# Critical temperature of an anisotropic superconductor containing both nonmagnetic and magnetic impurities

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The combined effect of both nonmagnetic and magnetic impurities on the superconducting transition temperature is studied theoretically within the BCS model. An expression for the critical temperature as a function of potential and spin-flip scattering rates is derived for a two-dimensional superconductor with arbitrary in-plane anisotropy of the superconducting order parameter, ranging from isotropic $s$ wave to $d$ wave (or any pairing state with nonzero angular momentum) and including anisotropic $s$ wave and mixed $(d+s)$ wave as particular cases. This expression generalizes the well-known Abrikosov-Gor'kov formula for the critical temperature of impure superconductors. The effect of defects and impurities in high-temperature superconductors is discussed. [S0163-1829(98)10037-1]

## I. INTRODUCTION

The mechanism of high-$T_c$ superconductivity still remains unknown. It is generally believed that elucidation of the symmetry of the superconducting order parameter $\Delta(\mathbf{p})$ in high-$T_c$ superconductors (HTSCs) could narrow the list of pairing mechanisms debated. Since the $CuO_2$ layers are thought to be responsible for the superconducting pairing in HTSCs, the in-plane symmetry of $\Delta(\mathbf{p})$ is of primary interest. However, in spite of strong evidence provided for $d_{x^2-y^2}$ in-plane symmetry of $\Delta(\mathbf{p})$, experimental data are somewhat controversial[1–16] (though the recent research seems to resolve the contradiction in favor of a $d_{x^2-y^2}$ wave[17]). The situation is complicated by orthorhombic distortion of $CuO_2$ layers in some HTSCs, resulting in probable admixture of the $s$-wave component to otherwise pure $d$ wave $\Delta(\mathbf{p})$.[18,19] Since HTSCs differ structurally, the form of $\Delta(\mathbf{p})$ may appear to be material dependent.

One indirect way to distinguish a pure $d$ wave from a highly anisotropic $s$ wave or a mixed $(d+s)$ wave is to study the response of HTSCs to intentionally incorporated impurities or radiation-induced defects. Depending on the symmetry of $\Delta(\mathbf{p})$, clear differences are predicted for the variation of experimentally accessible characteristics such as the critical temperature $T_c$, the superfluid density, etc.[20–32] For example, the rate of $T_c$ degradation by defects and impurities in a two-dimensional superconductor is determined by the value of the Fermi surface (FS) average $\langle\Delta(\mathbf{p})\rangle_{FS}$ (Ref. 20) and should be different in $d$-wave superconductors with $\langle\Delta(\mathbf{p})\rangle_{FS}=0$ and anisotropic $s$-wave or mixed $(d+s)$-wave ones with $\langle\Delta(\mathbf{p})\rangle_{FS}\neq 0$, the specific value of $\langle\Delta(\mathbf{p})\rangle_{FS}$ being dictated by the degree of $\Delta(\mathbf{p})$ anisotropy or by the relative contributions of $d$-wave and $s$-wave components to $\Delta(\mathbf{p})$.

Numerous experimental studies give evidence for $T_c$ degradation by impurity doping[33–48] or radiation damage[49–58] of HTSCs, the two effects being remarkably analogous if the dependence of $T_c$ on the in-plane residual resistivity $\rho_0$ is considered.[59] Impurity-induced scattering of charge carriers in doped HTSCs and their scattering by displaced host atoms in irradiated HTSCs are believed to be the main reasons for the suppression of superconductivity and the increase in $\rho_0$.[35,36,44,45,51,52,54,55]

The comparison of experimental curves $T_c(\rho_0)$ with theoretical ones reveals that the observed reduction of $T_c$ by impurities and radiation defects is more gradual than predicted theoretically for $d$-wave superconductors.[31,50,53,56,57,59–61] A critical value of $\rho_0^c$ at which $T_c=0$ ranges from 200 to 1500 $\mu\Omega$ cm depending on the type of disorder and the kind of HTSC material,[34,40,42,44,47,52,53,55,57–59] while for a $d$-wave superconductor with $T_c\approx 100$ K the theory gives $\rho_0^c \approx 50\ \mu\Omega$ cm.[31,59–61] To reconcile the experimental findings with the $d$-wave symmetry of $\Delta(\mathbf{p})$ in HTSCs, a number of suggestions have been made, including the anisotropy of impurity scattering in the momentum space,[62,63] an ''intermediate'' (between Cooper pairs and local bosons) state of paired electrons,[64] a depletion of the hole density due to the oxygen vacancies in the $CuO_2$ planes,[65] an anomalously small value of the plasma frequency,[56,58] the spatial variation of the order parameter,[66] etc.[67]

Another way is to abandon the $d$-wave hypothesis in favor of anisotropic $s$-wave or mixed $(d+s)$-wave models[27,28,30,31,68] (it must be emphasized that if the relative weight of the isotropic $s$-wave component in the mixed $(d+s)$-wave $\Delta(\mathbf{p})$ is large, then the symmetry of such an order parameter can in fact be viewed as the anisotropic $s$-wave one[53]). However, while the initial slope of an experimentally observed $T_c(\rho_0)$ curve in HTSCs can actually be explained by anisotropic $s$-wave symmetry of $\Delta(\mathbf{p})$,[31,53,57] the theory faces problems when explaining the complete suppression of superconductivity at a finite value of $\rho_0$. Indeed, the theory predicts[20,27,30,31] that $T_c$ of a non-$d$-wave anisotropic two-dimensional superconductor does not vanish at a certain critical value of $\rho_0^c$ [as it does in the case of pure $d$-wave symmetry of $\Delta(\mathbf{p})$], but instead asymptotically goes to zero as $\rho_0$ increases. This contradicts the experiments mentioned above. Besides, the experimentally observed form of the $T_c(\rho_0)$ curve is usually close to linear,[42,44,47,52,53,55,57–59] while the theory predicts a positive

 

curvature of the $T_c(\rho_0)$ curve in a non-$d$-wave superconductor.

Note, however, that experimentally determined values of $\rho_0$ reflect the contribution from different scattering channels, while a theoretical analysis of $T_c$ degradation by defects and impurities in HTSCs is usually made for the specific case of spin-independent scattering potential.[20,23,25,27–32,53,55,57,60–62,64–67] Meanwhile a lot of experiments give evidence for the presence of magnetic scatterers (along with nonmagnetic ones) in nonstoichiometric HTSCs, e.g., in oxygen-deficient, doped, or irradiated samples.[36,45,69–78] For example, the oxygen vacancies or excess oxygen atoms carry or induce local magnetic moments and hence play a role of paramagnetic centers.[69–71] Furthermore, doping by Zn induces local magnetic moments residing probably on the nearest-neighbor Cu sites.[72–78] This is supported by studies of nonmagnetic impurities in Heisenberg antiferromagnets[79–81] and by numerical calculations within the two-dimensional $t$-$J$ model.[82] Besides, there are intrinsic (host) magnetic atoms in some stoichiometric HTSCs, e.g., in $GdBa_2Cu_3O_7$.[83]

Though the estimated moment-carrier exchange energy $J$ may appear to be too small to solely account for suppression of superconductivity in disordered HTSCs,[84] an intriguing possibility of the magnetic pair-breaking scattering as the common origin of the significant decrease in $T_c$ remains.[85] Moreover, it is suggested[86] that interaction between Cooper pairs and localized magnetic moments in ''optimally'' doped HTSCs leads to a depression of $T_c$ relative to its ''intrinsic'' value. Hence, among other things, the understanding of the role of magnetic scattering in HTSCs is important from the viewpoint of search for new materials with higher $T_c$.

