# Phases of random antiferromagnetic spin-1 chains

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We formulate a real-space renormalization scheme that allows a study of the effects of bond randomness in the Heisenberg antiferromagnetic spin-1 chain. There are four types of bonds that appear during the renormalization flow. We implement the decimation procedure numerically. We give a detailed study of the probability distributions of all these bonds in the phases that occur when the strength of the disorder is varied. Approximate flow equations are obtained in the weak-disorder regime as well as in the strong-disorder case, where the physics is that of the random-singlet phase. [S0163-1829(98)05025-5]

## I. INTRODUCTION

The effect of quenched impurities on the physics of one-dimensional spin systems is an important and unsolved problem. Many spin chains can be doped chemically, and this creates some kind of disorder in the system. In addition, the spin-$\frac{1}{2}$ chain is equivalent to a system of spinless fermions through the Jordan-Wigner transformation. This means that the problem of interacting spinless fermions in a disordered potential is equivalent to a random spin chain problem. There are not many techniques that allow a study of these systems. The real-space renormalization group is prominent among them. Some time ago a pioneering study by Ma and Dasgupta[1] showed that the spin-$\frac{1}{2}$ Heisenberg antiferromagnetic chain with bond randomness is in a so-called random-singlet phase. In this phase, the spins are locked into singlets that extend over arbitrarily long distances, in a pattern dictated by the bond distribution. It has recently been realized that the results of their renormalization procedure are in fact *exact*.[2] This random-singlet phase may capture the physics of higher-dimensional disordered systems.[3]

In the spin-$\frac{1}{2}$ case, the random-singlet phase appears for various kind of disorder and in a wide regions of the phase diagram when one adds $XXZ$ anisotropy. This results from the study of the weak-disorder regime by bosonizing the spin chain.[4]

The spin-1 Heisenberg antiferromagnetic chain has a physics which is vastly different in the pure case. There is a gap for spin excitations, and a finite spin-spin correlation length. These features can be best understood by consideration of a hidden topological order.[5,6] In fact, the ground state of the spin-1 chain has a hidden long-range order that can be measured only by use of a nonlocal correlation function, the so-called string order parameter. It is a natural question to ask what happens to these peculiar features under the influence of disorder. In fact the original Ma-Dasgupta renormalization scheme requires a broad enough bond distribution to work.[7] Thus more complex schemes have been proposed.[8–10] As a function of the disorder strength, it has been established that there is a phase transition between a low-disorder gapless phase with hidden order and a strong-disorder phase which is the random-singlet phase of Ma and Dasgupta (gapless and no hidden order).

In this paper, we give a detailed construction of a renormalization scheme suited to the study of the spin-1 chain. We generalize the Ma-Dasgupta decimation procedure by keeping more degrees of freedom. A brief account was given in Ref. 10. Here we obtain explicit flow equations that are valid deep inside each of the phases that appear. We are able to follow the spin populations as a function of the renormalization scale as well as the evolution of distribution functions of the various kinds of bonds that appear. In Sec. II, we define the renormalization scheme. In Sec. III, we study the weak-disorder phase of the spin-1 chain. Section IV contains our results for the strong-disorder regime. The critical regime is studied in Sec. V, and Sec. VI contains our conclusions.

## II. REAL-SPACE RENORMALIZATION PROCEDURE FOR DISORDERED ANTIFERROMAGNETIC SPIN-1 CHAIN

In this section, we explain how to obtain a real-space renormalization scheme adequate to study the disordered antiferromagnetic spin-1 chain.

### A. Ma-Dasgupta real-space renormalization in the spin-$\frac{1}{2}$ case

Ma and Dasgupta introduced a real-space renormalization procedure for the random antiferromagnetic spin-$\frac{1}{2}$ chain described by the Hamiltonian

$$H=\sum_i J_i \vec{S}_i \cdot \vec{S}_{i+1}, \tag{2.1}$$

where $\{\vec{S}_i\}$ is a quantum spin-$\frac{1}{2}$ operator, and $\{J_i\}$ a positive random variable distributed with some probability distribution $P_0(J)$. Suppose that $J_1$ is the largest coupling in the chain. The one-bond Hamiltonian

$$h_0=J_1\vec{S}_1\cdot\vec{S}_2=\frac{J_1}{2}[(\vec{S}_1+\vec{S}_2)^2-\vec{S}_1^2-\vec{S}_2^2]=\frac{J_1}{2}\left[(\vec{S}_1+\vec{S}_2)^2-\frac{3}{2}\right] \tag{2.2}$$

admits two energy levels labeled by $s=0$ and 1,

$$e_s=\frac{J_1}{2}\left[s(s+1)-\frac{3}{2}\right], \tag{2.3}$$

the level $e_s$ being $(2s+1)$ times degenerate: $e_0=-\frac{3}{4}J_1$ represents the singlet, and $e_1=\frac{1}{4}J_1$ the triplet. At energies much lower than $J_1$, the spins $\vec{S}_1$ and $\vec{S}_2$ will therefore be frozen into the singlet state $s=0$. The decimation procedure consists in eliminating the spins $\vec{S}_1$ and $\vec{S}_2$, and in replacing the four spin segment Hamiltonian $H_{0,1,2,3}$ involving the decimated spins $\vec{S}_1$ and $\vec{S}_2$ ,

$$H_{0,1,2,3}=h_0+h_1 \quad \text{where} \quad h_1=J_0\vec{S}_0\cdot\vec{S}_1+J_2\vec{S}_2\cdot\vec{S}_3, \tag{2.4}$$

by the effective Hamiltonian for the remaining spins $\vec{S}_0$ and $\vec{S}_3$ ,

$$H_{0,3}^{\text{eff}}=E'_{0,3}+J'_0\vec{S}_0\cdot\vec{S}_3, \tag{2.5}$$

which is meant to reproduce the four low-energy states of $H_{0,1,2,3}$ which are separated from the other 12 states of $H_{0,1,2,3}$ by a large gap of order $J_1$. Using second-order perturbation theory to treat $h_1$ gives

$$E'_{0,3}=-\frac{3}{4}J_1-\frac{3}{16J_1}(J_0^2+J_2^2) \tag{2.6}$$

and

$$J'_0=\frac{J_0J_2}{2J_1}. \tag{2.7}$$

The same procedure may be iterated and successively applied to the strongest bond of the chain. This defines a flow for the probability distribution of couplings $P(J,\Omega)$, where $\Omega$ is the current strongest coupling[1]

$$-\frac{\partial P(J,\Omega)}{\partial\Omega}=P(\Omega,\Omega)\int_0^\Omega dJ_a\int_0^\Omega dJ_bP(J_a,\Omega) \times P(J_b,\Omega)\ \ \delta\left(J-\frac{J_aJ_b}{2\Omega}\right). \tag{2.8}$$

This flow equation has to be supplied by some initial condition $P(J,\Omega_0)$. Fisher showed[2] that, for generic initial conditions, in the reduced variables $\Gamma=\ln(\Omega_0/\Omega)$ and $z=1/\Gamma\ \ln(\Omega/J)$, the probability distribution $R(z,\Gamma)$ of the variable $z$ flows towards the unique fixed point $R^*(z)$,

$$R(z,\Gamma)\underset{\Gamma\to\infty}{\to}R^*(z)\equiv\theta(z)e^{-z}, \tag{2.9}$$

where $\theta$ is the Heaviside step function. This so-called random-singlet fixed point corresponds to a power-law distribution in the original variables

$$P^*(J,\Omega)=\theta(\Omega-J)\ \ \frac{\alpha(\Omega)}{\Omega}\left(\frac{J}{\Omega}\right)^{\alpha(\Omega)-1},$$

where

$$\alpha(\Omega)\underset{\Omega\ll\Omega_0}{\simeq}\frac{1}{\ln\left(\frac{\Omega_0}{\Omega}\right)}, \tag{2.10}$$

for which two typical bonds are typically much weaker than the strongest one $\Omega$. The approximation involved in the use of perturbation theory to obtain rule (2.7) therefore becomes better and better as the decimation proceeds, and the whole procedure is therefore completely consistent even if the initial distribution is not broad. The Ma-Dasgupta renormalization scheme is moreover very appealing because it gives an interesting physical picture of the random spin-$\frac{1}{2}$ chain: at low energy, the chain is made of pairs of spins that are coupled together into singlets over arbitrarily long distances, the long singlets bonds being typically much weaker than the smaller ones.

