Coulomb interaction at the superconductor-to-Mott-insulator transition

Jinwu Ye
Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218
(Received 22 January 1998)

We reexamine the effects of long-range Coulomb interaction on the onset of superconductivity. We use the model of N complex scalar fields with the Coulomb interaction studied first by Fisher and Grinstein (FG). We find that near $d=3$ space dimension, the system undergoes second order phase transition if $N \geqslant 55.39$, but undergoes possible fluctuation driven first order transition if $N < 55.39$. We give the detailed derivation of the field theory renormalization group (RG) of this model to one loop. Our RG results disagree with those of FG near $d=3$. A possible scenario at $d=2$ is proposed.
[S0163-1829(98)00633-X]

I. INTRODUCTION

Superconductor to insulator (SI) transitions have been observed in disordered thin films by systematically varying the thickness of the films[1] or by tuning the magnetic fields.[2] A closely related SI transition has also been studied in artificially constructed $2d$ Josephson-junction arrays by systematically varying the ratio of charging energy and Josephson coupling energy[3] or by tuning the magnetic fields.[4] Universal conductivities have been measured right at the SI transitions in these systems. More recently, SI transition was demonstrated in $3d$ granular Al-Ge samples.[5] Many theoretical efforts[6–12] have been made to investigate the SI transitions in these experimental systems. It is generally accepted that the transition is correctly described by a model of *interacting* charge-$2e$ bosons moving in a $2d$ random potential. A related, but much simpler, model is Boson-Hubbard model which consists of bosons hopping on a periodic lattice. When the boson density is commensurate with the lattice, this model displays a quantum transition from Mott insulator to a superconductor. Neglecting disorder as a first approximation, a lot of authors studied the SI transition and its universal conductivity in this simplified model.[7,11–13] Near the quantum critical point, the effective low energy theory of this model is simply a ϕ^4 theory:

$$\mathcal{S}=\int d^dx d\tau \left[|\partial_0\phi_m|^2+|\partial_i\phi_m|^2+r|\phi_m|^2+\frac{u}{4}|\phi_m|^4 \right], \tag{1}$$

where ϕ_m are $m=1...N$ species of charge $e^*=2e$ complex bosons.

In order to confront with experiments, several important effects such as disorder, dissipation, and long-range Coulomb interaction should be taken into account. From general scaling arguments, Fisher, Grinstein and Girvin argued that the long range Coulomb interaction between the charged bosons will render the dynamic exponent z to be 1 in any dimension.[6] This is indeed what was observed on the magnetic field tuned SI transition in thin two-dimensional (2D) films.[2] The measurements on magnetic field-induced SI transitions in Josephson-junction arrays are also consistent with $z=1,\ \nu>1$. Therefore, it is very important to investigate the effects of Coulomb interaction on SI transition in any detail. In Ref. 7, Fisher and Grinstein (FG) studied the model of charged bosons with Coulomb interaction hopping on a lattice. By taking the continuum limit, they found that the low energy effective theory is described by

$$\begin{aligned}\mathcal{S}=\int d^dx d\tau \Big[&|(\partial_0-ie^*A_0)\phi_m|^2+|\partial_i\phi_m|^2 \\ &+r|\phi_m|^2+\frac{u}{4}|\phi_m|^4 \Big] \\ &+\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}k^\sigma|A_0(\vec{k},\omega)|^2,\end{aligned} \tag{2}$$

where A_0 is the time component of the U(1) gauge fields and is responsible for the long-range interaction between the charged bosons. The $1/r$ Coulomb interaction corresponds to $\sigma=d-1$.

The model in Eq. (2) should not be confused with the model studied in Refs. 14, 15, and 17. The crucial difference is that only the time component A_0 of gauge fields is involved in Eq. (2), therefore, the Lorentz invariance is broken. However, in the scalar electrodynamics studied in Refs. 14, 15, and 17, both time and transverse components are involved, and the Lorentz invariance is respected.

In principle, the coupling of the order parameter to the spatial components of the U(1) gauge fields should also be included in the above equation. However, at 2D films, the penetration depth $\lambda_{2d}=\lambda^2/d$ with λ the *bulk* penetration depth and d the film thickness. In the experimental systems,[1] $\lambda \sim 100$ Å, $d \sim 5$ Å, so $\lambda_{2d} \sim 2000$ Å, the spatial coupling will not manifest itself until experimentally unobservably close to the SI transition.[7,9,18] Therefore, it is safe to neglect the spatial coupling in Eq. (2). In 3D samples investigated in Ref. 5, we assume that they are in the extreme type-II limit. In this limit, the spatial coupling can also be neglected.

FG did renormalization group (RG) analysis of the model by performing a double expansion in $\epsilon=3-d$ and $\epsilon_\sigma=2-\sigma$. The $1/r$ Coulomb interaction corresponds to $\epsilon=\epsilon_\sigma$. They concluded that depending on parameters, at two space

0163-1829/98/58(14)/9450(10)/$15.00

dimension, the transition can be either first order or second order with Coulomb coupling marginally irrelevant at the 3D XY critical point.

In this paper, we provide a detailed RG analysis of the model in Eq. (2) using the field theory method. We do not perform a double expansion in ϵ and ϵ_σ. Instead, for simplicity, we fix σ to be 2 (namely $\epsilon_\sigma \equiv 0$) and perform a $\epsilon = 3-d$ expansion. $\sigma \equiv 2$ corresponds to $1/r$ Coulomb interaction at $d=3$ and *logarithmic* instead of $1/r$ interaction at $d=2$. We find that *near* 3 space dimension, the system undergoes a second order phase transition if $N \geqslant 55.39$, but undergoes a *possible* fluctuation-driven first order transition if $N < 55.39$. We calculate the dynamic exponent z, the boson propagator exponent η, the gauge field (Coulomb interaction) exponent η_A, and the correlation exponent ν. We give the detailed derivation of field theory RG of this model, because the RG method we used is interesting in its own right. We expect the structure brought out by our RG procedures to have some general impact on how to perform correct RG on other zero temperature quantum critical phenomena.[19] The same method has been applied successfully to study the quantum transition from fractional quantum hall state to insulating state in a periodic potential.[20]

If putting ϵ_σ to zero in Eq. (7) of Ref. 7, we find our RG results disagree with those of FG. Although no details are given in Ref. 7, we suspect that (1) the anisotropy between space and time (namely the dynamic exponent z) was not treated correctly by FG and (2) the Ward identity [Eq. (6) in Sec. II] is violated in the momentum shell method employed by FG.