Since, first, there exist two channels of carrier scattering by magnetic impurities (potential and spin-flip ones) and, second, in general, both magnetic and nonmagnetic scatterers are present in HTSCs, there is a need for a theoretical model that could describe the effects of nonmagnetic and magnetic scattering on equal footing. For an isotropic $s$-wave superconductor this is the Abrikosov-Gor'kov theory[87] that predicts a rapid $T_c$ suppression by magnetic impurities and insensitivity of $T_c$ to nonmagnetic scattering, in accordance with the Anderson theorem[88] (a discussion about the validity of the Abrikosov-Gor'kov approach[89,90] seems to be resolved in favor of the standard Green's functions technique). However, the Abrikosov-Gor'kov formula for $T_c$ versus scattering rate is not applicable to anisotropic superconductors, no matter what the specific symmetry of $\Delta(\mathbf{p})$ is [$d$ wave, ($d+s$) wave, anisotropic $s$ wave, or something else]. Hence, this formula cannot be used to describe impurity effects in HTSCs. On the other hand, theoretical considerations of impurity scattering in anisotropic superconductors are commonly restricted to nonmagnetic scatterers only.[20,21,23–25,27–32,60,62,64] Such a status of the theory of impure superconductors results in situations when the experiments on the $T_c$ reduction by impurities or radiation-induced defects in HTSCs are compared with *either* the Abrikosov-Gor'kov formula for $T_c$ of an *isotropic* $s$-wave superconductor containing *magnetic* impurities[38,47,51,70] *or* the formula for $T_c$ of an *anisotropic* superconductor, that, however, contains *nonmagnetic* impurities only.[41,51,53–55] In the latter case an *a priori* suggestion is often made about the pure $d$-wave symmetry of $\Delta(\mathbf{p})$.[50,56,58]

In a recent study,[91] the Abrikosov-Gor'kov theory has been generalized to the case of a multiband superconductor with arbitrary anisotropy of the interband order parameter and arbitrary strength of magnetic and/or nonmagnetic impurity scattering. Note, however, that the concept of multiband superconductivity (arising, e.g., from $CuO_2$ plains and Cu-O chains) is hardly probable to be applicable to HTSCs. Although it was argued in Ref. 91 that the mathematical formalism had been proven to be the same for a multiband superconductor and a superconductor with a general angular anisotropy of the order parameter,[92] an explicit formula for the critical temperature of a one-band anisotropic superconductor containing both nonmagnetic and magnetic impurities was not offered in Ref. 91.

The goal of this paper is to work out a theoretical framework for a description of the *combined* effect of nonmagnetic and magnetic scatterers on the $T_c$ of a two-dimensional superconductor with anisotropic $\Delta(\mathbf{p})$ (preliminary results have been presented in Ref. 93). We seek to obtain a rather simple (free from needless theoretical complications) Abrikosov-Gor'kov-like formula for $T_c$ that included physically meaningful parameters and could be compared with available experimental data. Within the weak-coupling limit of the BCS model and without specifying the microscopic mechanism of superconducting pairing, we derive the expression that relates $T_c$ to relaxation rates of charge carriers by nonmagnetic and magnetic scatterers, as well as to the numerical coefficient $\chi=1-\langle\Delta(\mathbf{p})\rangle_{FS}^2/\langle\Delta^2(\mathbf{p})\rangle_{FS}$, which is a measure of the degree of in-plane anisotropy of $\Delta(\mathbf{p})$ on the FS. The range $0\leqslant\chi\leqslant1$ covers the cases of isotropic $s$-wave ($\chi=0$), $d$-wave ($\chi=1$), anisotropic $s$-wave ($0<\chi<1$), and mixed ($d+s$)-wave ($0<\chi<1$) symmetries of $\Delta(\mathbf{p})$. In two particular cases of (i) both nonmagnetic and magnetic scattering in an isotropic $s$-wave superconductor ($\chi=0$) and (ii) nonmagnetic scattering only in a superconductor with arbitrary anisotropy of $\Delta(\mathbf{p})$ ($0\leqslant\chi\leqslant1$), our expression for $T_c$ reduces to the well-known formulas.[20,87]

The paper is organized as follows. The BCS model for an impure anisotropic superconductor containing both nonmagnetic and magnetic scatterers is described in Sec. II along with the theoretical formalism. The expression for the critical temperature as a function of potential and spin-flip relaxation times of charge carriers is derived in Sec. III for a superconductor with an arbitrary degree of the order parameter anisotropy. The results obtained are discussed in Sec. IV. In Sec. V concluding remarks are given.

## II. MODEL AND FORMALISM

Within the framework of the BCS model, the Hamiltonian of a superconductor containing both nonmagnetic and magnetic impurities is as follows:

$$\hat{H}=\sum_{\mathbf{p},\sigma}\xi(\mathbf{p})\hat{a}^+_{\mathbf{p}\sigma}\hat{a}_{\mathbf{p}\sigma}+\sum_{\mathbf{p},\mathbf{p}',\sigma,\sigma'}U(\mathbf{p},\sigma;\mathbf{p}',\sigma')\hat{a}^+_{\mathbf{p}\sigma}\hat{a}_{\mathbf{p}'\sigma'}+\sum_{\mathbf{p},\mathbf{p}'}V(\mathbf{p},\mathbf{p}')\hat{a}^+_{\mathbf{p}\uparrow}\hat{a}^+_{-\mathbf{p}\downarrow}\hat{a}_{-\mathbf{p}'\downarrow}\hat{a}_{\mathbf{p}'\uparrow}, \quad (1)$$

where the operator $\hat{a}^{+}_{\mathbf{p}\sigma}(\hat{a}_{\mathbf{p}\sigma})$ creates (annihilates) an electron with the quasimomentum $\mathbf{p}$ and the spin projection on the $z$ axis $\sigma=\uparrow$ or $\downarrow$, $\xi(\mathbf{p})=\epsilon(\mathbf{p})-\mu$ is the (spin-independent) quasiparticle energy measured from the chemical potential $\mu$, $U(\mathbf{p},\sigma;\mathbf{p}',\sigma')$ is the matrix element for electron scattering by randomly distributed impurities (defects) from the state $(\mathbf{p}',\sigma')$ to the state $(\mathbf{p},\sigma)$, and $V(\mathbf{p},\mathbf{p}')$ is the BCS pair potential.

Let the following sum,

$$U(\mathbf{r})=U_n(\mathbf{r})+U_m(\mathbf{r}), \tag{2}$$

be the total interaction between a conduction electron at a point $\mathbf{r}$ and *all* impurities present in the sample, $U_n(\mathbf{r})$ and $U_m(\mathbf{r})$ being the interaction components due to nonmagnetic and magnetic impurities, respectively:

$$U_n(\mathbf{r})=\sum_{\alpha} u_n(\mathbf{r}-\mathbf{R}_\alpha), \quad U_m(\mathbf{r})=\sum_{\beta} u_m(\mathbf{r}-\mathbf{R}_\beta), \tag{3}$$

where $u_n(\mathbf{r}-\mathbf{R}_\alpha)$ is the interaction between an electron at $\mathbf{r}$ and a nonmagnetic impurity at $\mathbf{R}_\alpha$, while $u_m(\mathbf{r}-\mathbf{R}_\beta)$ is the interaction between an electron at $\mathbf{r}$ and a magnetic impurity at $\mathbf{R}_\beta$. Since, in general, magnetic impurities give rise to both potential and exchange scattering,[87] one has

$$u_m(\mathbf{r}-\mathbf{R}_\beta)=u_m^{pot}(\mathbf{r}-\mathbf{R}_\beta)+u_m^{ex}(\mathbf{r}-\mathbf{R}_\beta), \tag{4}$$

where $u_m^{pot}(\mathbf{r}-\mathbf{R}_\beta)$ is the spin-independent potential component, and

$$u_m^{ex}(\mathbf{r}-\mathbf{R}_\beta)=J(\mathbf{r}-\mathbf{R}_\beta)\mathbf{s}\mathbf{S}_\beta \tag{5}$$

is the exchange interaction. Here $J(\mathbf{r}-\mathbf{R}_\beta)$ is the exchange energy, $\mathbf{S}_\beta$ is the spin of magnetic impurity located at $\mathbf{R}_\beta$, and $\mathbf{s}=\boldsymbol{\sigma}/2$ is the operator of electron spin (the three components of $\sigma$ are the Pauli matrices $\sigma_x$, $\sigma_y$, $\sigma_z$). We shall assume that orientations of the paramagnetic spins $\mathbf{S}_\beta$ are fixed and remain unchanged upon electron scattering (taking into account dynamic transitions of the impurity spin between $2S+1$ magnetic sublevels have little effect on the results[94]).