### B. Renormalization of an AF bond between two spins $S=1$

The one-bond Hamiltonian

$$h_0=J_1\vec{S}_1\cdot\vec{S}_2=\frac{J_1}{2}[(\vec{S}_1+\vec{S}_2)^2-\vec{S}_1^2-\vec{S}_2^2] =\frac{J_1}{2}[(\vec{S}_1+\vec{S}_2)^2-4] \tag{2.11}$$

admits three energy levels labeled by $s=0$, 1, and 2,

$$e_s=\frac{J_1}{2}[s(s+1)-4], \tag{2.12}$$

the level $e_s$ being $(2s+1)$ times degenerate: $e_0=-2J_1$ represents the singlet, $e_1=-J_1$ the triplet, and $e_2=J_1$ the quintuplet.

In the Ma-Dasgupta procedure, there are only two levels, and ''projecting onto the lowest level'' is equivalent to ''projecting out the highest level.'' Here these two possibilities are not equivalent. The first possibility has already been considered in Refs. 8 and 9 where it is shown that the generalization of Eq. (2.7), describing the effective coupling between $\vec{S}_0$ and $\vec{S}_3$ resulting from the projection onto the singlet formed by $\vec{S}_1$ and $\vec{S}_2$, reads

$$J'_0=\frac{4}{3}\frac{J_0J_2}{J_1}. \tag{2.13}$$

The coefficient $\frac{4}{3}$ being larger than 1, this rule is not automatically consistent: indeed, the inequalities $J_0<J_1$ and $J_2<J_1$ are not sufficient to imply that new coupling $J'_0$ is smaller than the decimated coupling $J_1$, in contrast with rule (2.7) concerning spin-$\frac{1}{2}$ chains. This procedure can, however, be considered as qualitatively correct for very broad initial randomness, where the cases which would produce a coupling $J'_0$ larger than the decimated coupling $J_1$ are statistically negligible. So the strongly disordered antiferromagnetic spin-1 chains are described by the same random-singlet fixed point already found in the study of disordered spin-$\frac{1}{2}$ chains.

For weak initial randomness, however, this naive procedure cannot be made consistent. We thus generalize the Ma-Dasgupta procedure with the interpretation of ''projecting out the highest level'' instead of ''projecting onto the lowest

level.'' More precisely, for the antiferromagnetic bond described by the Hamiltonian $h_0$, we project out the quintuplet $e_2$ but keep the singlet $e_0$ and the triplet $e_1$ by replacing the two spins $S=1$ $\vec{S}_1$ and $\vec{S}_2$ by two spins $S=\frac{1}{2}$ $\vec{S}'_1$ and $\vec{S}'_2$, and by replacing $h_0$ by the effective Hamiltonian

$$h_0^{\text{eff}}=-\frac{5J_1}{4}+J_1\vec{S}'_1\cdot\vec{S}'_2. \tag{2.14}$$

The four-spin-segment Hamiltonian $H_{0,1,2,3}$, containing the old spins $\vec{S}_1$ and $\vec{S}_2$,

$$H_{0,1,2,3}=h_0+h_1 \quad \text{where} \quad h_1=J_0\vec{S}_0\cdot\vec{S}_1+J_2\vec{S}_2\cdot\vec{S}_3, \tag{2.15}$$

has to be replaced by an effective Hamiltonian involving the spins $\vec{S}_0$, $\vec{S}'_1$, $\vec{S}'_2$, and $\vec{S}_3$:

$$H_{0,1,2,3}^{\text{eff}}=h_0^{\text{eff}}+h_1^{\text{eff}}. \tag{2.16}$$

If we use a first-order perturbation theory to treat $h_1$, we find that the singlet of $h_0$ remains unchanged

$$\langle s_{1,2}=0|h_1|s_{1,2}=0\rangle=0, \tag{2.17}$$

whereas the degeneracy of the triplet is lifted by the perturbation $h_1$. Using the Wigner-Eckart theorem for vectorial operators, we find, more explicitly,

$$\langle 1,m|h_1|1,m'\rangle=(\tfrac{1}{2}J_0\vec{S}_0+\tfrac{1}{2}J_2\vec{S}_3)\cdot\langle 1,m|(\vec{S}_1+\vec{S}_2)|1,m'\rangle. \tag{2.18}$$

We can reproduce these matrix elements by choosing the effective Hamiltonian

$$h_1^{\text{eff}}=J_0\vec{S}_0\cdot\vec{S}'_1+J_2\vec{S}'_2\cdot\vec{S}_3. \tag{2.19}$$

This does not reproduce the matrix elements mixing singlet and triplets like $\langle 0|h_1|1,m\rangle$.

In fact, as noted by Hyman,[8] the exact first-order effective Hamiltonian is

$$h_1^{\text{exact}}=J_0\vec{S}_0\cdot\left(\frac{1+\alpha}{2}\vec{S}'_1-\frac{\alpha-1}{2}\vec{S}'_2\right)+J_2\left(-\frac{\alpha-1}{2}\vec{S}'_1+\frac{1+\alpha}{2}\vec{S}'_2\right)\cdot\vec{S}_3, \tag{2.20}$$

where $\alpha=\sqrt{8/3}$. It contains ferromagnetic next-nearest-neighbor couplings between $\vec{S}_0$ and $\vec{S}'_2$, and between $\vec{S}'_1$ and $\vec{S}_3$, but they are *nonfrustrating*, and, as such, they are not expected to lead to qualitatively new physics. We use the simpler approximate effective Hamiltonian $h_1^{\text{eff}}$ that preserves the structure of the chain and that reproduces most of the matrix elements. It also means that we ignore multiplicative factors of $\alpha=\sqrt{8/3}$, but these will not matter in regimes in which the probability distribution is broad. In fact one can check explicitly in the spin-$\frac{1}{2}$ case that the coefficient $\frac{1}{2}$ in the decimation rule [Eq. (2.7)] plays no role at the fixed point. Such approximations will of course change nonuniversal quantities like the precise value of the critical disorder between the two regimes of the spin-1 chain.

We have now enlarged the initial space, since the chain now contains not only $S=1$ spins but also $S=\frac{1}{2}$ spins. However it is possible to define a decimation procedure that is ''closed'' inside a particular set of spin chains, as we will see in the following.