In this paper, we do not even intend to perform the double expansion in $\epsilon = 3-d$ and $\epsilon_\sigma = 2-\sigma$ because we believe that when *gauge field* fluctuations are involved, extrapolating to the physical case $\epsilon = \epsilon_\sigma = 1$ can only lead to misleading results.

The rest of the paper is organized as follows. In Sec. II, we introduce the model and set up the formulation to perform RG. In Sec. III, we give the detailed RG analysis and bring out its elegant structure. We find that our results disagree with those of FG and point out the possible mistakes made by FG. In Sec. IV, we give more discussions and point out the possible scenario at $d=2$ and some future directions. Finally, in the Appendix, we perform RG on the well-studied scalar electrodynamics in Feymann gauge as a nontrivial check on the correctness of our calculations in Sec. III.

II. THE MODEL AND FORMULATION

In order to perform RG analysis, we rewrite Eq. (2) in the following form:[16]

$$\mathcal{S} = \int d^d x\, d\tau \bigg[\alpha^2 |\partial_0 \phi_m|^2 + |\partial_i \phi_m|^2 + r|\phi_m|^2 + \frac{u}{4}\mu^\epsilon \alpha |\phi_m|^4 - ie\mu^{\epsilon/2}\alpha^{3/2} A_0(\partial_0 \phi_m^\dagger \phi_m - \phi_m^\dagger \partial_0 \phi_m) + e^2 \mu^\epsilon \alpha A_0^2 \phi_m^\dagger \phi_m + \frac{1}{2}(\partial_i A_0)^2 \bigg], \tag{3}$$

where $x_i(\tau)$ are spatial (temporal) coordinates with $\partial_0 \equiv \partial_\tau$, $\partial_i \equiv \partial_{x_i}$. We are working in $d=3+\epsilon$ spatial dimensions and μ is a renormalization scale. The parameter α is introduced to allow for anisotropic renormalization between space and time.[21,22]

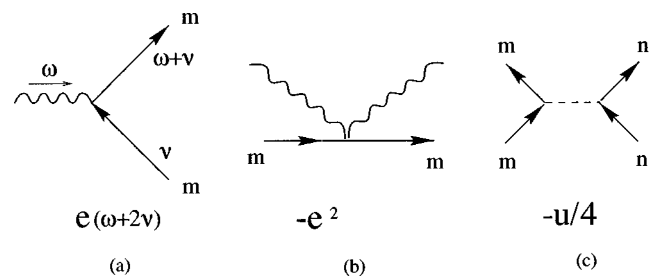

FIG. 1. The Feymann rules for three kinds of vertices in Eq. (3): (a) the type 1 vertex, (b) the type 2 (seagull) vertex, and (c) the quartic coupling of bosons.

It is easy to see Eq. (3) is invariant under the *space-independent* gauge transformation

$$\phi \to \phi e^{i\Lambda(\tau)}, \quad A_0 \to A_0 - \frac{\sqrt{\alpha}}{e}\partial_0 \Lambda(\tau). \tag{4}$$

In the perturbation theory, the three kinds of vertices in Fig. 1 are needed.

The loop expansion requires counterterms to account for ultraviolet divergences in momentum integrals; we write the counterterms as

$$\begin{aligned}\mathcal{L}_{CT} = {} & \alpha^2(Z_\alpha - 1)|\partial_0 \phi_m|^2 + (Z_2-1)|\partial_i \phi_m|^2 \\ & + \frac{u}{4}(Z_4-1)\mu^\epsilon \alpha |\phi_m|^4 \\ & - i(Z_1-1)e\mu^{\epsilon/2}\alpha^{3/2}A_0(\partial_0 \phi_m^\dagger \phi_m - \phi_m^\dagger \partial_0 \phi_m) \\ & + (Z_1-1)e^2\mu^\epsilon \alpha A_0^2 \phi_m^\dagger \phi_m + \frac{1}{2}(Z_3-1)(\partial_i A_0)^2.\end{aligned} \tag{5}$$

The Ward identity following from the gauge invariance dictates

$$Z_1 = Z_\alpha . \tag{6}$$

Using the identity, we relate the bare fields and couplings in $\mathcal{S}$ to the renormalized quantities by

$$\begin{aligned}\phi_{mB} &= Z_2^{1/2}\phi_m, \\ A_{0B} &= Z_3^{1/2} A_0, \\ \alpha_B &= (Z_\alpha/Z_2)^{1/2}\alpha, \\ e_B &= e\mu^{\epsilon/2}(Z_\alpha/Z_2)^{1/4} Z_3^{-1/2}, \\ u_B &= u\mu^\epsilon Z_4 Z_2^{-3/2} Z_\alpha^{-1/2}.\end{aligned} \tag{7}$$

The dynamic critical exponent z is related to the renormalization of α by[22]

$$z = 1 - \mu\frac{d}{d\mu}\ln\, \alpha = 1 - \frac{1}{2}\mu\frac{d}{d\mu}\ln\frac{Z_2}{Z_\alpha}. \tag{8}$$

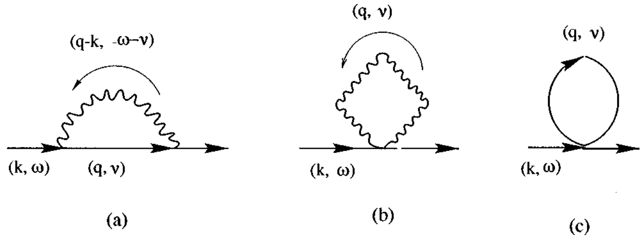

FIG. 2. The self-energy diagram of bosons.

We define the exponent η by the *equal-time* correlation function

$$\langle \phi(x,\tau)\phi^{\dagger}(0,\tau)\rangle \sim x^{-(d+z-2+\eta)}. \tag{9}$$

It is easy to see this correlation function is invariant under the gauge transformation Eq. (4), therefore, η is indeed a gauge invariant quantity and given by

$$\eta=\mu\frac{d}{d\mu}\ln Z_2. \tag{10}$$

The anomalous dimension η_A of the gauge field is also a gauge invariant quantity:

$$\eta_A=\mu\frac{d}{d\mu}\ln Z_3. \tag{11}$$

Near the critical point, the gauge field propagator is

$$\langle A_0(-\vec{k},-\omega)A_0(\vec{k},\omega)\rangle \sim \frac{1}{k^{2-\eta_A}}. \tag{12}$$

Physically, it means that at the critical point, at d spatial dimension, the long-range interaction takes the form $r^{-(d-2+\eta_A)}$.

Finally, the critical exponent ν is related to the anomalous dimension of the composite operator $\phi_m^{\dagger}\phi_m$ by

$$\nu^{-1}-2=\mu\frac{d}{d\mu}\ln Z_{\phi^{\dagger}\phi}. \tag{13}$$

The renormalization constant $Z_{\phi^{\dagger}\phi}$ can be calculated by inserting the operator into the boson self-energy diagrams.