The electron interactions $u_n(\mathbf{r}-\mathbf{R}_\alpha)$ and $u_m^{pot}(\mathbf{r}-\mathbf{R}_\beta)$ with nonmagnetic and magnetic impurities, respectively, are spin-independent and hence contribute to the scattering matrix elements $U(\mathbf{p},\sigma;\mathbf{p}',\sigma)$ only. In their turn, exchange interactions of electron with magnetic impurities, $u_m^{ex}(\mathbf{r}-\mathbf{R}_\beta)$, result in both spin-conserving and spin-flip scattering events and hence contribute to $U(\mathbf{p},\sigma;\mathbf{p}',\sigma)$ as well as to $U(\mathbf{p},\sigma;\mathbf{p}',-\sigma)$. Let us write down the matrix element $U(\mathbf{p},\sigma;\mathbf{p}',\sigma')$ as

$$U(\mathbf{p},\sigma;\mathbf{p}',\sigma')=U_1(\mathbf{p},\mathbf{p}',\sigma)\,\delta_{\sigma,\sigma'}+U_2(\mathbf{p},\mathbf{p}',\sigma)\,\delta_{\sigma,-\sigma'}, \tag{6}$$

where

$$\begin{aligned} U_1(\mathbf{p},\mathbf{p}',\sigma)&=u_n(\mathbf{p},\mathbf{p}')\sum_{\alpha} e^{-i(\mathbf{p}-\mathbf{p}')\mathbf{R}_\alpha} \\ &+u_m^{pot}(\mathbf{p},\mathbf{p}')\sum_{\beta} e^{-i(\mathbf{p}-\mathbf{p}')\mathbf{R}_\beta} \\ &+\frac{1}{2}J(\mathbf{p},\mathbf{p}')\gamma_\sigma\sum_{\beta} e^{-i(\mathbf{p}-\mathbf{p}')\mathbf{R}_\beta}S_\beta^z, \end{aligned} \tag{7}$$

$$U_2(\mathbf{p},\mathbf{p}',\sigma)=\frac{1}{2}J(\mathbf{p},\mathbf{p}')\sum_{\beta} e^{-i(\mathbf{p}-\mathbf{p}')\mathbf{R}_\beta}(S_\beta^x-i\gamma_\sigma S_\beta^y).$$

Here $u_n(\mathbf{p},\mathbf{p}')$, $u_m^{pot}(\mathbf{p},\mathbf{p}')$, and $J(\mathbf{p},\mathbf{p}')$ are the components of the matrix element for electron scattering by an isolated impurity; $\gamma_\sigma=+1$ and $-1$ for $\sigma=\uparrow$ and $\downarrow$, respectively. We do not consider the direct effect of nonmagnetic disorder on the magnetic pair breaking (this effect has been studied in Refs. 94 and 95 for the case of an isotropic $s$-wave pairing).

In order to account for anisotropy of the superconducting state, we assume a factorizable phenomenological pairing interaction $V(\mathbf{p},\mathbf{p}')$ of the form (see, e.g., Ref. 20)

$$V(\mathbf{p},\mathbf{p}')=-V_0\phi(\mathbf{n})\phi(\mathbf{n}'), \tag{8}$$

where $V_0$ is the pairing energy; $\mathbf{n}=\mathbf{p}/p$ is a unit vector along the momentum. Then the order parameter $\Delta(\mathbf{p})$ is[20]

$$\Delta(\mathbf{p})=-\sum_{\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\langle\hat{a}_{-\mathbf{p}'\downarrow}\hat{a}_{\mathbf{p}'\uparrow}\rangle=\Delta_0\phi(\mathbf{n}), \tag{9}$$

where $\Delta_0$ depends on the temperature. The function $\phi(\mathbf{n})$ specifies the anisotropy of $\Delta(\mathbf{p})$ in the momentum space [e.g., $\phi(\mathbf{n})\equiv 1$ for isotropic $s$-wave pairing]. We assume that $\phi(\mathbf{n})$ is temperature independent.

The self-consistent equation for $\Delta(\mathbf{p})$ can be derived by means of Green's functions technique (see, e.g., Ref. 96). We define the normal and anomalous temperature Green's functions

$$\begin{aligned} G(\mathbf{p},\sigma;\mathbf{p}',\sigma';\tau)&=-\langle T_\tau\hat{a}_{\mathbf{p}\sigma}(\tau)\hat{a}^{+}_{\mathbf{p}'\sigma'}(0)\rangle, \\ F(\mathbf{p},\sigma;\mathbf{p}',\sigma';\tau)&=\langle T_\tau\hat{a}^{+}_{-\mathbf{p}-\sigma}(\tau)\hat{a}^{+}_{\mathbf{p}'\sigma'}(0)\rangle, \\ \tilde{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\tau)&=-\langle T_\tau\hat{a}^{+}_{-\mathbf{p}-\sigma}(\tau)\hat{a}_{-\mathbf{p}'-\sigma'}(0)\rangle, \\ \tilde{F}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\tau)&=\langle T_\tau\hat{a}_{\mathbf{p}\sigma}(\tau)\hat{a}_{-\mathbf{p}'-\sigma'}(0)\rangle, \end{aligned} \tag{10}$$

and their Fourier transforms $G(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$, $F(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$, $\tilde{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$, $\tilde{F}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$, where angular brackets stand for the statistical averaging with the Hamiltonian (1), the symbol $T_\tau$ denotes the time ordering, $\tau$ is the imaginary time, and $\omega=\pi T(2n+1)$ are Matsubara frequencies (we set $\hbar=k_B=1$ throughout the paper).

It is convenient to introduce the matrix Green's function $\hat{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$ in the Nambu representation:

$$\hat{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega) = \begin{pmatrix} G(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega) & -\tilde{F}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega) \\ -F(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega) & \tilde{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega) \end{pmatrix}. \tag{11}$$

We stress that $\hat{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$ is nondiagonal in spin space (since there is spin-flip scattering of electrons on magnetic impurities) as well as in momentum space (until averaged over impurities coordinates). The matrix equation for $\hat{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)$ can be written as

$$\hat{G}_0^{-1}(\mathbf{p},\sigma;\mathbf{k},\lambda;\omega)\hat{G}(\mathbf{k},\lambda;\mathbf{p}',\sigma';\omega) - \hat{U}(\mathbf{p},\sigma;\mathbf{k},\lambda)\hat{G}(\mathbf{k},\lambda;\mathbf{p}',\sigma';\omega) = \hat{1}\,\delta_{\mathbf{p},\mathbf{p}'}\delta_{\sigma,\sigma'}, \tag{12}$$

where $\hat{G}_0(\mathbf{p},\sigma;\mathbf{k},\lambda;\omega) = \hat{G}_0(\mathbf{p},\sigma,\omega)\,\delta_{\mathbf{p},\mathbf{k}}\delta_{\sigma,\lambda}$ is the Green's function of a clean sample,

$$\hat{G}_0(\mathbf{p},\sigma,\omega) = -\frac{1}{\omega^2+\xi(\mathbf{p})^2+|\Delta(\mathbf{p})|^2} \times \begin{pmatrix} i\omega+\xi(\mathbf{p}) & \gamma_\sigma\Delta(\mathbf{p}) \\ \gamma_\sigma\Delta^*(\mathbf{p}) & i\omega-\xi(\mathbf{p}) \end{pmatrix}, \tag{13}$$

$\hat{1}$ is the unit matrix $2\times 2$, and the matrix $\hat{U}(\mathbf{p},\sigma;\mathbf{k},\lambda)$ that describes the effect of impurity scattering has the form

$$\hat{U}(\mathbf{p},\sigma;\mathbf{k},\lambda) = \begin{pmatrix} U(\mathbf{p},\sigma;\mathbf{k},\lambda) & 0 \\ 0 & -U(\mathbf{p},-\lambda;\mathbf{k},-\sigma) \end{pmatrix}. \tag{14}$$

The summation over repeated indices in Eq. (12) and below is implied. We note that $\hat{G}_0(\mathbf{p},\sigma,\omega)$ depends on $\sigma$ through $\gamma_\sigma$.

In order to avoid needless mathematical complications and to express the final results in terms of as few parameters as possible, we make several simplifying assumptions: (i) we consider the short-range scattering potentials, so that the matrix elements $u_n(\mathbf{p},\mathbf{p}')$, $u_m^{pot}(\mathbf{p},\mathbf{p}')$, $J(\mathbf{p},\mathbf{p}')$ are momentum independent and equal to $u_n$, $u_m^{pot}$, $J$, respectively ($s$-wave impurity scattering); (ii) we treat the impurity scattering in the Born limit; (iii) we restrict the momenta of the electron self-energy and BCS pair potential to the FS.