### C. Real-space renormalization procedure

We consider the enlarged set of spin chains described by the Hamiltonian

$$H=\sum_i J_i\vec{S}_i\cdot\vec{S}_{i+1}, \tag{2.21}$$

where the spin $\vec{S}_i$ is a spin operator of size $s_i=\frac{1}{2}$ or $s_i=1$, and where the couplings $\{J_i\}$ can be either positive or negative, but have to satisfy the following constraint: for any pair $\{i,j\}$ such that $i<j$, the classical magnetization of the classical ground state of the segment $(i,j)$, must be smaller or equal to one in absolute value

$$|m_{i,j}|\leqslant 1, \tag{2.22}$$

where the quantity $m_{i,j}$ reads

$$m_{i,j}=s_i+\sum_{n=i+1}^{j} s_n\times\text{sgn}\left[\prod_{p=i}^{n-1}(-J_p)\right]. \tag{2.23}$$

This condition for $j=i+1$ gives immediately that there are exactly four types of bonds: (1) link of type 1: Ferromagnetic bond between two spin-$\frac{1}{2}$ chains. (2) link of type 2: Antiferromagnetic (AF) bond between two spin-$\frac{1}{2}$ chains. (3) link of type 3: Antiferromagnetic bond between one spin-1 chain and one spin-$\frac{1}{2}$ chain. (4) link of type 4: Antiferromagnetic bond between two spin-1 chains. Our decimation procedure is the following.

To each bond $(\vec{S}_i,\vec{S}_{i+1},J_i)$ we associate the energy difference between the higher state and lower states of the reduced Hamiltonian $J_i\vec{S}_i\cdot\vec{S}_{i+1}$:

$$\Delta_i=-J_i \quad \text{if the bond } i \text{ is of type 1,} \tag{2.24}$$

$$\Delta_i=J_i \quad \text{if the bond } i \text{ is of type 2,} \tag{2.25}$$

$$\Delta_i=\tfrac{3}{2}J_i \quad \text{if the bond } i \text{ is of type 3,} \tag{2.26}$$

$$\Delta_i=3J_i \quad \text{if the bond } i \text{ is of type 4.} \tag{2.27}$$

We pick up the bond $(\vec{S}_{i_1},\vec{S}_{i_2},J_{i_1})$ corresponding to the strongest $\Delta_i$ of the chain. To define the renormalization rule for this bond, we again divide the four-spin Hamiltonian into

$$H_{i_0,i_1,i_2,i_3}=h_0+h_1$$

where

$$h_0=J_{i_1}\vec{S}_{i_1}\vec{S}_{i_2} \quad \text{and} \quad h_1=J_{i_0}\vec{S}_{i_0}\cdot\vec{S}_{i_1}+J_{i_2}\vec{S}_{i_2}\cdot\vec{S}_{i_3}, \tag{2.28}$$

and treat $h_1$ as a perturbation of $h_0$ to find the effective Hamiltonian replacing $H_{i_0,i_1,i_2,i_3}$ when the highest-energy state of $h_0$ is removed. We now have to distinguish the four types of bonds.

Rule (1) $F$ bond between two spins $S=\frac{1}{2}$.

The Hamiltonian $h_0 = J_{i_1}\vec{S}_{i_1}$ and $\vec{S}_{i_2}$ admits two energy levels: the triplet $e_1 = -|J_{i_1}|/4$ and the singlet $e_0 = 3|J_{i_1}|/4$. The perturbation $h_1$ lifts the degeneracy of the triplet, and using Wigner-Eckart theorem, we find that $h_1$ is equivalent at first order of perturbation theory to

$$h_1^{\rm eq} = (\tfrac{1}{2}J_{i_0}\vec{S}_{i_0} + \tfrac{1}{2}J_{i_2}\vec{S}_{i_3})\cdot(\vec{S}_{i_1} + \vec{S}_{i_2}). \tag{2.29}$$

To eliminate the singlet state and only keep the triplet state of $h_0$, we remove the two spin-$\frac{1}{2}$ $\vec{S}_{i_1}$ and $\vec{S}_{i_2}$ and replace them by a single spin-1 $\vec{S}'_{i_1}$, and we replace $H_{i_0,i_1,i_2,i_3}$ by

$$H^{\rm eff}_{i_0,i'_1,i_3} = -\frac{|J_{i_1}|}{4} + \frac{1}{2}J_{i_0}\vec{S}_{i_0}\vec{S}'_{i_1} + \frac{1}{2}J_{i_2}\vec{S}'_{i_1}\vec{S}_{i_3}. \tag{2.30}$$

Rule (2) AF bond between two spins $S = \frac{1}{2}$.

Here, we directly apply the Ma-Dasgupta procedure discussed in Sec. II A: we remove the two spin-$\frac{1}{2}$ $\vec{S}_{i_1}$ and $\vec{S}_{i_2}$ chains, and replace $H_{i_0,i_1,i_2,i_3}$ by

$$H^{\rm eff}_{i_0,i_3} = -\frac{3}{4}J_{i_1} - \frac{3}{16J_{i_1}}(J_{i_0}^2 + J_{i_2}^2) + \frac{J_{i_0}J_{i_2}}{2J_{i_1}}\vec{S}_{i_0}\cdot\vec{S}_{i_3}. \tag{2.31}$$

Rule (3) AF bond between one spin-1 and one spin-$\frac{1}{2}$.

Suppose that $s_{i_1} = 1$ and $s_{i_2} = \frac{1}{2}$. The Hamiltonian $h_0 = J_{i_1}\vec{S}_{i_1}\vec{S}_{i_2}$ admits two energy levels: the doublet $e_{1/2} = -J_{i_1}$ and the quadruplet $e_{3/2} = J_{i_1}/2$. At first order perturbation theory, Wigner-Eckart theorem gives that, within the subspace of the doublet $s = \frac{1}{2}$, the perturbation $h_1$ is equivalent to

$$h_1^{\rm eq} = (\alpha_1 J_{i_0}\vec{S}_{i_0} + \alpha_2 J_{i_2}\vec{S}_{i_3})\cdot(\vec{S}_{i_1} + \vec{S}_{i_2}), \tag{2.32}$$

where the constants $\alpha_1$ and $\alpha_2$ read

$$\alpha_1 = \frac{1}{2}\left[1 + \frac{s_{i_1}(s_{i_1}+1) - s_{i_2}(s_{i_2}+1)}{s(s+1)}\right] = \frac{4}{3}$$

and

$$\alpha_2 = 1 - \alpha_1 = -\frac{1}{3}. \tag{2.33}$$

The renormalization rule is therefore the following: we eliminate the spins $\vec{S}_{i_1}$ and $\vec{S}_{i_2}$, and replace them by a single spin-$\frac{1}{2}$ $\vec{S}'_{i_1}$, and we replace $H_{i_0,i_1,i_2,i_3}$ by the effective Hamiltonian

$$H^{\rm eff}_{i_0,i'_1,i_3} = -J_{i_1} + \tfrac{4}{3}J_{i_0}\vec{S}_{i_0}\vec{S}'_{i_1} - \tfrac{1}{3}J_{i_2}\vec{S}'_{i_1}\vec{S}_{i_3}. \tag{2.34}$$

Rule (4) AF bond between two spins $S = 1$.