III. THE PERTURBATIVE RG CALCULATION AT ONE LOOP

A. The boson self-energy

First, we consider the boson self-energy diagrams in Fig. 2. Using the Feymann rules listed in Fig. 1, we can bring out the divergent structure of Fig. 2(a) by the conventional dimensional regularization (DR) method

$$\begin{aligned}(2a)&=e^2\mu^{\epsilon}\int\frac{d^d\vec{q}}{(2\pi)^d}\int\frac{d\nu}{2\pi}\frac{(\nu+\omega)^2}{(\vec{q}-\vec{k})^2(q^2+\nu^2)}\\
&=e^2\mu^{\epsilon}\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{(\vec{q}-\vec{k})^2}\int\frac{d\nu}{2\pi}\\
&\quad+e^2\mu^{\epsilon}\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{\omega^2-q^2}{2q(\vec{q}-\vec{k})^2}\\
&=e^2\mu^{\epsilon}\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{(\vec{q}-\vec{k})^2}\int\frac{d\nu}{2\pi}+\frac{e^2}{4\pi^2\epsilon}\left(\omega^2-\frac{k^2}{3}\right)\\
&\quad+\cdots,\end{aligned} \tag{14}$$

where $\cdots$ means all the finite terms.

In the above equation, we scaled all the frequencies by the anisotropy parameter α, so α does not appear explicitly in the above and in the following.

Similarly, we can evaluate Figs. 2(b) and 2(c)

$$2(\mathrm{b})=-e^2\mu^{\epsilon}\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{q^2}\int\frac{d\nu}{2\pi},$$

$$2(\mathrm{c})=-\frac{u}{4}\mu^{\epsilon}\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{2q}=0. \tag{15}$$

In the last equation, the convention of DR is used.

Adding all the contributions from Fig. 2, we get

$$2(\mathrm{a})+2(\mathrm{b})+2(\mathrm{c})=\frac{e^2}{4\pi^2\epsilon}\left(\omega^2-\frac{k^2}{3}\right)+\cdots. \tag{16}$$

It is easy to see that we encounter the UV frequency divergences in both Fig. 2(a) and Fig. 2(b). However, they carry opposite signs, therefore, cancel each other. Similar cancellations also appear in the Dirac fermion model studied in Ref. 20 and will appear in the following calculations of other Feymann diagrams. We expect this cancellation of UV divergences in *frequency* is a general feature of *nonrelativistic* quantum field theories describing zero-temperature quantum phase transitions.

From Eq. (16), we can identify the two constants Z_α, Z_2 in Eq. (5) in the minimal subtraction scheme

$$Z_\alpha=1+\frac{e^2}{4\pi^2\epsilon},\quad Z_2=1-\frac{e^2}{12\pi^2\epsilon}. \tag{17}$$

Note $Z_\alpha\neq Z_2$. This is due to the *lack* of Lorentz invariance of our model.

B. The type 1 vertex

We turn to the evaluation of the renormalization of the type 1 vertex in Fig. 3. Applying the Feymann rules to Fig. 3(a), we have

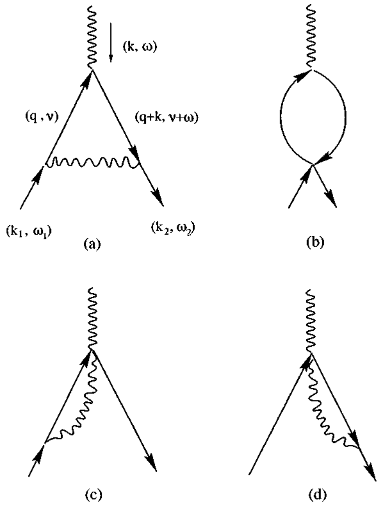

FIG. 3. The one-loop diagrams of type 1 vertex.

$$
\begin{aligned}
3(\mathrm{a}) &= (e\mu^{\epsilon/2})^3 \int \frac{d^d\vec{q}}{(2\pi)^d} \\
&\quad \times \int \frac{d\nu}{2\pi} \frac{(\nu+\omega_1)(2\nu+\omega)(\nu+\omega+\omega_2)}{(\vec{q}-\vec{k}_1)^2(q^2+\nu^2)[(\vec{q}+\vec{k})^2+(\nu+\omega)^2]} \\
&= \frac{e^2}{4\pi^2\epsilon}(\omega_1+\omega_2)+\cdots . \qquad (18)
\end{aligned}
$$

It is easy to see Fig. 3(b) vanishes identically and Fig. 3(c) and Fig. 3(d) are

$$3(\mathrm{c}) = -\frac{e^2}{2\pi^2\epsilon}\omega_1+\cdots,$$

$$3(\mathrm{d}) = -\frac{e^2}{2\pi^2\epsilon}\omega_2+\cdots,$$

$$3(\mathrm{c})+3(\mathrm{d}) = -\frac{e^2}{2\pi^2\epsilon}(\omega_1+\omega_2)+\cdots . \qquad (19)$$

Overall, we find for the type 1 vertex

$$3(\mathrm{a})+3(\mathrm{b})+3(\mathrm{c})+3(\mathrm{d}) = -\frac{e^2}{4\pi^2\epsilon}(\omega_1+\omega_2)+\cdots . \qquad (20)$$

The constant Z_1 in Eq. (3) can be identified

$$Z_1 = 1+\frac{e^2}{4\pi^2\epsilon}. \qquad (21)$$

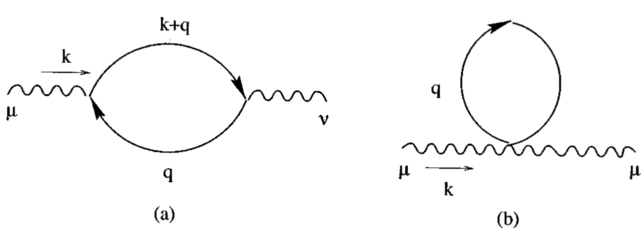

FIG. 4. The vacuum polarization of gauge fields.

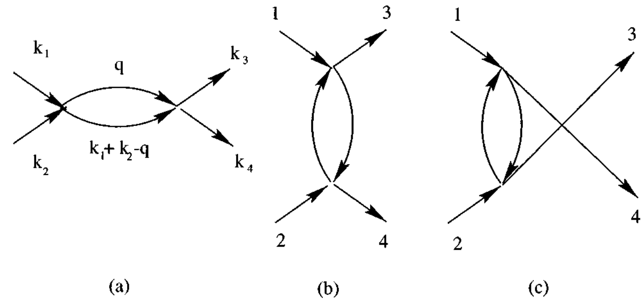

FIG. 5. The renormalization from the quartic coupling.