After averaging Eq. (12) over impurity configurations and directions of impurity spins one has

$$\langle \hat{G}(\mathbf{p},\sigma;\mathbf{p}',\sigma';\omega)\rangle_{imp} = \hat{G}(\mathbf{p},\sigma,\omega)\,\delta_{\mathbf{p},\mathbf{p}'}\delta_{\sigma,\sigma'}, \tag{15}$$

where

$$\hat{G}^{-1}(\mathbf{p},\sigma,\omega) = \hat{G}_0^{-1}(\mathbf{p},\sigma,\omega) - \hat{M}(\mathbf{p},\sigma,\omega), \tag{16}$$

$$\hat{M}(\mathbf{p},\sigma,\omega) = \langle \hat{U}(\mathbf{p},\sigma;\mathbf{k},\lambda)\hat{G}(\mathbf{k},\lambda,\omega)\hat{U}(\mathbf{k},\lambda;\mathbf{p},\sigma)\rangle_{imp} = \begin{pmatrix} (c_n|u_n|^2+c_m|u_m^{pot}|^2+c_m|u_m^{ex}|^2)\sum_{\mathbf{k}} G(\mathbf{k},\sigma,\omega) & (c_n|u_n|^2+c_m|u_m^{pot}|^2-c_m|u_m^{ex}|^2)\sum_{\mathbf{k}} \tilde{F}(\mathbf{k},\sigma,\omega) \\ (c_n|u_n|^2+c_m|u_m^{pot}|^2-c_m|u_m^{ex}|^2)\sum_{\mathbf{k}} F(\mathbf{k},\sigma,\omega) & (c_n|u_n|^2+c_m|u_m^{pot}|^2+c_m|u_m^{ex}|^2)\sum_{\mathbf{k}} \tilde{G}(\mathbf{k},\sigma,\omega) \end{pmatrix}. \tag{17}$$

Here $c_n$ and $c_m$ are the concentrations of nonmagnetic and magnetic impurities, respectively, and we have designated $|u_m^{ex}|^2 = |J|^2 S(S+1)/4$. Note that $|u_m^{ex}|^2$ includes contributions from both spin-flip and spin-conserving scattering of electrons due to their exchange interaction with magnetic impurities, Eq. (5). This is because the matrix element of the spin-conserving exchange scattering depends on the orientation of electron spin through $\gamma_\sigma$; see Eq. (7). The coefficient $c_m|u_m^{ex}|^2$ in Eq. (17) enters into the factors at the normal and anomalous Green's functions with opposite signs, while the coefficients $c_n|u_n|^2$ and $c_m|u_m^{pot}|^2$, which are due to the scattering by nonmagnetic impurities and the potential component of the scattering by magnetic impurities, respectively, appear in Eq. (17) with the same signs.

Making use of Eqs. (13), (16), and (17), we obtain

$$\hat{G}(\mathbf{p},\sigma,\omega,\Delta) = \hat{G}_0(\mathbf{p},\sigma,\omega',\Delta_\omega), \tag{18}$$

where

$$\omega' = \omega - i(c_n|u_n|^2+c_m|u_m^{pot}|^2+c_m|u_m^{ex}|^2) \times \sum_{\mathbf{k}} \frac{i\omega'+\xi(\mathbf{k})}{\omega'^2+\xi^2(\mathbf{k})+|\Delta_\omega(\mathbf{k})|^2}, \tag{19}$$

$$\Delta_\omega(\mathbf{p}) = \Delta(\mathbf{p}) + (c_n|u_n|^2+c_m|u_m^{pot}|^2-c_m|u_m^{ex}|^2) \times \sum_{\mathbf{k}} \frac{\Delta_\omega(\mathbf{k})}{\omega'^2+\xi^2(\mathbf{k})+|\Delta_\omega(\mathbf{k})|^2}. \tag{20}$$

Then one has from Eq. (9),

$$\Delta(\mathbf{p}) = -T\sum_{\omega}\sum_{\mathbf{p}'} V(\mathbf{p},\mathbf{p}')\frac{\Delta_\omega(\mathbf{p}')}{\omega'^2+\xi^2(\mathbf{p}')+|\Delta_\omega(\mathbf{p}')|^2}. \tag{21}$$

For further considerations it is convenient to express the coefficients $c_n|u_n|^2$, $c_m|u_m^{pot}|^2$, and $c_m|u_m^{ex}|^2$ in terms of electron relaxation times $\tau_n$, $\tau_m^{pot}$, and $\tau_m^{ex}$ for scattering by non-

magnetic impurities, potential scattering by magnetic impurities, and exchange scattering by magnetic impurities, respectively:

$$\frac{1}{\tau_n}=2\pi c_n|u_n|^2N(0),\quad \frac{1}{\tau_m^{pot}}=2\pi c_m|u_m^{pot}|^2N(0),\quad \frac{1}{\tau_m^{ex}}=2\pi c_m|u_m^{ex}|^2N(0), \tag{22}$$

where $N(0)$ is the density of electron states at the Fermi level. The electron relaxation time $\tau_m$ due to magnetic impurities is given by the expression

$$\frac{1}{\tau_m}=\frac{1}{\tau_m^{pot}}+\frac{1}{\tau_m^{ex}}, \tag{23}$$

while the total electron relaxation time $\tau$ due to all impurities present in the sample can be found as

$$\frac{1}{\tau}=\frac{1}{\tau_n}+\frac{1}{\tau_m}=\frac{1}{\tau_n}+\frac{1}{\tau_m^{pot}}+\frac{1}{\tau_m^{ex}}. \tag{24}$$

We note that Eqs. (22) allow one to express the final results in terms of three relaxation times ($\tau_n$, $\tau_m^{pot}$, $\tau_m^{ex}$) instead of a large number of unknown parameters such as impurity concentrations and scattering matrix elements. Besides, the relaxation times are associated with the residual resistivity. This facilitates a comparison between the theory and experiment.

## III. CRITICAL TEMPERATURE OF AN IMPURE ANISOTROPIC SUPERCONDUCTOR

The critical temperature $T_c$ can be found from Eqs. (19), (20), and (21) as the temperature at which the order parameter goes to zero, i.e., $\Delta_0\to 0$ in Eq. (9). Setting $|\Delta_\omega(\mathbf{k})|^2=0$ in the denominators of Eqs. (19) and (20) and taking Eqs. (22) into account, we have at $T\to T_c$,

$$\omega'=\omega+\frac{1}{2}(1/\tau_n+1/\tau_m^{pot}+1/\tau_m^{ex})\,\mathrm{sgn}(\omega), \tag{25}$$

$$\Delta_\omega(\mathbf{p})=\Delta(\mathbf{p})+\frac{1}{2|\omega'|}(1/\tau_n+1/\tau_m^{pot}-1/\tau_m^{ex})\langle\Delta_\omega(\mathbf{p})\rangle_{FS}, \tag{26}$$

where the angular brackets $\langle\ldots\rangle_{FS}$ stand for a FS average:

$$\langle\ldots\rangle_{FS}=\int_{FS}(\ldots)\frac{d\Omega_\mathbf{p}}{|\partial\xi(\mathbf{p})/\partial\mathbf{p}|}\Bigg/\int_{FS}\frac{d\Omega_\mathbf{p}}{|\partial\xi(\mathbf{p})/\partial\mathbf{p}|}. \tag{27}$$

Substituting Eqs. (25) and (26) in Eq. (21), setting $|\Delta_\omega(\mathbf{p}')|^2=0$ in the denominator of Eq. (21), and taking Eqs. (8) and (9) into account, we have

$$\frac{1}{\lambda}=\pi T_c\sum_\omega\ \frac{1}{|\omega|+\frac{1}{2}(1/\tau_n+1/\tau_m^{pot}+1/\tau_m^{ex})}\times\left[\langle\phi^2(\mathbf{n})\rangle_{FS}+\langle\phi(\mathbf{n})\rangle_{FS}^2\,\frac{1/\tau_n+1/\tau_m^{pot}-1/\tau_m^{ex}}{2(|\omega|+1/\tau_m^{ex})}\right], \tag{28}$$

where $\lambda=V_0N(0)$ is the electron-boson coupling constant. The equation for the critical temperature $T_{c0}$ in the absence of impurities (i.e., at $1/\tau_n=1/\tau_m^{pot}=1/\tau_m^{ex}=0$) reads

$$\frac{1}{\lambda}=\pi T_{c0}\langle\phi^2(\mathbf{n})\rangle_{FS}\sum_\omega\ \frac{1}{|\omega|}. \tag{29}$$