In this case we apply the rule explained at the beginning of this section [see Eqs. (2.14)–(2.19)]: we replace the two spin-1 $\vec{S}_1$ and $\vec{S}_2$ by two spin-$\frac{1}{2}$ $\vec{S}'_{i_1}$ and $\vec{S}'_{i_2}$, and we replace $H_{0,1,2,3}$ by an effective Hamiltonian

$$H^{\rm eff}_{i_0,i_1,i_2,i_3} = -\frac{5J_{i_1}}{4} + J_{i_0}\vec{S}_{i_0}\cdot\vec{S}'_{i_1} + J_{i_1}\vec{S}'_{i_1}\cdot\vec{S}'_{i_2} + J_{i_2}\vec{S}'_{i_2}\cdot\vec{S}_{i_3}. \tag{2.35}$$

This renormalization procedure is entirely consistent from the point of view of the progressive elimination of the highest-energy degrees of freedom: it is easy to show that in the four cases of renormalization of a bond described above, all the energy scales $\Delta_i$ of the new bonds are always smaller than the energy scale $\Delta_{i1}$ of the bond that we renormalize.

It is also easy to check that this renormalization procedure is ''closed'' inside the set of spin chains defined by condition (2.22): if we apply this procedure to an initial chain belonging to this space, such as the random antiferromagnetic spin-1 chain we are interested in, the effective chain always belongs to this set of spin chains. In particular, spins higher than 1 cannot appear through this renormalization scheme.

However, since this renormalization procedure is not purely based on complete decimation of bonds, it introduces correlations between bonds, so that it is impossible to write exact closed flow equations for the probability distributions of couplings, in contrast with the Ma-Dasgupta procedure. To study the properties of this renormalization scheme, we have therefore performed numerical simulations on spin-1 chains containing $N$ sites with periodic boundary conditions ($N = 2^{22}$ for example), whose initial couplings $J_i$ are distributed according to probability distributions of the following form:

$$P_d(J) = \frac{1}{d} \quad \text{for} \quad 1 \leqslant J \leqslant 1+d$$

$$\text{and} \quad P_d(J) = 0 \quad \text{elsewhere} \tag{2.36}$$

The parameter $d$ represents the strength of the initial disorder of the couplings. For a given number of sites $N$, and a given initial strength $d$ of the disorder, we have numerically implemented the renormalization rules on a given number (typically 100) of initial independent samples, to compute averaged quantities over these different realizations of the initial disorder. It is convenient to use the variable

$$\Gamma = \ln\frac{\Omega_0}{\Omega}, \tag{2.37}$$

where $\Omega$ is the current strongest $\Delta$ [see Eq. (2.27)] and $\Omega_0$ the initial strongest $\Delta$. We have studied the flow of the following quantities: the number $N(\Gamma)$ of effective spins $S = \frac{1}{2}$ and 1 still present at scale $\Gamma$; the proportion $\{N_{(S=1)(\Gamma)}/N(\Gamma)\}$ of spins $S = 1$ among the effective spins at scale $\Gamma$; the proportions $\rho_i(\Gamma) = \{N_i(\Gamma)/N(\Gamma)\}$ of bonds of type $i = 1$, 2, 3, and 4 at scale $\Gamma$; the probability distributions $P_i(J,\Omega)$ of the coupling $J$ at scale $\Omega$ for the four types of bonds $i = 1$, 2, 3, and 4. It is in fact more convenient to study the probability distributions $\mathcal{P}_i(x,\Gamma)$ of the reduced variable,

$$x = \ln\left(\frac{\Omega}{\Delta(J)}\right), \tag{2.38}$$

where $\Delta(J)$ is defined as in Eq. (2.27),

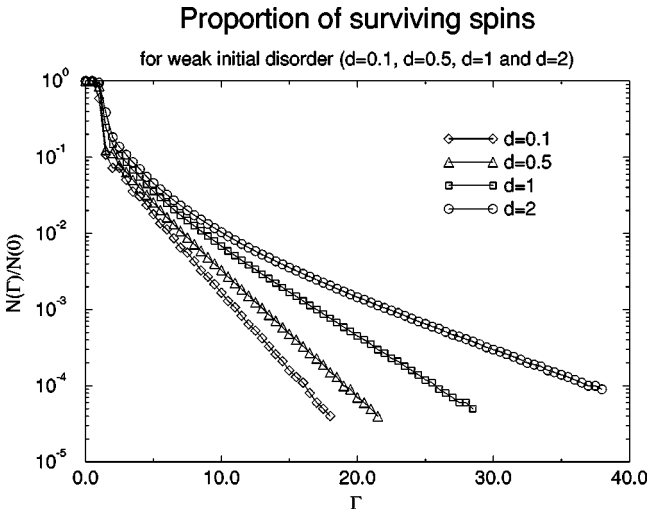



FIG. 1. Linear-log plot of the proportion $N(\Gamma)/N(0)$ of effective spins at scale $\Gamma$, for weak initial disorder $d=0.1$, 0.5, 1, and 2: this proportion decays exponentially [Eq. (3.1)].

$$\Delta(J)=-J \quad \text{for bonds of type 1,} \tag{2.39}$$

$$\Delta(J)=J \quad \text{for bonds of type 2,} \tag{2.40}$$

$$\Delta(J)=\tfrac{3}{2}J \quad \text{for bonds of type 3,} \tag{2.41}$$

$$\Delta(J)=3J \quad \text{for bonds of type 4,} \tag{2.42}$$

so that the random variable $x$ varies in $(0,\infty)$ for any type of bonds.

## III. WEAK-DISORDER PHASE

### A. Numerical results

In the weak-disorder phase, we find that the number $N(\Gamma)$ of effective spins decays exponentially (see Fig. 1),

$$N(\Gamma)\propto_{\Gamma\to\infty} e^{-\alpha(d)\Gamma}, \tag{3.1}$$

where $\alpha(d)$ is a decreasing function of the disorder $d$ that vanishes in the limit $d\to d_c^-$. As a consequence, the magnetic susceptibility at temperature $T$ can be computed by summing Curie laws for the free spins at scale $\Omega=T$. So we have

$$\chi\propto\frac{1}{T^{1-\alpha(d)}}. \tag{3.2}$$

The proportions $\rho_i(\Gamma)$ of the four types of bonds reach a stationary regime characterized by (see Fig. 2)

$$\rho_1(\Gamma)\simeq 0.25, \quad \rho_2(\Gamma)\simeq 0.75, \quad \rho_3(\Gamma)\simeq 0, \quad \rho_4(\Gamma)\simeq 0. \tag{3.3}$$

There are asymptotically only bonds of type 1 and bonds of type 2. This means in particular that there are only effective spin-$\frac{1}{2}$ in the chain, and no more spins $S=1$. Since two bonds of type 1 cannot be neighbors according to constraint (2.22), the even and odd bonds are not equivalent, as in the effective model of Hyman and Yang:[9] the ''even'' bonds are all antiferromagnetic, whereas the ''odd'' bonds are either ferromagnetic or antiferromagnetic with equal probability.

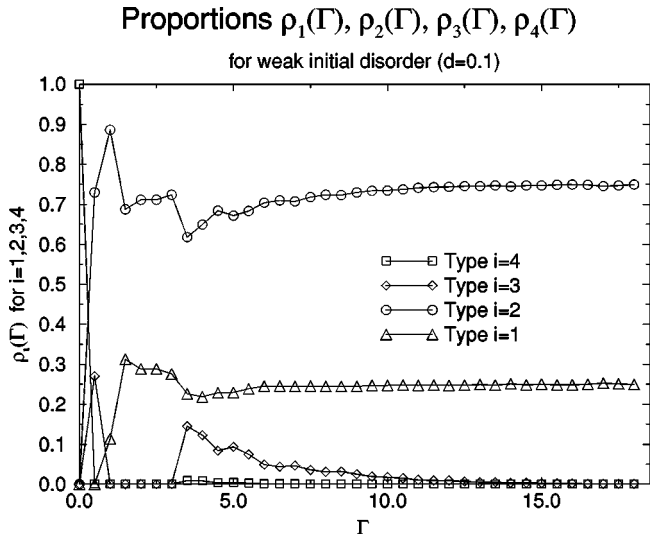



FIG. 2. The proportions $\rho_i(\Gamma)$ of the four types $i=1$, 2, 3, and 4 of bonds at scale $\Gamma$, for weak initial disorder $d=0.1$: they reach the asymptotic regime (3.3).

