Force-free configurations of vortices in high-temperature superconductors near the melting transition

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We study force-free configurations of Abrikosov flux lines in the line-liquid and line-crystal limit, near the melting transition at H_m. We show that the condition for zero force configurations can be solved by appealing to the structure of chiral liquid crystalline phases. [S0163-1829(98)08138-7]

In high temperature superconductors, the extremely large ratio of the London penetration depth λ to the coherence length ξ suggests that the most important degrees of freedom are Abrikosov vortex excitations. The configuration of flux lines in applied currents and fields thus becomes of great interest. It is therefore useful to construct a theory of the flux lines themselves which may be used to study their conformations.

We start by considering the London equation for a superconductor, which relates the current density $\mathbf{j}$ to the magnetic field $\mathbf{B}$:

$$\nabla\times\mathbf{j}=-\frac{c}{4\pi\lambda^2}\,\mathbf{B}, \tag{1}$$

where λ is the London penetration depth. This is supplemented by Maxwell's equation:

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\,\mathbf{j}. \tag{2}$$

These two equations predict much of the phenomenology of superconductors. In particular, if $\mathbf{B}$ is along the $\hat{z}$ axis, then the London equation predicts that screening currents will circulate in the *xy* plane. If the magnetic field is confined within flux tubes parallel to the $\hat{z}$ axis then the currents will circulate about these confined regions. These flux tubes will form the Abrikosov flux-line lattice.

Under an applied current, flux lines will adopt a steady state configuration in which there is no net force. In the absence of pinning the Lorentz force per unit length on the vortices, $\mathbf{F}/l=\mathbf{B}\times\mathbf{j}$, will be balanced by repulsive vortex-vortex interactions. In a "force-free" configuration[1] it is necessary that $\mathbf{j}$ be parallel to $\mathbf{B}$:

$$\alpha(\mathbf{x})\mathbf{B}(\mathbf{x})=\frac{c}{4\pi}\,\mathbf{j}(\mathbf{x})=\nabla\times\mathbf{B}(\mathbf{x}), \tag{3}$$

where the last equality follows from Maxwell's equation and $\alpha(\mathbf{x})$ is a spatially varying scalar. In a superconductor, the magnetic field is confined to be near flux tubes and along their tangents. If the flux lines trace out the curves $\mathbf{R}_i(s)$, where s is their arclength, then

$$\mathbf{m}(\mathbf{x})=\int ds\sum_i \frac{d\mathbf{R}_i}{ds}\,\delta^3[\mathbf{R}_i(s)-\mathbf{x}] \tag{4}$$

is the local tangent density of flux lines. Since flux lines cannot end we have $\nabla\cdot\mathbf{m}=0$. Using standard techniques for treating topological defects[2] we have

$$[1-\lambda^2\nabla^2]\mathbf{B}=\Phi_0\mathbf{m}, \tag{5}$$

where Φ_0 is the flux quantum. It is useful to decompose $\mathbf{m}$ as the product of a unit vector $\hat{\mathbf{n}}$ and an areal density ρ, $\mathbf{m}=\rho\hat{\mathbf{n}}$.[3] If α varies on a lengthscale long compared with the penetration depth then by applying the operator $[1-\lambda^2\nabla^2]$ to Eq. (3) we find:

$$\nabla\rho\times\hat{\mathbf{n}}+\rho\nabla\times\hat{\mathbf{n}}\approx\alpha\rho\hat{\mathbf{n}}. \tag{6}$$

If we consider the system near H_m where the flux lines are dense, we can take $\rho\approx\rho_0$, a constant and Eq. (6) becomes $\nabla\times\hat{\mathbf{n}}=\alpha\hat{\mathbf{n}}$ while conservation of flux becomes $\nabla\cdot\hat{\mathbf{n}}=0$. Together, these equations appear in the study of liquid crystals: the flux lines adopt a configuration with no splay ($\nabla\cdot\hat{\mathbf{n}}=0$), no bend ($\hat{\mathbf{n}}\times[\nabla\times\hat{\mathbf{n}}]=0$), but with twist ($\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}=\alpha$). We will pursue this analogy with liquid crystals. Of course, the flux-line density does not need to be uniform. The liquid crystal analogy will allow us, however, to consider a class of paradigmatic vortex configurations which do not require density variations and are thus of low energy.