Following the standard procedure, we obtain from Eqs. (28) and (29) the equation for the critical temperature $T_c$ as

$$\ln\left(\frac{T_{c0}}{T_c}\right)=\pi T_c\sum_\omega\ \frac{1}{|\omega|+\frac{1}{2}(1/\tau_n+1/\tau_m^{pot}+1/\tau_m^{ex})}\times\left[\frac{1}{2|\omega|}(1/\tau_n+1/\tau_m^{pot}+1/\tau_m^{ex})-\frac{\langle\phi(\mathbf{n})\rangle_{FS}^2}{\langle\phi^2(\mathbf{n})\rangle_{FS}}\,\frac{1/\tau_n+1/\tau_m^{pot}-1/\tau_m^{ex}}{2(|\omega|+1/\tau_m^{ex})}\right]. \tag{30}$$

At this stage it is convenient to introduce the coefficient $\chi$ of anisotropy of the order parameter on the FS (Refs. 20 and 31)

$$\chi=1-\frac{\langle\phi(\mathbf{n})\rangle_{FS}^2}{\langle\phi^2(\mathbf{n})\rangle_{FS}}=1-\frac{\langle\Delta(\mathbf{p})\rangle_{FS}^2}{\langle\Delta^2(\mathbf{p})\rangle_{FS}}. \tag{31}$$

For isotropic $s$-wave pairing we have $\Delta(\mathbf{p})\equiv$const on the FS; therefore, $\langle\Delta(\mathbf{p})\rangle_{FS}^2=\langle\Delta^2(\mathbf{p})\rangle_{FS}$, and $\chi=0$. For a two-dimensional superconductor with $d$-wave pairing we have $\chi=1$ since $\langle\Delta(\mathbf{p})\rangle_{FS}=0$. The range $0<\chi<1$ corresponds to anisotropic $s$-wave or mixed $(d+s)$-wave in-plane pairing. The higher the in-plane anisotropy of $\Delta(\mathbf{p})$ (e.g., the greater the partial weight of a $d$-wave in the case of mixed pairing), the closer to unity is the value of $\chi$.

Note that $\chi=1$ holds not only for the $d$-wave pairing state, but also for any pairing state with angular momentum $l>0$, e.g., for $p$-wave state ($l=1$), see Eq. (31). In its turn, the range $0<\chi<1$ generally corresponds to mixing of $s$-wave state with some higher angular harmonic state. Hence, while this paper focuses primarily on $s$-wave, $d$-wave, and $(d+s)$-wave states, one should keep in mind that the results obtained are more general and may be applied to superconductors with other symmetries of the order parameter.

Making use of the definition (31) and the formula[97]

$$\sum_{k=0}^{\infty}\left(\frac{1}{k+x}-\frac{1}{k+y}\right)=\Psi(y)-\Psi(x), \tag{32}$$

where $\Psi$ is the digamma function, we obtain from Eq. (30),

$$\ln\left(\frac{T_{c0}}{T_c}\right)=(1-\chi)\left[\Psi\left(\frac{1}{2}+\frac{1}{2\pi T_c\tau_m^{ex}}\right)-\Psi\left(\frac{1}{2}\right)\right]+\chi\left\{\Psi\left[\frac{1}{2}+\frac{1}{4\pi T_c}\left(\frac{1}{\tau_n}+\frac{1}{\tau_m^{pot}}+\frac{1}{\tau_m^{ex}}\right)\right]-\Psi\left(\frac{1}{2}\right)\right\}. \tag{33}$$

In two particular cases of (i) both nonmagnetic and magnetic scattering in an isotropic $s$-wave superconductor ($\chi=0$) and (ii) nonmagnetic scattering only in a superconductor with arbitrary in-plane anisotropy of $\Delta(\mathbf{p})$ ($1/\tau_m^{ex}=1/\tau_m^{pot}=0$, $0\leqslant\chi\leqslant1$), Eq. (33) reduces to the well-known expressions[87,20]

$$\ln\left(\frac{T_{c0}}{T_c}\right)=\Psi\left(\frac{1}{2}+\frac{1}{2\pi T_c\tau_m^{ex}}\right)-\Psi\left(\frac{1}{2}\right) \tag{34}$$

and

$$\ln\left(\frac{T_{c0}}{T_c}\right)=\chi\left[\Psi\left(\frac{1}{2}+\frac{1}{4\pi T_c\tau_n}\right)-\Psi\left(\frac{1}{2}\right)\right], \tag{35}$$

respectively.

Now let us consider the limiting cases of low and high impurity concentration ($T_{c0}-T_c\ll T_{c0}$ and $T_c\to0$, respectively). At $1/4\pi T_{c0}\tau_n\ll1$, $1/4\pi T_{c0}\tau_m^{pot}\ll1$, and $1/4\pi T_{c0}\tau_m^{ex}\ll1$ (low impurity concentration) one has from Eq. (33),

$$T_{c0}-T_c\approx\frac{\pi}{4}\left[\frac{\chi}{2}\left(\frac{1}{\tau_n}+\frac{1}{\tau_m^{pot}}\right)+\frac{1-\chi/2}{\tau_m^{ex}}\right]. \tag{36}$$

In particular cases (i) and (ii) considered above, Eq. (36) reduces to the expressions[87,20]

$$T_{c0}-T_c\approx\frac{\pi}{4\tau_m^{ex}} \tag{37}$$

and

$$T_{c0}-T_c\approx\frac{\pi\chi}{8\tau_n} \tag{38}$$

for initial $T_c$ suppression by magnetic (at $\chi=0$) or nonmagnetic (at arbitrary value of $\chi$) scatterers, respectively.

As to the high impurity concentration, we recall that in the BCS theory, nonmagnetic scattering alone is insufficient for the non-$d$-wave two-dimensional superconductivity ($0\leqslant\chi<1$) to be destroyed completely;[20] at $1/\tau_m^{ex}=0$, the value of $T_c$ asymptotically goes to zero as $1/\tau_n$ increases. On the other hand, $T_c$ of a $d$-wave superconductor with $\chi=1$ vanishes at a critical value $1/\tau_{n,c}=\pi T_{c0}/\gamma\approx1.764T_{c0}$, with $\gamma=e^C\approx1.781$, where $C$ is the Euler constant. In its turn, magnetic scattering in the absence of nonmagnetic scattering ($1/\tau_n=0$) is known to suppress the isotropic $s$-wave superconductivity with $\chi=0$ at a critical value $1/\tau_{m,c}^{ex}=\pi T_{c0}/2\gamma\approx0.882T_{c0}$ (Ref. 87).

On the basis of Eq. (33), it is straightforward to derive the general condition for impurity (defect) suppression of $T_c$ for a superconductor having an arbitrary in-plane anisotropy coefficient $\chi$ and containing both nonmagnetic and magnetic scatterers:

$$\frac{1}{\tau_{eff,c}}=\frac{\pi}{\gamma}2^{\chi-1}T_{c0}, \tag{39}$$

where $\tau_{eff,c}$ is the critical value of the effective relaxation time $\tau_{eff}$, defined as

$$\frac{1}{\tau_{eff}}=\left(\frac{1}{\tau_m^{ex}}\right)^{1-\chi}\left(\frac{1}{\tau_n}+\frac{1}{\tau_m^{pot}}+\frac{1}{\tau_m^{ex}}\right)^{\chi}. \tag{40}$$

From Eqs. (39) and (40) one can see that $1/\tau_{eff,c}$ increases monotonically with $1/\tau_n$, $1/\tau_m^{pot}$, and $1/\tau_m^{ex}$ at any value of $\chi$, with the exception of the case $\chi=0$, where $1/\tau_{eff,c}$ does not depend on $1/\tau_n$ and $1/\tau_m^{pot}$; see Eq. (40). If $\chi$ is close to unity [$\Delta(\mathbf{p})$ with strong in-plane anisotropy], then $1/\tau_{eff}\approx1/\tau_n+1/\tau_m^{pot}+1/\tau_m^{ex}$, i.e., the contribution of nonmagnetic and magnetic scattering to pair breaking is about the same. If $\chi\ll1$ [almost isotropic $\Delta(\mathbf{p})$], then $1/\tau_{eff}\approx1/\tau_m^{ex}$, i.e., $\tau_{eff}$ is determined primarily by magnetic scattering. The higher the anisotropy coefficient $\chi$, the greater is the relative contribution of nonmagnetic scatterers to $T_c$ suppression as compared to magnetic scatterers.