From Eqs. (17) and (21), it is evident that the Ward identity [Eq. (6)] indeed holds. The Ward identity can also be shown by evaluating the renormalization of the seagull vertex.

C. The polarization of gauge field

Next, we calculate the vacuum polarization graph of the U(1) gauge field A_μ in Fig. 4. The fermion bubble has Lorentz invariance, therefore, the momenta in Fig. 4 are ($D = d+1 = 4-\epsilon$)-dimensional momenta

$$
\begin{aligned}
\Pi_{\mu,\nu}(k) &= e^2\mu^\epsilon N \int \frac{d^Dq}{(2\pi)^D} \frac{(k+2q)_\mu(k+2q)_\nu}{q^2(k+q)^2} \\
&\quad - 2e^2\mu^\epsilon N \int \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2} \\
&= -\frac{Ne^2}{24\pi^2\epsilon}(k^2\delta_{\mu\nu}-k_\mu k_\nu)+\cdots , \qquad (22)
\end{aligned}
$$

where N comes from the summation over the boson suffix m.

In Eq. (3), only the time component of the gauge field A_μ is involved. Putting $\mu=\nu=0$ in the above equation we find

$$\Pi_{0,0}(\vec{k},\omega) = -\frac{Ne^2}{24\pi^2\epsilon}k^2+\cdots , \qquad (23)$$

where k is the d dimensional space momentum. The constant Z_3 in Eq. (3) can be identified as

$$Z_3 = 1-\frac{Ne^2}{24\pi^2\epsilon}. \qquad (24)$$

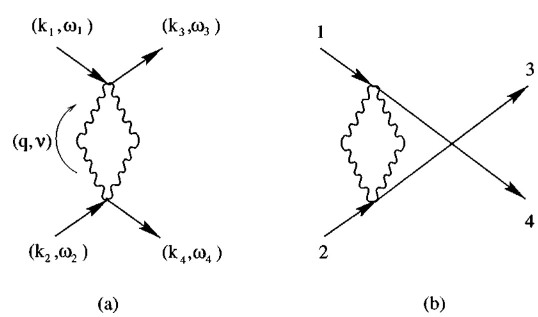

FIG. 6. The renormalization from the seagull term.

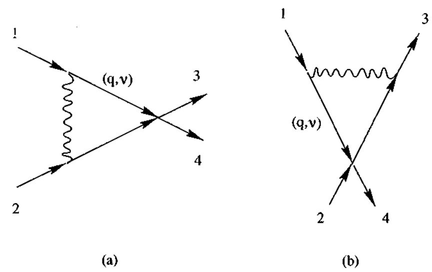

FIG. 7. The renormalization from the combination of the type 1 term + the quartic term.

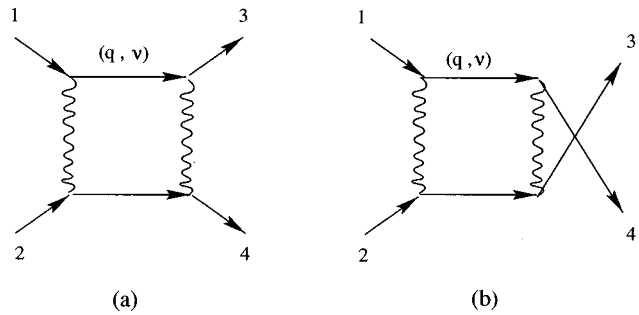

FIG. 8. The renormalization from the type 1 term.

D. The renormalization of the quartic term

In this subsection, we compute the renormalization of the quartic term. First, let us look at the contributions from the quartic coupling in Fig. 5. As in Fig. 4, the momenta in the above figure are also D-dimensional momenta.

$$5(\mathrm{a})=\frac{u^2\mu^{2\epsilon}}{2}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2(k_1+k_2-q)^2}=\frac{u^2}{2}\frac{1}{8\pi^2\epsilon}+\cdots\,. \tag{25}$$

Similarly, we can calculate the contributions from Figs. 5(b) and 5(c),[23]

$$5(\mathrm{b})=5(\mathrm{c})=\frac{(N+3)u^2}{4}\frac{1}{8\pi^2\epsilon}+\cdots\,. \tag{26}$$

Adding all the contributions from Fig. 5, we get

$$5(\mathrm{a})+5(\mathrm{b})+5(\mathrm{c})=\frac{(N+4)u^2}{16\pi^2\epsilon}+\cdots\,. \tag{27}$$

Second, let us look at the contributions from the seagull term in Fig. 6:

$$6(\mathrm{a})=2(e^2\mu^\epsilon)^2\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{q^2(q+k_1-k_3)^2}\int\frac{d\nu}{2\pi},$$

$$6(\mathrm{b})=2(e^2\mu^\epsilon)^2\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{q^2(q+k_1-k_4)^2}\int\frac{d\nu}{2\pi},$$

$$6(\mathrm{a})+6(\mathrm{b})=2(e^2\mu^\epsilon)^2\int\frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{(\vec{q}-\vec{k}_1)^2}\times\left(\frac{1}{(\vec{q}-\vec{k}_3)^2}+\frac{1}{(\vec{q}-\vec{k}_4)^2}\right)\int\frac{d\nu}{2\pi}. \tag{28}$$

As in Eqs. (14) and (15), we encounter the UV divergences in the frequency integrals. We leave them alone at this moment and go ahead to calculate the diagrams in Fig. 7. The corresponding expressions are

$$7(\mathrm{a})=-u(e\mu^{\epsilon/2})^2\int\frac{d^d\vec{q}}{(2\pi)^d}\int\frac{d\nu}{2\pi}\frac{(\nu+\omega_1)(\omega_1+2\omega_2-\nu)}{(\vec{q}-\vec{k}_1)^2(q^2+\nu^2)[(\vec{k}_1+\vec{k}_2-\vec{q})^2+(\omega_1+\omega_2-\nu)^2]}=\frac{ue^2}{8\pi^2\epsilon}+\cdots\,,$$

$$7(\mathrm{b})=-u(e\mu^{\epsilon/2})^2\int\frac{d^d\vec{q}}{(2\pi)^d}\int\frac{d\nu}{2\pi}\frac{(\nu+\omega_1)(2\omega_3-\omega_1+\nu)}{(\vec{q}-\vec{k}_1)^2(q^2+\nu^2)[(\vec{k}_3-\vec{k}_1+\vec{q})^2+(\omega_3-\omega_1+\nu)^2]}=-\frac{ue^2}{8\pi^2\epsilon}+\cdots\,. \tag{29}$$