For simplicity, we consider a superconductor in a magnetic field, applied along the z axis. The Abrikosov flux lattice can be modeled as an elastic medium:[4]

$$F_{\rm Lattice}=\frac{1}{2}\int d^3x\{c_{11}u_{ii}^2+2c_{66}[u_{ij}^2-u_{ii}^2]+c_{44}(\partial_z\mathbf{u})^2\}, \tag{7}$$

where $\mathbf{u}$ is the two-dimensional displacement vector (perpendicular to the average flux-line direction), u_{ij} is the two-dimensional strain tensor $u_{ij}=(\partial_i u_j+\partial_j u_i)/2$, and we have used the elastic constants c_{ij} as defined in Ref. 5. The equilibrium conformation will minimize the elastic free energy while maintaining a force-free configuration.

First we consider the case just above H_m where c_{66} vanishes—the flux liquid.[6] In this case, when the flux lines are aligned by an external magnetic field, we can directly show that a current parallel to the field tends to twist the flux-line tangents, as in a cholesteric liquid crystal. We employ the duality mapping between the superfluid and the superconductor[7] under an applied local current. We write $\mathbf{j}^0=\rho_e e\mathbf{v}^0$ where $\mathbf{v}$ is the Cooper-pair velocity, ρ_e is the pair density, and e is the pair charge. The partition function for the London theory in an applied field $\mathbf{H}$ is:

$$Z=\int [d\mathbf{v}][d\mathbf{A}]\delta[\nabla\cdot\mathbf{A}]\exp\left\{-\int d^3x\,\frac{m\rho_e e^2}{2}(\mathbf{v}-\mathbf{v}^0-\mathbf{A})^2+\frac{1}{2}(\nabla\times\mathbf{A})^2+\mathbf{H}\cdot\nabla\times\mathbf{A}\right\}, \tag{8}$$

where m is the mass of the Cooper pair. Writing the velocity in Fourier space in terms of longitudinal and transverse components,

$$\mathbf{v}(\mathbf{k})=i\mathbf{k}\phi+\frac{i\mathbf{k}\times\mathbf{m}}{|\mathbf{k}|^2}, \tag{9}$$

where $\mathbf{m}$ is the density of flux vortices pointing in the $\hat{\mathbf{m}}$ direction. Since flux lines cannot begin or end in the sample $\nabla\cdot\mathbf{m}=0$. Upon substituting Eq. (9) into Eq. (8) and integrating out ϕ and $\mathbf{A}$ we have (to leading order in momentum):

$$Z=\int [d\mathbf{m}]\delta[\nabla\cdot\mathbf{m}]\times\exp\left\{-\int d^3x\left[\frac{1}{2}\mathbf{m}^2-\mathbf{m}\cdot\mathbf{H}-\mathbf{v}^0\cdot\nabla\times\mathbf{H}+\mathbf{v}^0\cdot\nabla\times\mathbf{m}\right]\right\}, \tag{10}$$

where we have omitted terms independent of $\mathbf{m}$ and $\mathbf{v}$. The first two terms are responsible for the presence of vortices in the superconductor—they favor $\mathbf{m}=\mathbf{H}$. The next term induces the screening current in the Meissner phase. The last term is the new interaction which tends to twist the vortices around the applied current. When $\mathbf{H}\|\hat{z}$, it is natural to write $\mathbf{m}\approx\rho\hat{\mathbf{z}}+\rho_0\delta\mathbf{n}$, where $\delta\mathbf{n}$ is the projection of the average tangent onto the xy plane. Then $\nabla\cdot\mathbf{m}=0$ becomes:

$$\partial_z\delta\rho+\rho_0\nabla_\perp\cdot\delta\mathbf{n}=0. \tag{11}$$

This constraint can be solved[8] by introducing a two-dimensional vector field $\mathbf{u}$ and writing $\delta\rho=-\rho_0\nabla_\perp\cdot\mathbf{u}$ and $\delta\mathbf{n}=\partial_z\mathbf{u}$. In terms of this field $\mathbf{u}$,

$$F_{\text{total}}=\int d^3x\left\{\frac{c_{11}}{2}u_{ii}^2+\frac{c_{44}}{2}(\partial_z\mathbf{u})^2-\frac{m\rho_0}{2e^2\rho_c}j\cdot\nabla_\perp\times\partial_z\mathbf{u}\right\}, \tag{12}$$