We note however that while the concept of the effective relaxation time $\tau_{eff}$ can be used for evaluation of the *critical level* of nonmagnetic and magnetic disorder, it is not possible to express $T_c$ in terms of $\tau_{eff}$ in the *whole range* $0\leqslant T_c\leqslant T_{c0}$; see Eq. (33). In other words, the combined effect of nonmagnetic and magnetic scattering on $T_c$ cannot be described by a single universal parameter depending on the values of $\tau_n$, $\tau_m^{pot}$, $\tau_m^{ex}$, and $\chi$; see Ref. 93 for more details. Hence, while the quantity $1/\tau_{eff,c}$ characterizes the critical strength of impurity scattering corresponding to $T_c=0$, the quantity $1/\tau_{eff}$ (when it is less than $1/\tau_{eff,c}$) does not determine the value of $T_c$ unequivocally.

Based on Eqs. (39) and (40), it is possible to derive the following expression for the critical value of $1/\tau_n$ in the presence of magnetic scattering:

$$\frac{1}{\tau_{n,c}}=\frac{1}{\tau_m^{ex}}\left[2\left(\frac{\pi T_{c0}\tau_m^{ex}}{2\gamma}\right)^{1/\chi}-1\right]-\frac{1}{\tau_m^{pot}}. \tag{41}$$

This expression is valid as long as its right-hand side is positive, since otherwise the superconductivity is completely suppressed solely by magnetic impurities. The value of $1/\tau_{n,c}$ decreases as $1/\tau_m^{pot}$ and $1/\tau_m^{ex}$ increase at constant $\chi$ or as $\chi$ increases at constant $1/\tau_m^{pot}$ and $1/\tau_m^{ex}$.

To conclude this section, it is interesting to note that $T_c$ does not depend on $\chi$ provided that $1/\tau_m^{ex}=1/\tau_n+1/\tau_m^{pot}$; see Eq. (33).

## IV. DISCUSSION

Equation (33) is obviously more general than Eqs. (34) and (35), which are commonly used for the analysis of experimental data on $T_c$ suppression by defects and impurities in HTSCs; see references in the Introduction. In fact, making use of Eq. (34) or Eq. (35) one assumes *a priori* that either (i) the order parameter in HTSCs is isotropic in momentum space, or (ii) magnetic scatterers are completely absent in HTSCs. In our opinion, the experimental dependencies of $T_c$ versus impurity concentration or radiation dose should be analyzed within the framework of the theory presented

above; see Eq. (33). One should not *guess* as to the degree of in-plane anisotropy of $\Delta(\mathbf{p})$ and the type of scatterers, but try to *determine* the value of $\chi$ and relative weights of magnetic and nonmagnetic components in electron scattering through comparison of theoretical predictions with available or specially performed experiments.

We recall that Eq. (33) has been derived within the weak-coupling limit of the BCS model. Note however that the exact solution of the Eliashberg equations for a particular case of a $d$-wave superconductor containing nonmagnetic impurities only indicates[60] that the analytical $T_c/T_{c0}$ versus $1/\tau_n$ curve falls near the numerically calculated $T_c/T_{c0}$ versus $1/\tau_n^*$ curve, where $1/\tau_n^*$ is the scattering rate renormalized by the strong-coupling effects (it is $1/\tau_n^*$ that enters the formula for the experimentally determined in-plane residual electrical resistivity $\rho_0$). We believe therefore that Eq. (33) is also valid beyond the weak-coupling approximation implying that $\tau_n$, $\tau_m^{pot}$, and $\tau_m^{ex}$ in Eq. (33) are the *renormalized* relaxation times that govern the experimentally measured physical quantities. It would be interesting to check this by direct numerical solution of the Eliashberg equations for an anisotropic superconductor with nonmagnetic and magnetic impurities.

In order to compare the predictions of theory with experiment, it is convenient to represent the electron scattering time, Eq. (24), in terms of the in-plane residual resistivity $\rho_0$. Following Radtke *et al.*,[60] we have

$$\frac{1}{\tau_n}+\frac{1}{\tau_m^{pot}}+\frac{1}{\tau_m^{ex}}=\frac{\omega_{pl}^2}{4\pi}\rho_0, \tag{42}$$

where $\omega_{pl}$ is the plasma frequency. Note that spin-independent $(1/\tau_n+1/\tau_m^{pot})$ and spin-dependent $(1/\tau_m^{ex})$ scattering rates variously appear in Eqs. (33) and (42) for the critical temperature and residual resistivity. Hence, for a given degree of anisotropy of the order parameter (i.e., for a given value of $\chi$), the *universal* dependence of $T_c/T_{c0}$ on $\rho_0$ cannot be obtained, as opposed to the case of a $d$-wave or anisotropic $s$-wave superconductor containing nonmagnetic impurities only.[60,31]

Let us express $\rho_0$ as

$$\rho_0=\rho_0^{nm}+\rho_0^{ex}, \tag{43}$$

where $\rho_0^{nm}$ is due to electron scattering by nonmagnetic impurities and potential scattering by magnetic impurities, while $\rho_0^{ex}$ is due to exchange scattering by magnetic impurities:

$$\frac{1}{\tau_n}+\frac{1}{\tau_m^{pot}}=\frac{\omega_{pl}^2}{4\pi}\rho_0^{nm}, \tag{44}$$

$$\frac{1}{\tau_m^{ex}}=\frac{\omega_{pl}^2}{4\pi}\rho_0^{ex}. \tag{45}$$

From Eqs. (22) and (42)–(45) we have

$$\rho_0^{nm}=(1-\alpha)\rho_0, \quad \rho_0^{ex}=\alpha\rho_0, \tag{46}$$

where

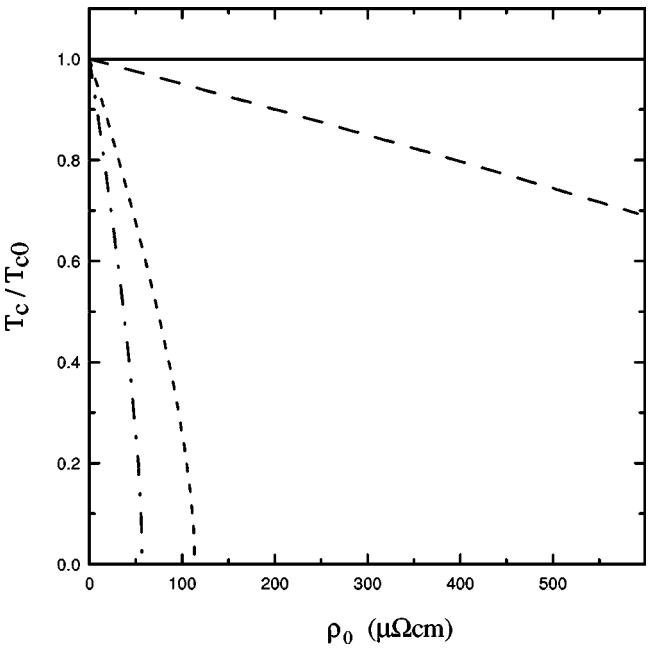


FIG. 1. Dependence of the normalized critical temperature $T_c/T_{c0}$ on the residual resistivity $\rho_0$ due to nonmagnetic and magnetic impurities in a superconductor with $T_{c0}=100$ K and $\chi=0$ (isotropic $s$-wave pairing) for different values of the coefficient $\alpha$ specifying the relative contribution to $\rho_0$ from exchange scattering. $\alpha=0$ (solid curve), 0.04 (long-dashed curve), 0.5 (short-dashed curve), 1 (dot-dashed curve). The critical value $\rho_0^c=1.42$ mΩ cm for $\alpha=0.04$. The plasma frequency is taken to be $\omega_{pl}=1$ eV. One can go to the other values of $T_{c0}$ and $\omega_{pl}$ through replacing $\rho_0$ by $\rho_0(T_{c0}/100)\omega_{pl}^{-2}$, where $T_{c0}$ is measured in K, and $\omega_{pl}$ is measured in eV.

$$\alpha=\frac{|u_m^{ex}|^2}{(c_n/c_m)|u_n|^2+|u_m^{pot}|^2+|u_m^{ex}|^2}. \tag{47}$$

The value of $\alpha$ depends, first, on the scattering strengths of individual nonmagnetic and magnetic impurities (through matrix elements $u_n$, $u_m^{pot}$, $u_m^{ex}$) and, second, on the ratio of impurity concentrations $c_n/c_m$. The latter is expected to remain constant under doping or irradiation, at least at relatively low (but sufficient to destroy the superconductivity) doping level or radiation dose. For example, low-energy irradiation of $Yba_2Cu_3O_{7-x}$ was found to induce nonmagnetic defects only,[50] i.e., $c_m/c_n=0$, and hence $\alpha=0$.