It is necessary to introduce the probability distribution $\mathcal{P}_2^{\text{even}}(x,\Gamma)$ for the couplings of the even bonds of type 2, and the probability distribution $\mathcal{P}_2^{\text{odd}}(x,\Gamma)$ for the couplings of the odd bonds of type 2. We find that $\mathcal{P}_2^{\text{even}}(x,\Gamma)$ becomes stationary for large enough $\Gamma$, and takes the form of an exponential distribution

$$\mathcal{P}_2^{\text{even}}(x,\Gamma)\simeq\alpha_e e^{-\alpha_e x}, \tag{3.4}$$

where $\alpha_e$ is independent of $\Gamma$, but depends on the value $d$ of the disorder, and is numerically very close to the parameter $\alpha(d)$ characterizing the decay of $N(\Gamma)$ [Eq. (3.1)]. The probability distributions $\mathcal{P}_1(x,\Gamma)$ and $\mathcal{P}_2^{\text{odd}}(x,\Gamma)$ coincide (up to statistical fluctuations) and take the form of an exponential distribution (see Fig. 3)

$$\mathcal{P}_1(x,\Gamma)\simeq\mathcal{P}_2^{\text{odd}}(x,\Gamma)\simeq\alpha_o(\Gamma)e^{-\alpha_o(\Gamma)x}, \tag{3.5}$$

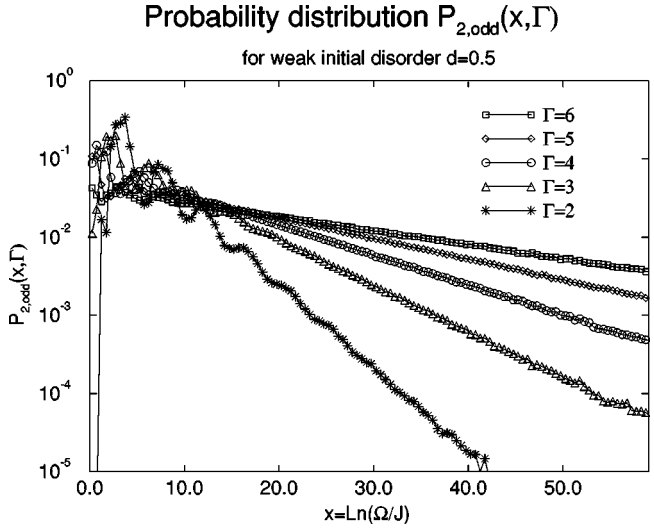



FIG. 3. Linear-log plot of the probability distribution $\mathcal{P}_2^{\text{odd}}(x,\Gamma)$ for $\Gamma=2$, 3, 4, 5, and 6, for weak initial disorder $d=0.5$: $\mathcal{P}_2^{\text{odd}}(x,\Gamma)$ is well described by the exponential form (3.5), with a parameter $\alpha_o(\Gamma)$ that is found to decay exponentially with $\Gamma$ [Eq. (3.6)].

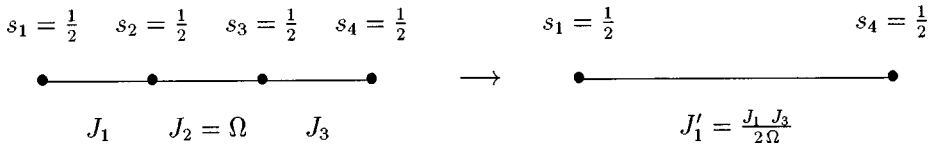
where the parameter $\alpha_o(\Gamma)$ decays exponentially:

$$\alpha_o(\Gamma) \propto e^{-\alpha_e \Gamma} \tag{3.6}$$

As a consequence, for large enough $\Gamma$, the bond of the chain of highest $\Delta$ (corresponding to smallest $x$), that is chosen to be renormalized, is always an even bond of type 2. In the renormalization operation (2), this even bond disappears together with its two odd neighbors, and a new weak odd bond is produced. This explains why the distribution $\mathcal{P}_2^{\rm even}(x,\Gamma)$ for even bonds remains stationary, whereas the distribution of couplings of odd bonds becomes broader and broader in the variable $x$. This weak disorder phase is therefore the same as the ''Haldane phase'' found by Hyman and Yang in their effective model introduced in Ref. 9, and is very similar to the random dimer phase found in the study of random dimerized antiferromagnetic spin-$\frac{1}{2}$ chains:[11] in the asymptotic regime, the chain is made of a set of nearly uncoupled dimers.

### B. Approximate flow equations

Assuming that the ''even'' bonds are all of type 2, that the ''odd'' bonds are either of type 1 or of type 2 with equal probability, and that the unique important process is the decimation of an even bond according to rule (2),




it is possible to write approximate flow equations for the probability distributions of the couplings are normalized according to

$$1 = \int_0^\Omega dJ\ P_2^{\rm even}(J,\Omega) = \int_0^\Omega dJ\ P_2^{\rm odd}(J,\Omega)$$

$$= \int_{-\Omega}^0 dJ\ P_1(J,\Omega). \tag{3.7}$$

It is convenient to introduce the normalized distribution of all odd bonds

$$P^{\rm odd}(J,\Omega) = \tfrac{1}{2}[P_2^{\rm odd}(J,\Omega) + P_1(J,\Omega)] \quad \text{for} \quad -\Omega < J < \Omega. \tag{3.8}$$

The approximate flow equations for the probability distributions $P_2^{\rm even}(J,\Omega)$ and $P^{\rm odd}(J,\Omega)$ then read

$$-\frac{\partial P_2^{\rm even}(J,\Omega)}{\partial \Omega} = P_2^{\rm even}(\Omega,\Omega)\ P_2^{\rm even}(J,\Omega), \tag{3.9}$$

$$-\frac{\partial P^{\rm odd}(J,\Omega)}{\partial \Omega} = -P_2^{\rm even}(\Omega,\Omega) P^{\rm odd}(J,\Omega),$$

$$+P_2^{\rm even}(\Omega,\Omega) \int_{-\Omega}^{\Omega} dJ_1 P^{\rm odd}(J_1,\Omega)$$

$$\times \int_{-\Omega}^{\Omega} dJ_3 P^{\rm odd}(J_3,\Omega)\ \delta\left(J - \frac{J_1 J_3}{2\Omega}\right). \tag{3.10}$$