where we have allowed for anisotropic elastic constants. This theory is simply the theory of polymer cholesterics, in the limit of small pitch.[3] For large deviations one might expect that the flux lines will rotate in a plane perpendicular to a pitch direction. This configuration was, in fact, proposed in the seminal work of Campbell and Evetts.[1] Moreover, an additional conformation is possible in a finite radius, cylindrical sample, namely a double-twist configuration as in the blue phase of chiral liquid crystals. This possible double-twist configuration is shown in Fig. 1. In this double-twist conformation the $\mathbf{B}$ field and $\mathbf{j}$ wrap around each other, simultaneously satisfying the Maxwell and London Eqs. (2) and (1). It was correctly noted in Ref. 1 that this configuration would be energetically unacceptable as the radius of the cylinder grew—the flux lines on the boundary of the sample would grow unacceptably long. However, as we shall see in the following, a defect-riddled state can allow local configurations similar to those shown in Fig. 1, with finite displacements of the flux lines. The handedness of the rotation of the flux lines is determined by the right-hand rule and the direction of the current. Note, however, that an equally acceptable force-free conformation would be *absolutely* straight flux lines parallel to the applied current. The difficulty with this is that thermal fluctuations will destabilize this state and lead to a helical instability of flux lines, as predicted by Clem[9] in 1977. As a consistency check, we note that if we were to consider the effect of the Lorentz forces acting on the individual flux lines that there is an instability also at any finite current[10] towards helical flux-line trajectories.

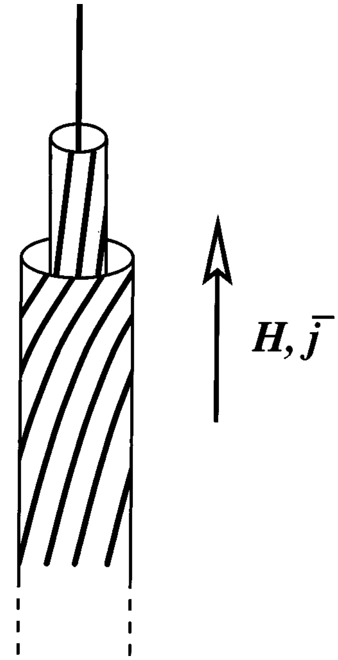

FIG. 1. Configuration of flux lines and current in a wire (both follow the heavy lines). The flux lines are parallel to the current everywhere and both wrap around the center of the wire. Note that there is a nonvanishing $\nabla\times\mathbf{B}$ and $\nabla\times\mathbf{j}$. The applied field $\mathbf{H}$ and the average current density $\bar{\mathbf{j}}$ are parallel.

We note that there is a certain duality between the current and the magnetic field in the London-Maxwell equations. In particular, the equations are invariant under

$$\mathbf{j}\rightarrow\frac{c}{4\pi\lambda}\mathbf{B},\qquad \mathbf{B}\rightarrow-\frac{4\pi\lambda}{c}\mathbf{j}. \tag{13}$$

It would thus be natural to consider the dual physical situation to the Abrikosov flux lattice. In this case, the current would flow along the $\hat{z}$ axis leading to a screening magnetic circulation in the xy plane. If the current were confined into regions, so would be the circulating magnetic field. Physically, this is accomplished via flux lines tracing out helical trajectories: the xy components of the flux-line tangents circulate in that plane, dragging the magnetic field with them providing the necessary magnetic field.

There is, however, an essential difference between the Abrikosov solution and its dual: in the original problem, quantum mechanics imposes a constraint on the amount of magnetic flux that could be confined in a flux tube—single valuedness of the wave function implies that the flux must be an integer multiple of the flux quantum $\Phi_0 = (2\pi\hbar c)/(2e)$. This constraint is responsible for the presence of a second-order transition between the Meissner state and the Abrikosov state. In the dual case, there is no equivalent quantization of current flux. It is easy to understand why in the dual language: a helical flux line can execute an arbitrarily long-pitched wobble which allows $[\nabla\times\mathbf{B}]_\perp$ to be arbitrarily small. This is what allows Clem's helical instability.

When $c_{66}\neq 0$ we are forced to consider a crystalline structure with a force-free conformation of the flux lines. This problem has been considered in the context of liquid crystals—namely, how a chiral line crystal minimizes its free energy in the presence of the two competing tendencies to twist and to have periodic order. These two tendencies frustrate each other and thus, as in the Renn-Lubensky twist-grain boundary (TGB) phase of smectic liquid crystals the frustration will be resolved via the introduction of topological defects—screw dislocations.