Thus the dependence of $T_c/T_{c0}$ on $\rho_0$ for a given value of $\chi$ is specified by the material-dependent and ''disorder-dependent'' dimensionless coefficient $\alpha$. The greater is the relative contribution from exchange scattering by magnetic impurities to $\rho_0$, the higher is the value of $\alpha$ ($\alpha$ ranges from 0 in the absence of exchange scattering to 1 in the absence of non-spin-flip scattering). Substituting Eqs. (44) and (45) in Eq. (33) and taking Eqs. (46) into account, we have

$$\ln\left(\frac{T_{c0}}{T_c}\right)=(1-\chi)\left[\Psi\left(\frac{1}{2}+\alpha\frac{\omega_{pl}^2}{8\pi^2T_c}\rho_0\right)-\Psi\left(\frac{1}{2}\right)\right]+\chi\left[\Psi\left(\frac{1}{2}+\frac{\omega_{pl}^2}{16\pi^2T_c}\rho_0\right)-\Psi\left(\frac{1}{2}\right)\right]. \tag{48}$$

Figures 1–4 show the plot of $T_c/T_{c0}$ versus $\rho_0$ in a su-

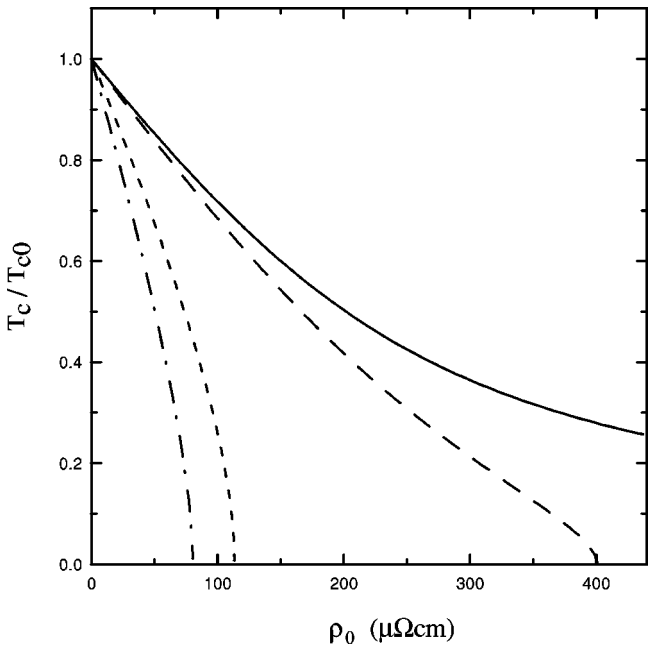


FIG. 2. Same as in Fig. 1 for $\chi=0.5$ [a specific case of anisotropic $s$-wave or $(d+s)$-wave in-plane pairing].

perconductor with $T_{c0}=100$ K and $\omega_{pl}=1$ eV for different values of $\chi$ and $\alpha$ ranging from 0 to 1. The choice of $T_{c0}$ and $\omega_{pl}$ is, to some extent, arbitrary (though these values of $T_{c0}$ and $\omega_{pl}$ are typical for HTSCs, e.g., for $YBa_2Cu_3O_7$). In order to go to the other values of $T_{c0}$ and $\omega_{pl}$ one should just replace $\rho_0$ in Figs. 1–4 by $\rho_0(T_{c0}/100)\omega_{pl}^{-2}$, where $T_{c0}$ is measured in K, and $\omega_{pl}$ is measured in eV.

From Figs. 1–4 one can see that at $\chi<1$ the rate of $T_c$ decrease with increase in $\rho_0$ becomes higher as $\alpha$ increases from 0 to 1, i.e., as the relative contribution of exchange scattering to $\rho_0$ increases. At $\chi=0$ (isotropic $s$-wave pairing) the value of $T_c$ does not depend on $\rho_0$ for $\alpha=0$, while

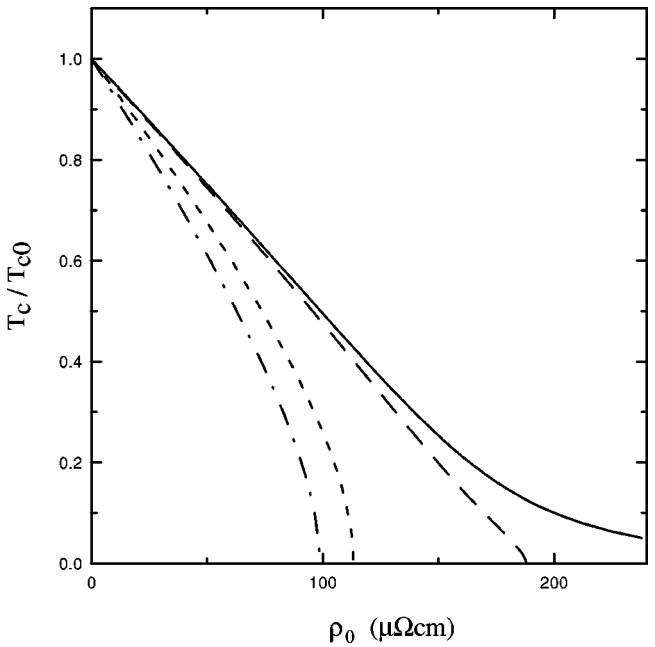


FIG. 3. Same as in Fig. 1 for $\chi=0.8$ [a specific case of anisotropic $s$-wave or $(d+s)$-wave in-plane pairing].

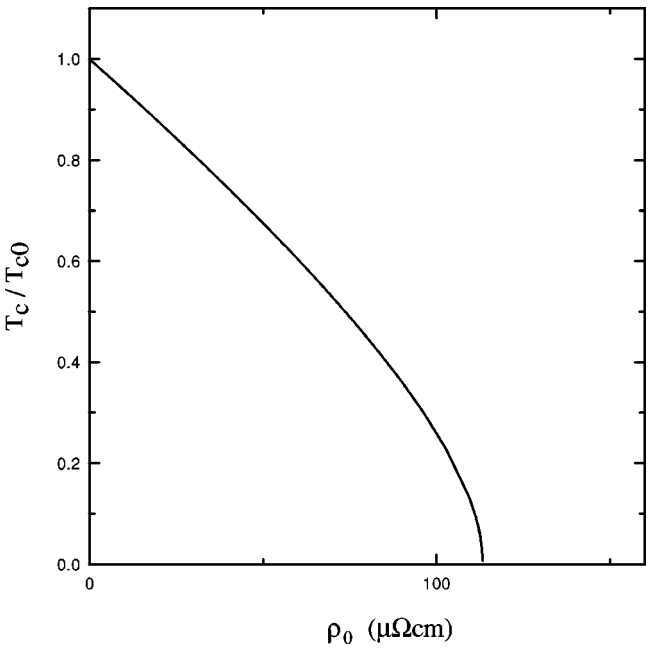


FIG. 4. Same as in Fig. 1 for $\chi=1$ ($d$-wave in-plane pairing). In this case the value of $T_c/T_{c0}$ at a given $\rho_0$ does not depend on $\alpha$; see Eq. (48).

the superconductivity is completely suppressed ($T_c=0$) at a critical value of $\rho_0^c=1.42$ m$\Omega$ cm, 113 $\mu\Omega$ cm, and 56.5 $\mu\Omega$ cm for $\alpha=0.04$, 0.5, and 1, respectively; see Fig. 1. At $\chi=0.5$ (a specific case of anisotropic $s$-wave or mixed $(d+s)$-wave in-plane pairing) the value of $T_c$ monotonously goes to zero as $\rho_0$ increases for $\alpha=0$, while $\rho_0^c=401$ $\mu\Omega$ cm, 113 $\mu\Omega$ cm, and 80 $\mu\Omega$ cm for $\alpha=0.04$, 0.5, and 1, respectively; see Fig. 2. At $\chi=0.8$ (strongly anisotropic $s$-wave or mixed $(d+s)$-wave in-plane pairing with a predominance of the $d$-wave component) one has $\rho_0^c=188$ $\mu\Omega$ cm, 113 $\mu\Omega$ cm, and 99 $\mu\Omega$ cm for $\alpha=0.04$, 0.5, and 1, respectively; see Fig. 3. The curves $T_c(\rho_0)$ for different $\alpha$ come closer together as the coefficient $\chi$ increases, i.e., as the order parameter becomes more anisotropic. At $\chi=1$ ($d$-wave in-plane pairing) all curves $T_c(\rho_0)$ merge together (see Fig. 4), i.e., the value of $T_c/T_{c0}$ at a given $\rho_0$ does not depend on $\alpha$, in accordance with Eq. (48), the critical value of $\rho_0^c$ being equal to 113 $\mu\Omega$ cm at any $\alpha$. Note that for $\alpha=0.5$ the curves $T_c(\rho_0)$ are the same at any value of $\chi$; see Figs. 1–4 and Eq. (48).