In the new variables $\Gamma = \ln(\Omega_0/\Omega)$ and $x = \ln(\Omega/J) \in [0,+\infty)$, the flow equation for $\mathcal{P}_2^{\rm even}(x)$ admits stationary solutions of exponential form

$$\mathcal{P}_2^{\rm even}(x) = \alpha_e e^{-\alpha_e x}, \tag{3.11}$$

with an undetermined constant $\alpha_e$, in agreement with our numerical result (3.4). With the last change of variables,

$$x \to z = \alpha_o(\Gamma) \ln\left(\frac{\Omega}{|J|}\right), \tag{3.12}$$

the flow equation for the corresponding probability distributions $\tilde{P}_1^{\rm odd}(z,\Gamma)$ and $\tilde{P}_2^{\rm odd}(z,\Gamma)$ admit the same stationary solution

$$\tilde{P}_1^{\rm odd}(z,\Gamma) \simeq \tilde{P}_2^{\rm odd}(z,\Gamma) \underset{\Gamma\to\infty}{\to} e^{-z}$$

with

$$\alpha_o(\Gamma) \underset{\Gamma\to\infty}{\propto} e^{-\alpha_e \Gamma}, \tag{3.13}$$

where $\alpha_e$ is the number characterizing $\mathcal{P}_2^{\rm even}(x)$ [Eq. (3.11)]. We may also write the flow equation for the total number $N(\Omega)$ of spins still present at scale $\Omega$,

$$-\frac{dN}{d\Omega} = -P_2^{\rm even}(\Omega,\Omega) N(\Omega), \tag{3.14}$$

so that we obtain the following asymptotic behavior in the variable $\Gamma$:

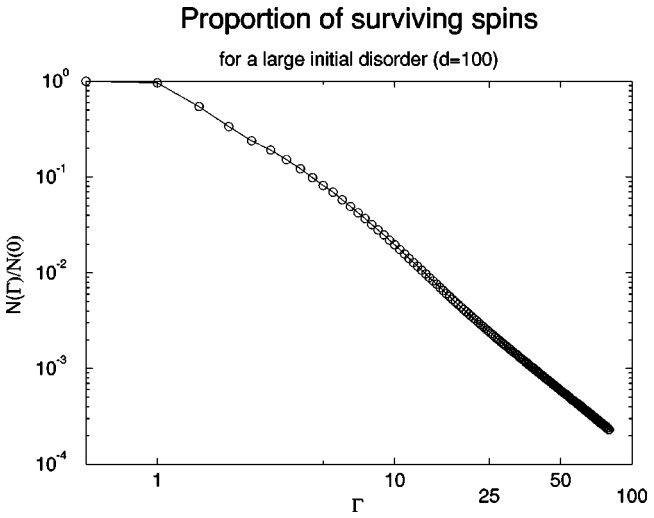



FIG. 4. Log-log plot of the proportion $N(\Gamma)/N(0)$ of effective spins at scale $\Gamma$ for strong initial disorder $d=100$: this proportion follows the power-law asymptotic behavior (4.1).

$$N(\Gamma) \underset{\Gamma\to\infty}{\propto} e^{-\alpha_e \Gamma}. \tag{3.15}$$

## IV. STRONG-DISORDER PHASE

### A. Numerical results

In the strong-disorder phase $d>d_c$, we find that the number $N(\Gamma)$ of effective spins decays as in the random-singlet theory for the disordered antiferromagnetic spin-$\frac{1}{2}$ chain (see Fig. 4):

$$N(\Gamma) \underset{\Gamma\to\infty}{\propto} \frac{1}{\Gamma^2}. \tag{4.1}$$

The magnetic susceptibility thus has the random-singlet behavior

$$\chi \propto \frac{1}{T \ln^2 T}. \tag{4.2}$$

The proportions $\rho_i(\Gamma)$ of the four types of bonds reach an asymptotic regime characterized by (see Fig. 5)

$$\rho_1(\Gamma)\sim 0, \quad \rho_2(\Gamma)\sim \epsilon(\Gamma)$$

$$\rho_3(\Gamma)\sim 2\epsilon(\Gamma), \quad \rho_4(\Gamma)\sim 1-3\epsilon(\Gamma), \tag{4.3}$$

where $\epsilon(\Gamma)$ slowly goes to 0 as $\Gamma\to\infty$. This means that there is a sea of bonds of type 4, with sometimes defects of structure {bond of type 3, bond of type 2, bond of type 3}. This defect structure is produced by the renormalization rule (4) for a bond of type 4 when its two neighbor bonds are also of type 4. The fact that there are no more bonds of type 1 in the asymptotic regime (4.3) shows that defects are destroyed by the renormalization of the central bond of type 2 and not by the bonds of type 3; this means that for the probability distribution $P_4(J,\Omega)$ at large enough $\Omega$, two typical couplings are much weaker than the larger one. We indeed find that $\mathcal{P}_4(x,\Gamma)$ is an exponential distribution (see Fig. 6),

$$\mathcal{P}_4(x,\Gamma) \simeq \alpha_4(\Gamma) e^{-\alpha_4(\Gamma)x}, \tag{4.4}$$

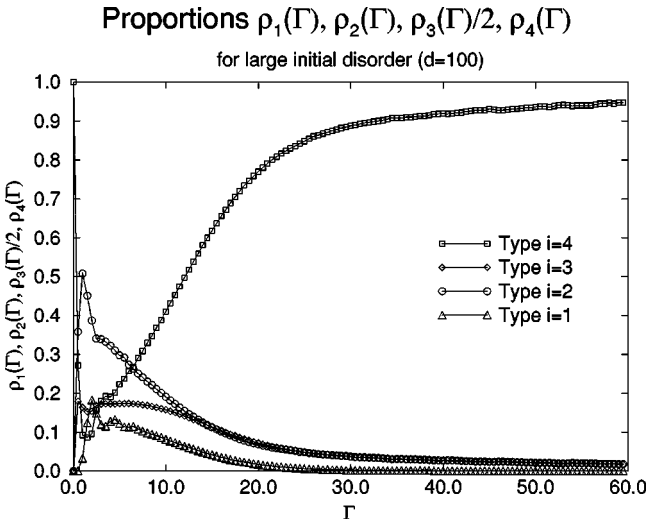



FIG. 5. The proportions $\rho_i(\Gamma)$ of the four types $i=1$, 2, 3, and 4 of bonds at scale $\Gamma$, for strong initial disorder $d=100$: they reach the asymptotic regime (4.3).

where the parameter $1/\alpha_4(\Gamma)$ follows the random-singlet behavior (see Fig. 7)

$$\frac{1}{\alpha_4(\Gamma)} \simeq \Gamma + \text{Cst.} \tag{4.5}$$

As a consequence, if a defect is produced at the renormalization energy scale $\Omega$, it survives until the energy scale $\Omega/3$, where it is decimated according to rule (2), and the whole defect of structure (bonds of type 3, bonds of type 2, bonds of type 3) entirely disappears to give one bond of type 4. Figure 8 indeed shows clearly that the probability distribution $\mathcal{P}_2(x,\Gamma)$ tends to concentrate on the interval $0<x<\ln 3$ as $\Gamma$ increases. That has to be contrasted with bonds

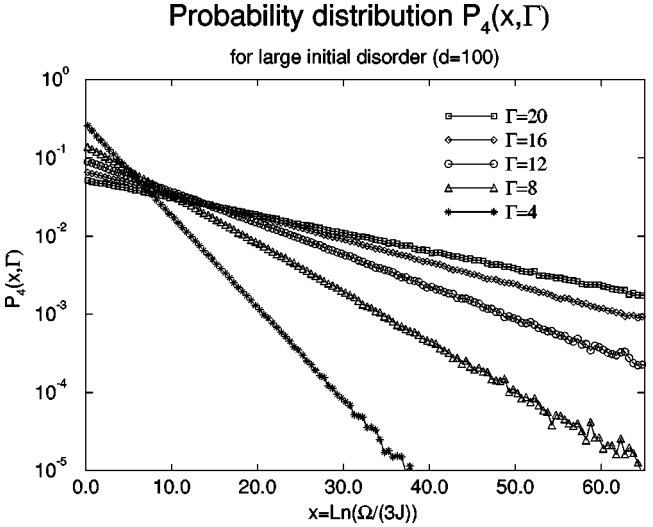



FIG. 6. Linear-log plot of the probability distribution $\mathcal{P}_4(x,\Gamma)$ for $\Gamma=4$, 8, 12, 16, and 20, for strong initial disorder $d=100$: $\mathcal{P}_4(x,\Gamma)$ is well described by the exponential form (4.4) with a parameter $\alpha_4(\Gamma)$ plotted in Fig. 7.

