2\xi^4$ in the zero to 2π range that occur symmetrically about $\theta = \pi$ at $\cos\theta = -\rho_0^2/\rho_B^2$. For $V_0 < 2/\rho_B^2$ we find the same (kink and) kink lattice solutions as discussed above. For $V_0 > 2/\rho_B^2$, however, the kink lattice solution does not exist. Instead, we find the following (nontopological) pulse lattice exact solution

$$\tan \frac{\theta_L(z)}{2} = \pm \frac{\rho_B}{\xi_p} \frac{q'}{\mathrm{dn}\left(\frac{z-z_0}{\xi_p}, q\right)}, \tag{11a}$$

with periodicity $d = 2K(q)\xi_p$, where $\mathrm{dn}(z,q)$ is a Jacobi elliptic function and $q' = \sqrt{1-q^2}$ is the complementary modulus. Here

$$\xi_p = \frac{\rho_B}{\tan \frac{\theta_2}{2}}, \quad q^2 = \frac{\tan^2 \frac{\theta_2}{2} - \tan^2 \frac{\theta_1}{2}}{\tan^2 \frac{\theta_2}{2}}, \tag{11b}$$

and $\tan^2\theta_1/2$ and $\tan^2\theta_2/2$ are the two appropriate roots of Eq. (9b) with $2/\rho_B^2 < V_0 < \rho_0^2/2\xi^4$. Specifically, $V_0 = \rho_0^2/2\xi^4$ and $V_0 = 2/\rho_B^2$ correspond to $q = 0$ and $q = 1$ (single pulse), respectively.

The energy of the pulse lattice is obtained in a similar way to that of the kink lattice. However, we do not present it here explicitly. The argument about the periodic deformation of the elastic cylinder applies in the presence of a pulse lattice as well. Note that the limit $q\to 1$ ($q'\to 0$) cannot be taken in Eq. (11). Instead, we directly obtain the single pulse solution (when $V_0=2/\rho_B^2$) from the equation of motion (Eq. 3),

$$\tan\frac{\theta}{2}=\pm\frac{\zeta}{\rho_B}\cosh\frac{z}{\zeta},\quad \zeta=\frac{\rho_0\rho_B}{\sqrt{\rho_B^2-\rho_0^2}},$$

where ζ is the characteristic length. The pulse energy

$$E_{pulse}=\frac{4}{\zeta}-\frac{4\rho_0}{\rho_B^2}\tanh^{-1}\frac{\rho_0}{\zeta}.$$

The cylinder will deform in the presence of a pulse soliton. The arguments presented after Eq. (5) apply in the present case as well.

In conclusion, we showed that the equation of motion in the continuum approximation for classical Heisenberg spins on a cylinder in an external magnetic field is the double sine-Gordon equation. We found exact single 2π kink, single pulse, kink lattice, and pulse lattice solutions. For an elastic cylinder we demonstrated that the cylinder will deform (pinch) in the region of the soliton. The kink and pulse lattices will cause a periodic pinch. The extent of deformation can be tuned by the magnitude of applied magnetic field. The present study provides an example of field-induced geometrical frustration caused by a mismatch of cylinder radius ρ_0, magnetic length scale ρ_B, and soliton characteristic length ξ (or ξ_L and the lattice periodicity d). We have previously studied examples of geometrical frustration due to internal factors such as (a) spin anisotropy,[21] (b) soliton interaction,[21] and (c) variable curvature[24] (e.g., elliptic cylinder or a spheroid).

Physically relevant systems, where the magnetoelastic effects predicted above can possibly be observed, include magnetically coated deformable (whether metallic, nonmetallic or organic) cylindrical thin films.[25] Microtubules[1] containing magnetorheological fluids[2] may also serve as a physical realization of such systems.

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