It is important to note that Figs. 7(a) and 7(b) have opposite signs. Actually, there are *two* diagrams in class 7(a) and *four* diagrams in class 7(b) corresponding to different ways to put the photon line, so the overall contributions are

$$2[7(\mathrm{a})]+4[7(\mathrm{b})]=-\frac{ue^2}{4\pi^2\epsilon}+\cdots\,. \tag{30}$$

Let us look at the contributions from the type 1 vertex in Fig. 8. The corresponding expressions are

$$8(\mathrm{a}) = (e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \int \frac{d\nu}{2\pi} \frac{(\nu+\omega_1)(\nu+\omega_3)(\omega_1+2\omega_2-\nu)(\omega_1+\omega_2+\omega_4-\nu)}{(\vec{q}-\vec{k}_1)^2(\vec{q}-\vec{k}_3)^2(q^2+\nu^2)[(\vec{k}_1+\vec{k}_2-\vec{q})^2+(\omega_1+\omega_2-\nu)^2]}$$

$$= (e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_1)^2(\vec{q}-\vec{k}_3)^2} \int \frac{d\nu}{2\pi} - \frac{3e^4}{8\pi^2\epsilon} + \cdots ,$$

$$8(\mathrm{b}) = (e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \int \frac{d\nu}{2\pi} \frac{(\nu+\omega_1)(\nu+\omega_4)(\omega_1+2\omega_2-\nu)(\omega_1+\omega_2+\omega_3-\nu)}{(\vec{q}-\vec{k}_1)^2(\vec{q}-\vec{k}_4)^2(q^2+\nu^2)[(\vec{k}_1+\vec{k}_2-\vec{q})^2+(\omega_1+\omega_2-\nu)^2]}$$

$$= (e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_1)^2(\vec{q}-\vec{k}_4)^2} \int \frac{d\nu}{2\pi} - \frac{3e^4}{8\pi^2\epsilon} + \cdots . \qquad (31)$$

Actually, there are two diagrams in class 8(a) and two diagrams in class 8(b) corresponding to different ways to put the photon line, so the overall contributions from Fig. 8 are

$$2[8(\mathrm{a})] + 2[8(\mathrm{b})] = 2(e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_1)^2} \left(\frac{1}{(\vec{q}-\vec{k}_3)^2} + \frac{1}{(\vec{q}-\vec{k}_4)^2} \right) \int \frac{d\nu}{2\pi} - \frac{3e^4}{2\pi^2\epsilon} + \cdots . \qquad (32)$$

Finally, we need to compute the diagrams in Fig. 9. The corresponding expressions are

$$9(\mathrm{a}) = -2(e\mu^{\epsilon/2})^2 e^2 \mu^{\epsilon} \int \frac{d^d\vec{q}}{(2\pi)^d} \times \int \frac{d\nu}{2\pi} \frac{(\nu+\omega_2)(\nu+\omega_4)}{(\vec{q}-\vec{k}_2)^2(\vec{q}-\vec{k}_4)^2(q^2+\nu^2)}$$

$$= -2(e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_2)^2(\vec{q}-\vec{k}_4)^2} \times \int \frac{d\nu}{2\pi} + \frac{e^4}{2\pi^2\epsilon} + \cdots ,$$

$$9(\mathrm{b}) = -2(e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_1)^2(\vec{q}-\vec{k}_3)^2} \times \int \frac{d\nu}{2\pi} + \frac{e^4}{2\pi^2\epsilon} + \cdots . \qquad (33)$$

Exchanging leg 3 with leg 4 of Figs. 9(a) and 9(b), we get another two diagrams [9(c) and 9(d)] which were not shown explicitly

$$9(\mathrm{c}) = -2(e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_2)^2(\vec{q}-\vec{k}_3)^2} \times \int \frac{d\nu}{2\pi} + \frac{e^4}{2\pi^2\epsilon} + \cdots ,$$

$$9(\mathrm{d}) = -2(e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_1)^2(\vec{q}-\vec{k}_4)^2} \times \int \frac{d\nu}{2\pi} + \frac{e^4}{2\pi^2\epsilon} + \cdots . \qquad (34)$$

The overall contributions are

$$9(\mathrm{a})+9(\mathrm{b})+9(\mathrm{c})+9(\mathrm{d}) = -4(e\mu^{\epsilon/2})^4 \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{(\vec{q}-\vec{k}_1)^2} \times \left(\frac{1}{(\vec{q}-\vec{k}_3)^2} + \frac{1}{(\vec{q}-\vec{k}_4)^2} \right) \int \frac{d\nu}{2\pi} + \frac{2e^4}{\pi^2\epsilon} + \cdots . \qquad (35)$$

Adding all the contributions from Eqs. (27), (29), (30), (32), and (35), we find the UV divergences in the frequency integrals indeed cancel and lead to the constant Z_4 in Eq. (3)

$$Z_4 = 1 + \frac{(N+4)u}{16\pi^2\epsilon} - \frac{e^2}{4\pi^2\epsilon} + \frac{e^4}{2\pi^2 u\epsilon} . \qquad (36)$$

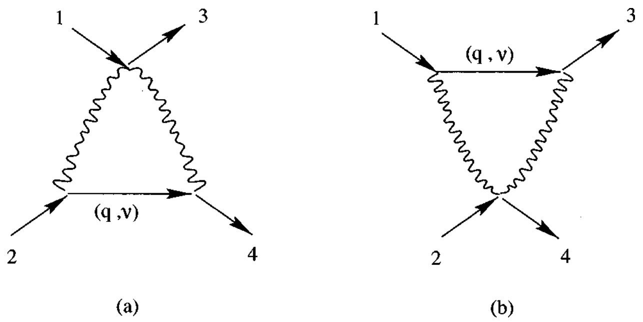

FIG. 9. The renormalization from the type 1 term + the seagull term.

E. The calculation of β function

In the previous subsections, we did the explicit calculations of the renormalization constants by considering a direct perturbative expansion in the Coulomb fine structure constant $w=e^2$ and the quartic coupling u. At one loop order, the values of the renormalization constants are summarized as

$$Z_\alpha = 1+\frac{w}{4\pi^2\epsilon},$$

$$Z_2 = 1-\frac{w}{12\pi^2\epsilon},$$

$$Z_3 = 1-\frac{Nw}{24\pi^2\epsilon},$$

$$Z_4 = 1+\frac{(N+4)u}{16\pi^2\epsilon}-\frac{w}{4\pi^2\epsilon}+\frac{w^2}{2\pi^2 u\epsilon}. \tag{37}$$

Substituting the above equation into Eq. (7), we find the β functions for w and u

$$\beta(w) = -\epsilon w+\frac{N+4}{24\pi^2}w^2,$$

$$\beta(u) = -\epsilon u+\frac{(N+4)u^2}{16\pi^2}-\frac{wu}{4\pi^2}+\frac{w^2}{2\pi^2}. \tag{38}$$

The fixed points are

$$w^* = \frac{24\pi^2}{N+4}\epsilon,$$

$$u^*_\pm = \frac{8\pi^2\epsilon}{N+4}\left[1+\frac{6}{N+4}\pm\frac{\Delta}{N+4}\right], \tag{39}$$

where $\Delta=\sqrt{N^2-52N-188}$.