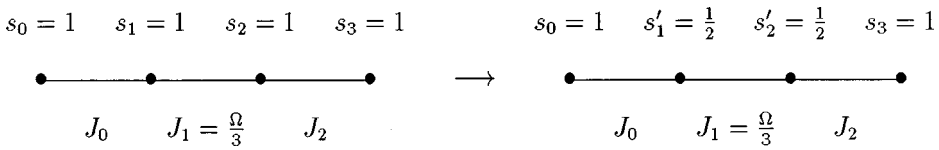
of type 3, which are characterized by a distribution $\mathcal{P}_3(x,\Gamma)$ that tends to coincide with $\mathcal{P}_4(x,\Gamma)$ for large enough $\Gamma$.

### B. Approximate flow equations phase

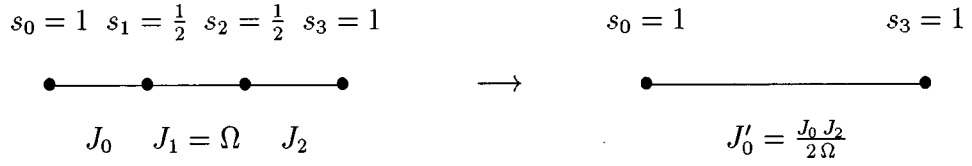
Assuming that there is a sea of bonds of type 4, with sometimes defects of structure (bond of type 3, bond of type 2, bond of type 3), it is possible to write approximate flow equations for the probability distributions of the couplings normalized according to

$$1=\int_0^{\Omega} dJ\ P_2(J,\Omega)=\int_0^{2\Omega/3} dJ\ P_3(J,\Omega)$$
$$=\int_0^{\Omega/3} dJ\ P_4(J,\Omega). \tag{4.6}$$

Assuming that the only two important renormalization processes are the production of the defect structure (bond of type 3, bond of type 2, bond of type 3) by the renormalization rule (4) for a bond of type 4 when its two neighbor bonds are also of type 4,




and the suppression of the defect structure by the decimation rule (2),




we obtain the following approximate flow equations for the three probability distributions:

$$-\frac{\partial P_2(J,\Omega)}{\partial\Omega}=P_2(\Omega,\Omega)P_2(J,\Omega)+\frac{1}{3}P_4\left(\frac{\Omega}{3},\Omega\right)\frac{N_4(\Omega)}{N_2(\Omega)}\left[\delta\left(J-\frac{\Omega}{3}\right)-P_2(J,\Omega)\right], \tag{4.7}$$

$$-\frac{\partial P_3(J,\Omega)}{\partial\Omega}=\frac{2}{3}P_4\left(\frac{\Omega}{3},\Omega\right)\frac{N_4(\Omega)}{N_3(\Omega)}\ [P_4(J,\Omega)-P_3(J,\Omega)], \tag{4.8}$$

$$-\frac{\partial P_4(J,\Omega)}{\partial\Omega}=\frac{1}{3}P_4\left(\frac{\Omega}{3},\Omega\right)\ P_4(J,\Omega)-\frac{N_2(\Omega)}{N_4(\Omega)}P_2(\Omega,\Omega)P_4(J,\Omega)+\frac{N_2(\Omega)}{N_4(\Omega)}P_2(\Omega,\Omega)\int_0^{2\Omega/3}dJ_0P_3(J_0,\Omega)\times\int_0^{2\Omega/3}dJ_2\ \ P_3(J_2,\Omega)\,\delta\left(J-\frac{J_0J_2}{2\Omega}\right), \tag{4.9}$$

together with the flow equations for the number $N_i(\Omega)$ of bonds of type $i=2$, 3, and 4:

$$-\frac{dN_2}{d\Omega}=-\frac{1}{2}\frac{dN_3}{d\Omega}=\frac{1}{3}P_4\left(\frac{\Omega}{3},\Omega\right)\ N_4(\Omega)-P_2(\Omega,\Omega)\ N_2(\Omega), \tag{4.10}$$

$$-\frac{dN_4}{d\Omega}=P_2(\Omega,\Omega)N_2(\Omega)-P_4\left(\frac{\Omega}{3},\Omega\right)\ N_4(\Omega), \tag{4.11}$$

so that the total number $N(\Omega)=N_2(\Omega)+N_3(\Omega)+N_4(\Omega)$ of bonds evolves according to

$$-\frac{dN}{d\Omega}=-2P_2(\Omega,\Omega)N_2(\Omega). \tag{4.12}$$

It is more convenient to write the flow equations for the probability distributions $\mathcal{P}_i(x,\Gamma)$ of the reduced variable $x=\ln[\Omega/\Delta(J)]$, where $\Delta(J)$ is defined by Eq. (2.42), so that the random variable $x$ varies in $(0,\infty)$ for any type of bonds

$$\frac{\partial\mathcal{P}_2(x,\Gamma)}{\partial\Gamma}=\frac{\partial\mathcal{P}_2(x,\Gamma)}{\partial x}+\mathcal{P}_2(0,\Gamma)\mathcal{P}_2(x,\Gamma)+\frac{N_4(\Gamma)}{N_2(\Gamma)}\ \mathcal{P}_4(0,\Gamma)[\,\delta(x-\ln 3)-\mathcal{P}_2(x,\Gamma)], \tag{4.13}$$

$$\frac{\partial \mathcal{P}_3(x,\Gamma)}{\partial \Gamma} = \frac{\partial \mathcal{P}_3(x,\Gamma)}{\partial x} + 2\frac{N_4(\Gamma)}{N_2(\Gamma)}\ \mathcal{P}_4(0,\Gamma) \times [\mathcal{P}_4(x-\ln 2,\Gamma) - \mathcal{P}_3(x,\Gamma)], \quad (4.14)$$

$$\frac{\partial \mathcal{P}_4(x,\Gamma)}{\partial \Gamma} = \frac{\partial \mathcal{P}_4(x,\Gamma)}{\partial x} + \left[\mathcal{P}_4(0,\Gamma) - \frac{N_2(\Gamma)}{N_4(\Gamma)}\ \mathcal{P}_2(0,\Gamma)\right]\mathcal{P}_4(x,\Gamma) + \frac{N_2(\Gamma)}{N_4(\Gamma)}\mathcal{P}_2(0,\Gamma)\int_0^\infty dx_1\ \mathcal{P}_3(x_1,\Gamma) \times \int_0^\infty dx_2\ \mathcal{P}_3(x_2,\Gamma)\,\delta\left(x - x_1 - x_2 - \ln\frac{3}{2}\right). \quad (4.15)$$