In Ref. 11 two types of crystal defect arrays were considered. One array consisted of a periodic arrangement of tilt-grain boundaries (TGB) which would change the local flux line direction. The other array was made of helicoidal-grain boundaries (HCB), each of which is a honeycomb lattice of screw dislocations lying in the *xy* plane. A single isolated HCB leads to a twisting of the crystalline order along the flux-line direction. If we were to consider stacking many HCB's together with some spacing d', this twisted moiré state would have both twisting of $\delta\mathbf{n}$ as well as twisting of the crystal directions. This is similar to the physics of blue phases in chiral liquid crystals.[12] In chiral liquid crystals, there is a tendency for the local director **n** to twist. However, in blue phases this twist manifests itself in double-twist cylinders. Taking the nematic director field as a local tangent vector density for lines, these double-twist cylinders become ropelike bundles of twisted lines. Analogously, a twisted bundle of flux lines will allow the magnetic field to circulate while keeping the flux lines, on average, along a single direction. While in the softer liquid crystal theory the elastic energy cost of this deformation is proportional to the angle of rotation,[11] in the flux-line system interactions between the screw dislocations of the vortex lattice will lead to logarithmic corrections to this energy. In any event, the energy of a grain boundary per unit area will be finite. We propose these defected states as paradigms for a flux-line lattice under an applied, parallel current.

Notice that we can have no twist if $\alpha=0$. However, $\mathbf{j}=\alpha\mathbf{B}$. If current flows through the superconductor then it must flow on the often-neglected boundaries of the sample. Thus, to study the energetics of this state, we must include the usual London energy for the supercurrent. We consider a current along the magnetic field direction $\hat{z}$ and assume that there is a flux-line lattice. If v is the Cooper-pair velocity and ρ_e is the density of Cooper pairs, then the total current is $I_z = e\rho_e A v$ where A is the cross-sectional area of the region in which current flows. If there are no defects in the flux lattice the current must flow within a penetration depth λ of the sample boundary and $A\approx 2\pi R\lambda$. The London energy for this current configuration is, per unit length along $\hat{z}$,

$$F/L = 2\pi\int_{R-\lambda}^{R} r\,dr\,m_e\rho_e v^2 = \frac{m_e}{2\pi R\lambda\rho_e e^2}\,I_z^2, \qquad (14)$$

where m_e is the mass of the electron.

If we allow the Abrikosov lattice to have defects then the current can flow through more of the cross section thus lowering the London energy. Of course, the energy decrease will be offset by the energy of the screw dislocations in the flux-line lattice. If we consider a moiré configuration which is reasonably dense then as a rough approximation we take $\nabla\times\delta\mathbf{n}$ to be uniform along $\hat{z}$. This implies that the current runs uniformly through the entire cross section of the sample. The decreased London energy is:

$$F_{\text{defects}}/L = \frac{m}{\pi R^2\rho_e e^2}\,I_z^2. \qquad (15)$$

In the moiré state we must add the energy of the dislocation lattice that produces the twisted configuration. In a crystalline lattice the energy in the strain field due to a single dislocation diverges logarithmically with system size. If we have a network of dislocations as in the moiré state,[11] however, the strain energy of the lattice is finite. In this case the largest contribution to the energy of a grain boundary is due to the logarithmic interactions of the screw dislocations. We take the energy per unit length of a screw dislocation to be $\epsilon_0\ln(d/\zeta)$ where d is the average defect spacing and ζ is the defect core size.[13] The energy cost per unit length along $\hat{z}$ of the sample is therefore:

$$E_{\text{defects}}/L = \epsilon_0\ln(d/\zeta)\,\frac{\pi R^2}{d^2}. \qquad (16)$$