It is easy to see that Δ is imaginary for $N<N_c=55.39$. This N_c is much smaller than the corresponding critical number $n_c=365$ in the scalar electrodynamics studied in Ref. 14. For $N<N_c$, u is complex for both fixed points and physically inaccessible. The only accessible fixed points are the Gaussian and Heisenberg fixed points, both of which are unstable to turning on Coulomb interaction. There is a runaway flow to the *negative* value of u which was interpreted as a fluctuation-driven first order transition in Ref. 14. For $N>N_c$, these fixed points have real values for u^* and are physically accessible.

In order to compare our β functions with those of FG, we put $N=1$ in Eq. (38)

$$\beta(w) = -\epsilon w+\frac{5}{24\pi^2}w^2,$$

$$\beta(u) = -\epsilon u+\frac{5u^2}{16\pi^2}-\frac{wu}{4\pi^2}+\frac{w^2}{2\pi^2}. \tag{40}$$

Setting $w=(8\pi^2/5)w_{\rm FG}$, $u=32\pi^2 u_{\rm FG}$, the above equations become

$$\beta(w_{\rm FG}) = -\epsilon w_{\rm FG}+\frac{1}{3}w_{\rm FG}^2,$$

$$\beta(u_{\rm FG}) = -\epsilon u_{\rm FG}+10u_{\rm FG}^2-\frac{2}{5}w_{\rm FG}u_{\rm FG}+\frac{1}{25}w_{\rm FG}^2. \tag{41}$$

Putting $\epsilon_\sigma=0$ in Eq. (7) of Ref. 7, we find the RG flow equations obtained by FG are different from ours. The β function should be gauge-independent at least at one-loop order (see also the discussions in the Appendix on this point). Since no details were given by FG, we can only guess the possible mistakes made by FG: (1) the anisotropy between space and time (namely the dynamic exponent z) was not treated correctly by FG; (2) the Ward identity in the time component Eq. (6) is violated in the momentum shell method employed by FG. The disagreement at $\epsilon_\sigma=0$ raises doubts of the conclusions reached by FG at Coulomb interaction case $\epsilon=\epsilon_\sigma$.

Let us look at the solution right at the critical dimension $d=3$. Putting $\epsilon=0$ in Eq. (38) and taking the ratio of the two equations, we find the trajectories of the RG flow satisfy the homogeneous differential equation:

$$\frac{du}{dw} = \frac{3}{2}\left(\frac{u}{w}\right)^2-\frac{6}{N+4}\frac{u}{w}+\frac{12}{N+4}. \tag{42}$$

If $N>N_c$, the solution of this equation is

$$C+\ln w = \frac{N+4}{\Delta}\ln\left|\frac{\frac{u}{w}-\left(\frac{2}{N+4}+\frac{1}{3}\right)-\frac{\Delta}{3(N+4)}}{\frac{u}{w}-\left(\frac{2}{N+4}+\frac{1}{3}\right)+\frac{\Delta}{3(N+4)}}\right|, \tag{43}$$

where C is an arbitrary constant. All the trajectories flow to the origin with two marginally irrelevant couplings w and u. At finite temperature, they will lead to logarithmic corrections to naive scaling functions.

If $N<N_c$, the solution of this equation is

$$C+\ln w = \frac{2(N+4)}{\Delta}\arctan\left[\frac{3(N+4)}{\Delta}\left(\frac{u}{w}-\left(\frac{2}{N+4}+\frac{1}{3}\right)\right)\right]. \tag{44}$$

All the trajectories flow to the negative values of u, indicating a fluctuation driven first order transition.

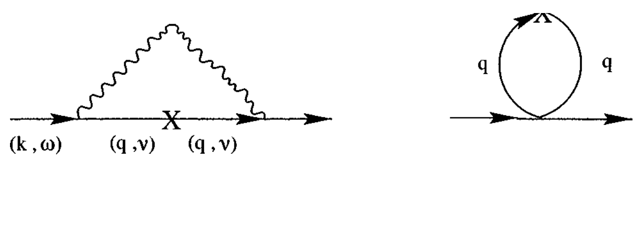

FIG. 10. The cross stands for the insertion of the composite operator at zero momentum.

F. The calculation of critical exponents

In this subsection, we calculate the critical exponents of the second order transition when $N>N_c$. From Eqs. (8), (37), and (39), we find the dynamic exponent z

$$z=1-\frac{4\epsilon}{N+4}. \tag{45}$$

From Eqs. (10), (37), and (39), we find the exponent η

$$\eta=\frac{2\epsilon}{N+4}. \tag{46}$$

From Eqs. (11), (37), and (39), we find the exponent η_A

$$\eta_A=\frac{N\epsilon}{N+4}. \tag{47}$$

The long-range interaction is modified from the form $\sim r^{-1+\epsilon}$ to the form $\sim r^{-1+4\epsilon/(N+4)}$. $Z_{\phi^\dagger\phi}$ can be calculated from Fig. 10. The corresponding expressions are

$$10(\mathrm{a})=e^2\mu^\epsilon\int\frac{d^d\vec{q}}{(2\pi)^d}\int\frac{d\nu}{2\pi}\frac{(\nu+\omega)^2}{(\vec{q}-\vec{k})^2(q^2+\nu^2+m^2)^2}$$

$$=\frac{e^2}{8\pi^2\epsilon}+\cdots,$$

$$10(\mathrm{b})=-\frac{u}{2}(N+1)\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+m^2)^2}$$

$$=-\frac{u(N+1)}{16\pi^2\epsilon}+\cdots, \tag{48}$$

where we put the mass of the boson m into the boson propagators in order to avoid the *infrared* divergence.