Magnetic scatterers in a non-$d$-wave superconductor, even if they are present in a small proportion ($\alpha\ll 1$), result in $\rho_0^c$ decrease as compared with $\rho_0^c$ of a sample containing nonmagnetic impurities only. The decrease in $\rho_0^c$ with $\alpha$ is more pronounced at low values of $\chi$, i.e., in superconductors having weakly anisotropic order parameter; see Figs. 1–3. At $\chi$ as high as 0.8, i.e., in a superconductor having strongly anisotropic (but different from a pure $d$ wave) order parameter, the value of $\rho_0^c$ for $\alpha=1$ is less than twice as low as that for $\alpha=0.04$; see Fig. 3. In such a superconductor, the role of a small amount of magnetic impurities is to suppress the superconductivity at a finite value of $\rho_0^c$ as opposed to the case when exchange scattering is absent ($\alpha=0$), though the curves $T_c(\rho_0)$ at $\alpha=0$ and 0.04 almost coincide in a very broad range of $T_c/T_{c0}$; see Fig. 3.

In our opinion, an argument in favor of other than pure $d$-wave in-plane symmetry of the order parameter in HTSCs (at least in some of them) is as follows. A pure $d$-wave two-dimensional superconductor with $\chi=1$ is characterized by the *universal* dependence of $T_c$ on $\rho_0$ that is the same at any value of $\alpha$, i.e., at any relative contribution of exchange scattering to the total value of $\rho_0$; see Eq. (48) and Fig. 4. Meanwhile, $T_c$ versus $\rho_0$ curves and the values of $\rho_0^c$ in HTSCs are material-dependent and disorder-dependent.[34,40,42,44,47,52,53,55,57–59] This fact attests that the value of $\chi$ varies (though, may be, slightly) from one HTSC to another, while the value of $\alpha$ depends both on the kind of HTSC material and on the type of impurities or radiation-induced defects.

Besides, the experimentally observed form of $T_c(\rho_0)$ curve in HTSCs is usually close to linear in a very broad range of critical temperatures.[42,44,47,52,53,55,57–59] The theoretical curve $T_c(\rho_0)$ has such a form if $\chi$ is close to unity (but $\chi\neq 1$) and $\alpha$ is much less than unity, e.g., at $\chi=0.8$ and $\alpha=0.04$; see Fig. 3. In contrast, the theory predicts the negative curvature of $T_c(\rho_0)$ curve for a pure $d$-wave superconductor (no matter how great is the contribution of exchange scattering to $\rho_0$) (see Fig. 4) as well as for a non-$d$-wave superconductor with strong exchange scattering, and the positive curvature of $T_c(\rho_0)$ curve for a non-$d$-wave superconductor containing nonmagnetic impurities only; see Figs. 1–3. So, we expect that the majority of HTSCs have the mixed $(d+s)$-wave order parameter with the predominance of a $d$-wave component $(1-\chi\ll 1)$ and that exchange scattering by magnetic impurities or radiation-induced defects contributes to $\rho_0$, though quite insignificantly $(\alpha\ll 1)$.

The admixture of $s$-wave component to a $d$-wave order parameter, e.g., $\Delta(\mathbf{p})=\Delta_d(\cos p_x-\cos p_y)+\Delta_s$, may be a consequence of orthorhombic distortion of $CuO_2$ planes in some HTSCs.[17–19] The value of the coefficient $\chi$ contains the information about the partial weight of that component in the order parameter, i.e., about the value of $\Delta_s/\Delta_d$. So, having determined the value of $\chi$ from experimental data on radiation-induced and impurity-induced reduction of the critical temperature, one can deduce the value of $\Delta_s/\Delta_d$ making use of Eq. (31).

Besides, it should be stressed that $(d+s)$-wave symmetry is only one of possible candidates for the symmetry of anisotropic pairing state in HTSCs. It is likely to occur in orthorhombic HTSCs. In what concerns purely tetragonal HTSCs, one may expect mixing of an isotropic $s$-wave state with the state having some higher even angular harmonic, e.g., with a $g$-wave state. Such a mixed $(g+s)$-wave state, just as a $(d+s)$-wave state, is also characterized by $\chi$ values in the range from 0 to 1, depending on the partial weights of $s$-wave and $g$-wave components in the order parameter. All the results obtained in this paper are therefore applicable to the case of $(g+s)$ pairing, as well as to the case of any other in-plane symmetry of the order parameter.

To conclude this section, we note that an assumption about the constancy of the parameter $\alpha$ (i.e., an assumption about the constancy of the ratio of the concentrations of nonmagnetic to magnetic scatterers) under doping or irradiation must be checked before detailed comparison of the theory presented in this paper to experimental data. If this assumption appears to be incorrect, Eq. (33) for the critical temperature can still be used, the scattering times being given by Eqs. (44) and (45). In that case, however, one faces an additional complication concerning the evaluation of contributions to the residual resistivity $\rho_0$ from magnetic and nonmagnetic scatterers $\rho_0^{ex}$ and $\rho_0^{nm}$, respectively.

## V. SUMMARY

The combined effect of nonmagnetic and magnetic defects and impurities on the critical temperatures of superconductors with different gap anisotropies was studied theoretically within the weak-coupling limit of the BCS model. For the case of short-range scattering potentials, an expression was derived that relates the critical temperature to the relaxation rates of charge carriers on nonmagnetic and magnetic scatterers as well as to the coefficient of in-plane anisotropy of the superconducting order parameter on the Fermi surface.

We note that the results obtained in this paper can be modified to include the effects of anisotropic (momentum-dependent) impurity scattering. For example, in the case of significant overlap between the anisotropy functions of scattering potential and that of the pair potential, the anisotropic superconductivity has been proven to become less sensitive to nonmagnetic impurities.[62,63,98] However, it is not clear if there is such an overlap in HTSCs.

Besides, numerical calculations within an extended Hubbard model point to the spatial variation of the order parameter in the vicinity of impurities in anisotropic superconductors.[66] As a result, suppression of $T_c$ is significantly weaker than that predicted by the Abrikosov-Gor'kov-type theory. This effect presumably is especially pronounced in superconductors with short coherence length. However, a complete theory of such an effect remains to be developed.

It is worth noting that impurity doping and irradiation generally result not only in a structural disorder but also in creation or annihilation of charge carriers. Thus the effects of carrier and impurity concentrations on $T_c$ of HTSCs should be considered on equal footing.[99] Moreover, since high-temperature superconductivity appears upon doping of parent insulators, a description of those effects should form the basis for the future theory of HTSCs.

In conclusion, the results obtained provide a basis for evaluation of the degree of anisotropy of the superconducting order parameter (e.g., for an estimate of the partial weight of an $s$ wave in a mixed $(d+s)$-wave order parameter) as well as the ratio between nonmagnetic and magnetic scattering rates in high-$T_c$ superconductors through careful comparison of theoretical predictions with the experiments on impurity-induced and radiation-induced reduction of the critical temperature. We hope that the present paper will serve as a stimulus for further experiments on the combined effect of nonmagnetic and magnetic scattering in the copper-oxide superconductors.

## ACKNOWLEDGMENTS

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