With a uniform current density Maxwell's equation gives $\nabla\times\mathbf{B}=4I_z/(cR^2)$. In turn, the defect density is determined by the amount of $\nabla\times\delta\mathbf{n}$ required to produce the uniform current. For a defect spacing d we estimate[14]

$$\nabla_\perp\times\delta\mathbf{n}\approx\frac{a_0}{2d^2}. \qquad (17)$$

Putting this together with Maxwell's equation, we get

$$\frac{\pi R^2}{d^2} = \frac{8\pi I_z}{c}\,\frac{1}{\sqrt{\Phi_0 H_z}}, \qquad (18)$$

where we have used $\rho_0 = H_z/\Phi_0$ and $a_0 = 1/\sqrt{\rho_0}$. Thus the defect energy is:

$$E_{\text{defects}}/L = \ln\left[\frac{c\sqrt{\Phi_0 H_z}\,\pi R^2}{8\pi I_z\zeta^2}\right]\frac{4\pi\epsilon_0 I_z}{c\sqrt{\Phi_0 H_z}}. \qquad (19)$$

Putting together all the energies we may compare the energy of the moiré structure with that of the untwisted Abrikosov structure. There will be an instability towards a twisted state when

$$\frac{m}{2\pi R\lambda\rho_c e^2}\,I_z^2 \geqslant \frac{m}{\pi R^2\rho_c e^2}\,I_z^2 + E_{\text{defects}}/L. \qquad (20)$$

If $\lambda \ll R$ we may neglect the first term on the right-hand side of Eq. (20) and find that the moiré state is favored when

$$I_z \geqslant \ln\left[\frac{c\sqrt{\Phi_0 H_z}\pi R^2}{8\pi I_z \zeta^2}\right]\frac{8\pi^2 R\lambda \rho_c e^2 \epsilon_0}{m_e c\sqrt{\Phi_0 H_z}}, \qquad (21)$$

or when the applied current density $j_z = I_z/(\pi R^2)$ is

$$j_z \geqslant \frac{\lambda}{R}\ln\left[\frac{c\sqrt{\Phi_0 H_z}}{8\pi j_z \zeta^2}\right]\frac{8\pi\rho_c e^2 \epsilon_0}{m_e c\sqrt{\Phi_0 H_z}}. \qquad (22)$$

Thus as the system size increases the current density necessary to go to the moiré state goes to zero. Thus the instability of Clem may be stabilized through the lattice structure and its screw dislocations. The moiré state that proposed here is not exactly force-free. In each cell of the honeycomb, the flux-line displacement is a z-dependent rotation: $u_i = Cz\epsilon_{ij}x_j$. This configuration has $\nabla\times\mathbf{B}$ parallel to $\hat{\mathbf{z}}$, *not* the local $\mathbf{B}$. However, when averaged over one cell, $\mathbf{B}$ is parallel to $\hat{\mathbf{z}}$ and so there is no net force on each bundle of vortices. It is clear, however, that small adjustments to the flux-line locations that do not change their topology can yield an entirely force-free configuration.

We comment on recent experiments[15] performed on thin film, $YBa_2Cu_3O_{7-\delta}$, *ab*-plane superconductors which have seen what are called ''vortex'' twisters, composed of thousands of flux lines. These twisters are formed by first applying a magnetic field along the long direction of the sample creating an Abrikosov flux-line lattice. An additional field $H_\perp$ is applied perpendicular to the plane of the flux lines. The anisotropy insures that the flux lines will remain in the *ab* plane if $H_\perp$ is small enough.[16] Lorentz forces from the screening current push the flux lines along the direction either parallel or antiparallel to the c axis depending on the sign of the force. The forces build up and the Bean critical state[17] is formed. For current to flow along the sides of the superconductor, the flux lines must twist about, giving a nonzero $\nabla\times\mathbf{B}$. Moreover, these vortex twisters can be made ever more stable and compact through ''work hardening'' by applying an ac component of $H_\perp$. The work-hardened bundles are stable for hours, while the unhardened bundles are not stable at all. It is unlikely that these vortex twisters form a perfect moiré structure. However, the local structure of these vortices might resemble the highly entangled moiré state. This could explain the work hardening: when there are many screw dislocations it is difficult for the flux-line lattice to relax, as the defects cannot cross the flux lines without cutting them. A flux bundle twisted as in Fig. 1 could relax easily as there are no topological impediments.

Along these lines, it would be interesting to study the dynamics of the flux-line lattice under an applied current. Just above H_m, the flux-line picture suggests a striking frequency dependence of the I-V curve for currents along the field axis. At high frequencies the flux-line lattice may act like a solid, unable to relax, while at low frequencies it could flow, reminiscent of viscoelasticity in polymer melts. Finally, if work hardening the Abrikosov lattice were possible, the critical current in the sample might be increased.[18]

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