In contrast to the self-energy diagrams (Fig. 2), there is *no* UV frequency divergence in the above equations. From the above equation, we can identify

$$Z_2Z_{\phi^\dagger\phi}=1+\frac{u(N+1)}{16\pi^2\epsilon}-\frac{e^2}{8\pi^2\epsilon}. \tag{49}$$

Substituting the value of Z_2 in Eq. (17), we get

$$Z_{\phi^\dagger\phi}=1+\frac{u(N+1)}{16\pi^2\epsilon}-\frac{w}{24\pi^2\epsilon}. \tag{50}$$

The correlation length exponent ν is given by

$$\nu^{-1}=2-\frac{N+1}{2(N+4)}\epsilon\left[\frac{N-1}{N+1}+\frac{6}{N+4}\pm\frac{\Delta}{N+4}\right]. \tag{51}$$

IV. DISCUSSIONS AND CONCLUSIONS

In this paper, we provide a rather detailed derivation of RG analysis of Eq. (2). We find the delicate cancellation of the UV *frequency integral* divergences; therefore, we only need to regularize the UV divergences in momenta integral. We believe that this is due to the space-independent gauge invariance [Eq. (4)] [or the Ward identity in the time component, Eq. (6)]. Similar cancellations also appear in the Dirac fermion model studied in Refs. 20 and 28. We expect this is a very general feature of zero temperature quantum critical phenomena.[19] This feature in *nonrelativistic* quantum field theory reminds us of the theorem in *relativistic* quantum field theory that infrared divergences must be cancelled out in any physical processes.

Here, we would like to make a brief comparison between the boson model studied in this paper and the Dirac fermion model studied in Refs. 20 and 28. In the Dirac fermion model, both Coulomb and Chern-Simon couplings become marginal at $d=2$; the competition between these two couplings produces a line of fixed points with $z=1$ and *nonvanishing* renormalized Coulomb coupling. The fermion quartic term is irrelevant at the noninteracting fixed point and *remains* irrelevant on this $z=1$ line of fixed points. In the boson model studied in this paper, both the Coulomb and the boson quartic couplings become marginal at $d=3$. Fixing σ to be 2 (namely $\epsilon_\sigma\equiv 0$), we performed a $\epsilon=3-d$ expansion. $\sigma\equiv 2$ corresponds to $1/r$ Coulomb interaction at $d=3$, $1/r^{d-2}$ interaction at space dimension d, and *logarithmic* instead of $1/r$ interaction at $d=2$. The Coulomb coupling was found to drive the quartic coupling to negative value, indicating a *possible* first order transition.

Halperin, Lubensky, and Ma[14] (HLM) investigated the model describing the superconducting to normal (SN) and nematic to smectic-A transition in the $d=3$ bulk system. By performing $\epsilon=4-d$ expansion, they found a new stable fixed point with nonvanishing charge if the number of the order parameter components $N>M_c=365.9/2$, but runaway RG if $N<M_c$ (see also the Appendix). HLM interpreted this runaway RG flow as indicating fluctuation-driven first-order transition. Later, Dasgupta and Halperin (DH) studied the model directly on a $3d$ lattice.[24] By using Monte Carlo and duality arguments, they found the transition should be a second order transition in the universality class of the inverted XY model instead of a first-order transition. DH's results indicate that the ϵ expansion may break down at $d=3$.

In this paper, we find the critical value N_c in the pure Coulomb case is much smaller than the corresponding value M_c in the $3d$ SN transition (more precisely $4d$ scalar electrodynamics). This shows that the fluctuation is considerably reduced in the pure Coulomb case. We believe that it is more likely the SI transition at $d=2$ should be governed by a nontrivial fixed point with nonvanishing Coulomb coupling and dynamic exponent $z=1$.

In order to understand the effects of $1/r$ interaction at $d=2$ near the 3D XY fixed point, two possible expansions can be used. One route is to fix at $d=2$ and perform large N

expansion. Unfortunately, as shown in this paper, the direct large N expansion can only probe the physics at $N>N_c$, therefore, is not very useful at physical case $N=1$ in *sharp contrast* to the Dirac fermion case as discussed in Refs. 20 and 28. Another route is the double expansion in $\epsilon=3-d$ and $\epsilon_\sigma=2-\sigma$ performed by FG. However, we believe that when *gauge field* fluctuations are involved, extrapolating to the physical case $\epsilon=\epsilon_\sigma=1$ can only lead to misleading results. A method that works directly at $D=2+1$ is needed.

In order to make serious attempts to compare with experiments,[1–5] disorder has to be incorporated into Eq. (2). As shown in Ref. 8, due to Griffith effects, even weak disorder presumably produces a gapless Bose glass phase between the Mott insulator and superconducting phase. It is certainly very important to study the effects of Coulomb interaction on the transition from superconductor to this boson glass. It was suggested that with Coulomb interaction, the dynamic exponent z should be equal to 1 at all d and the renormalized Coulomb coupling is *finite* at the transition.[6] This scenario was confirmed by extensive Monte Carlo simulations of bosons in a $2d$ disordered medium.[27] It will be welcome if we can establish this scenario *analytically* on this concrete model. This scenario was indeed established in a clean $2d$ lattice model displaying quantum transitions from quantum Hall state to insulating state.[20]

ACKNOWLEDGMENTS

I thank B. Halperin, S. Kivelson, A. Millis, G. Murthy, L. Radzihovsky, N. Read, S. Sachdev, and Z. Tesanovic for helpful discussions. I also thank A. M. Goldman for pointing out Ref. 5 to me. This work was supported by NSF Grant No. DMR-97-07701.

APPENDIX: THE CALCULATION OF SCALAR ELECTRODYNAMICS IN FEYMANN GAUGE

The Lagrangian studied for scalar electrodynamics in Refs. 14, 15, and 17 is

$$\mathcal{L}=|(\partial_\mu-ieA_\mu)\phi_m|^2+r|\phi_m|^2+\frac{u}{4}\mu^\epsilon|\phi_m|^4+\frac{1}{4}(F_{\mu,\nu})^2+\frac{1}{2\lambda}(\partial_\mu A_\mu)^2, \tag{A1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Equation (A1) describes transitions from superconductor to normal metal and from a smectic-A to a nematic in liquid crystals at $d=3$.

In all the previous work, the calculations were done in Landau gauge where $\lambda=0$. The advantage of the Landau gauge is that the divergent parts of Figs. 7, 8, and 9 all vanish. In this appendix, we do all the calculations in the Feymann gauge where $\lambda=1$. In this gauge, Figs. 7, 8, and 9 do make contributions, but the geometrical factors in front of all the integrals should be the *same* as those in the Coulomb interaction case; therefore, by comparing our final results with those obtained in the Landau gauge, we can make a nontrivial check on our RG results. Our final results show that (1) the β function is indeed gauge independent at least to one loop and (2) the calculations done in the Coulomb interaction case are correct.

The loop expansion requires counterterms to account for ultraviolet divergences in momentum integrals; we write the counterterms as

$$\mathcal{L}_{\rm CT}=(Z_2-1)|\partial_\mu\phi_m|^2+\frac{u}{4}(Z_4-1)\mu^\epsilon\alpha|\phi_m|^4-i(Z_1-1)e\mu^{\epsilon/2}A_\mu(\partial_\mu\phi_m^\dagger\phi_m-\phi_m^\dagger\partial_\mu\phi_m)+(Z_1-1)e^2\mu^\epsilon A_\mu^2\phi_m^\dagger\phi_m+\frac{1}{4}(Z_3-1)(F_{\mu,\nu})^2. \tag{A2}$$

Lorentz invariance requires $Z_\alpha=Z_2$, therefore, only one constant Z_2 is needed.

Equation (A1) is invariant under the gauge transformation

$$\phi\to\phi e^{i\Lambda(x)},\quad A_\mu\to A_\mu-\frac{1}{e}\partial_\mu\Lambda(x). \tag{A3}$$

The Ward identity following from the gauge invariance dictates

$$Z_1=Z_2. \tag{A4}$$

Using the identity, we relate the bare fields and couplings in $\mathcal{L}$ to the renormalized quantities by

$$\phi_{mB}=Z_2^{1/2}\phi_m,\quad A_{\mu B}=Z_3^{1/2}A_\mu,\quad e_B=e\mu^{\epsilon/2}Z_3^{-1/2},\quad u_B=u\mu^\epsilon Z_4Z_2^{-1/2},\quad \lambda_B=\lambda Z_3. \tag{A5}$$

At one loop order, the values of the renormalization constants are

$$Z_1=1+\frac{w}{4\pi^2\epsilon},\quad Z_3=1-\frac{Nw}{24\pi^2\epsilon},\quad Z_4=1+\frac{(N+4)u}{16\pi^2\epsilon}-\frac{w}{4\pi^2\epsilon}+\frac{3w^2}{2\pi^2u\epsilon},\quad Z_{\phi^\dagger\phi}=1+\frac{(N+1)u}{16\pi^2\epsilon}-\frac{3w}{8\pi^2\epsilon}. \tag{A6}$$

Substituting the above equation into Eq. (A5), we find

$$\beta(w)=-\epsilon w+\frac{N}{24\pi^2}w^2,\quad \beta(u)=-\epsilon u+\frac{(N+4)u^2}{16\pi^2}-\frac{3wu}{4\pi^2}+\frac{3w^2}{2\pi^2}. \tag{A7}$$

Equations (A6) and (A7) should be compared with the corresponding Eqs. (37) and (38) in the Coulomb interaction case. The differences should be noted.

After proper scalings of w and u, we find Eq. (A7) is the same as that derived in the Landau gauge where $\lambda=0$. The critical exponents are

$$\eta_F=-\frac{6}{N}\epsilon,$$

$$\eta_A=\epsilon,$$

$$\nu^{-1}=2-\frac{\epsilon}{2(N+4)}\left[(N+1)-\frac{54}{N}\pm\frac{N+1}{N}\Delta_a\right], \quad \text{(A8)}$$

where $\Delta_a=\sqrt{N^2-180N-540}$.

It is easy to see that η *differs* from the value computed in the Landau gauge $[\eta_L=-(9/N)\epsilon]$.

It is *not* expected to be, because the correlation function of ϕ is not gauge invariant anyway. The physical meaning of η is not clear. However, the η in Eq. (46) calculated in the Coulomb interaction case is well defined. The reader is encouraged to read Refs. 20 and 28 for similar discussions in the Dirac fermion case.

η_A,ν are the same as those computed in the Landau gauge, because they are gauge invariant quantities. $\eta_A=\epsilon$ indicates that the Coulomb interaction *remains* $1/r$ to order ϵ at $d=3-\epsilon$ in contrast to the pure Coulomb interaction case. Actually, it was shown that the equation $\eta_A=\epsilon$ is *exact*.[25,26] The critical number which divides the second order phase transition and the possible fluctuation-driven first order transition is $M_c=365.9/2$ which is of course gauge-independent.

[1] D. B. Haviland, Y. Liu, and A. M. Goldman, Phys. Rev. Lett. **62**, 2180 (1990).

[2] A. F. Hebard and M. A. Paalanen, Phys. Rev. Lett. **65**, 927 (1990).

[3] L. J. Geerligs *et al.*, Phys. Rev. Lett. **63**, 326 (1989).

[4] H. S. J. van der Zant *et al.*, Phys. Rev. Lett. **69**, 2971 (1992).

[5] A. Gerber *et al.*, Phys. Rev. Lett. **78**, 4277 (1997).

[6] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1988).

[7] M. P. A. Fisher and G. Grinstein, Phys. Rev. Lett. **60**, 208 (1988).

[8] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).

[9] M. P. A. Fisher, Phys. Rev. Lett. **65**, 923 (1990).

[10] E. Granato and J. M. Kosterlitz, Phys. Rev. Lett. **65**, 1267 (1990).

[11] X. G. Wen and A. Zee, Int. J. Mod. Phys. B **4**, 437 (1990).

[12] M. C. Cha *et al.*, Phys. Rev. B **44**, 6883 (1991).

[13] Kedar Damle and S. Sachdev, Phys. Rev. B **56**, 8714 (1997).

[14] B. I. Halperin, T. C. Lubensky, and S. K. Ma, Phys. Rev. Lett. **32**, 292 (1972).

[15] J. Chen, T. C. Lubensky, and D. R. Nelson, Phys. Rev. B **17**, 366 (1978).

[16] In the following, for the simplicity of notation, we replace e^* by e.

[17] S. Hikami, Prog. Theor. Phys. **62**, 226 (1979).

[18] M. R. Beasley, J. E. Mooij, and T. P. Orlando, Phys. Rev. Lett. **42**, 1165 (1979).

[19] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[20] Jinwu Ye and S. Sachdev, Phys. Rev. Lett. **80**, 5409 (1998).

[21] E. Brezin and J. Zinn-Justin, Phys. Rev. B **13**, 251 (1976).

[22] D. Boyanovsky and J. Cardy, Phys. Rev. B **26**, 154 (1982).

[23] Actually, the equality only holds up to finite terms. In order to get the right prefactor $N+3$, the faithful representation of the quartic term in Fig. 1(c) should be used.

[24] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981).

[25] I. Herbut and Z. Tesanovic, Phys. Rev. Lett. **76**, 4588 (1996).

[26] Z. Tesanovic, cond-mat/9801306 (unpublished).

[27] M. Wallin *et al.*, Phys. Rev. B **49**, 12115 (1994).

[28] Jinwu Ye and S. Sachdev (unpublished).