Since the singular term containing the $\delta$ function in Eq. (4.13) tends to develop a discontinuity in $\mathcal{P}_2(x,\Gamma)$ at $x = \ln 3$, it is convenient to set

$$\mathcal{P}_2(x,\Gamma) = [1-e(\Gamma)]\frac{\theta(\ln 3 - x)}{\ln 3} + e(\Gamma) f_2(x,\Gamma), \quad (4.16)$$

where $0 < e(\Gamma) < 1$, and $f_2(x,\Gamma)$ is a normalized probability distribution that is regular at $x = \ln 3$. Equation (4.13) will be satisfied if $e(\Gamma)$ and $f(x,\Gamma)$ satisfy

$$e(\Gamma) = 1 - (\ln 3)\frac{N_4(\Gamma)}{N_2(\Gamma)}\mathcal{P}_4(0,\Gamma), \quad (4.17)$$

$$\frac{de(\Gamma)}{d\Gamma} = -e(\Gamma)[1-e(\Gamma)]f(0,\Gamma), \quad (4.18)$$

$$\frac{\partial f(x,\Gamma)}{\partial \Gamma} = \frac{\partial f(x,\Gamma)}{\partial x} + f(0,\Gamma) f(x,\Gamma). \quad (4.19)$$

Obvious stationary solutions for $f(x,\Gamma)$ are simple exponentials

$$f(x,\Gamma) \underset{\Gamma\to\infty}{\simeq} \alpha_f\ e^{-\alpha_f x}, \quad (4.20)$$

in which case $e(\Gamma)$ vanishes exponentially,

$$e(\Gamma) \underset{\Gamma\to\infty}{\propto} e^{-\alpha_f \Gamma}, \quad (4.21)$$

so that $\mathcal{P}_2(x,\Gamma)$ converges towards the stationary solution

$$\mathcal{P}_2^*(x) = \frac{1}{\ln 3}\,\theta(\ln 3 - x). \quad (4.22)$$

This corresponds in the original variables to

$$P_2(J,\Omega) = \frac{1}{(\ln 3)J} \quad \text{for } \frac{\Omega}{3} < J < \Omega. \quad (4.23)$$

We also obtain the following equation in the asymptotic regime:

$$N_4(\Gamma)\ \mathcal{P}_4(0,\Gamma) \simeq N_2(\Gamma)\ \frac{1}{\ln 3}, \quad (4.24)$$

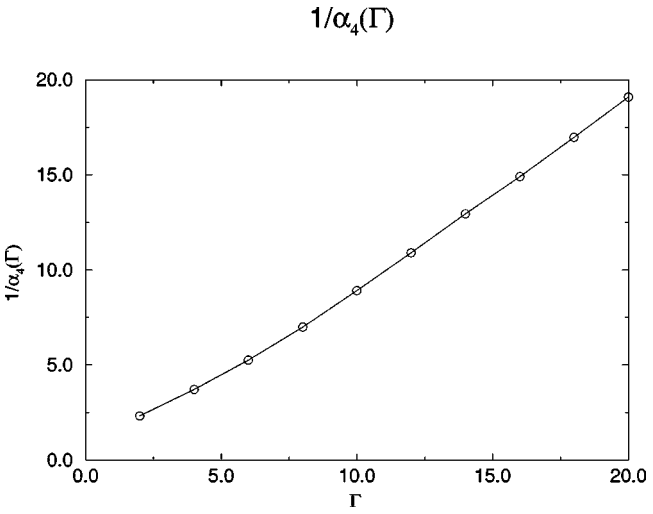



FIG. 7. Plot of the inverse of the parameter $\alpha_4(\Gamma)$ defined in Eq. (4.4) as a function of $\Gamma$: it follows the random-singlet behavior (4.5).

that we will use now to study the flow equations for $\mathcal{P}_3(x,\Gamma)$ and $\mathcal{P}_4(x,\Gamma)$

With the last change of variables

$$x \to z = \alpha_4(\Gamma)x, \quad (4.25)$$

we find that the flow equation for the corresponding probability distributions $\tilde{P}_4(z,\Gamma)$ and $\tilde{P}_3(z,\Gamma)$ admit the stationary solutions

$$\tilde{P}_4(z,\Gamma) \underset{\Gamma\to\infty}{\to} e^{-z} \quad \text{and} \quad \tilde{P}_3(z,\Gamma) \underset{\Gamma\to\infty}{\to} e^{-z}, \quad (4.26)$$

where

$$\alpha_4(\Gamma) \underset{\Gamma\to\infty}{\propto} \frac{1}{\Gamma}, \quad (4.27)$$

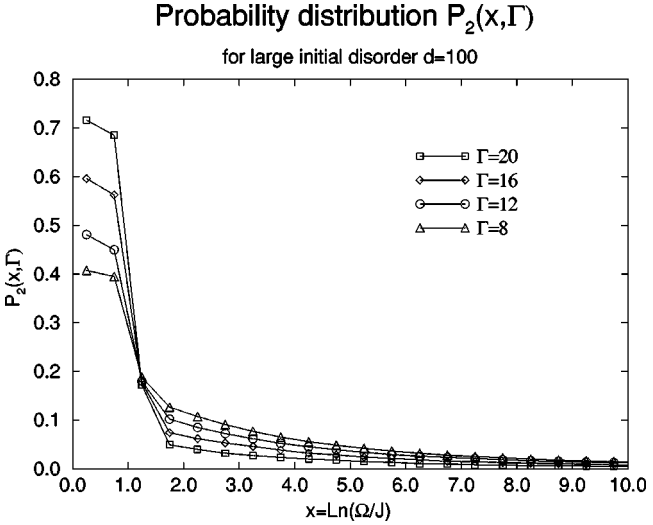



FIG. 8. Plot of the probability distribution $\mathcal{P}_2(x,\Gamma)$ for $\Gamma = 8$, 12, 16, and 20, for a strong initial disorder $d = 100$: $\mathcal{P}_2(x,\Gamma)$ tends to concentrate on the interval $0 < x < \ln 3$, as explained in the text.

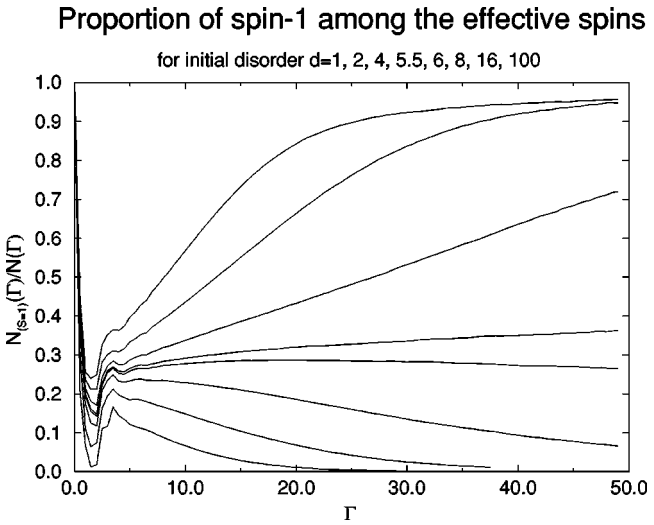



FIG. 9. Proportion of spins $S=1$ among the effective spins at scale $\Gamma$ for various values of the initial disorder $d=1$, 2, 3, 4, 5.5, 6, 8, 16, and 100: this proportion flows toward 0 in the weak-disorder phase, and toward 1 in the strong-disorder phase. Between these two attractive values, there is an unstable fixed point at $d_c \simeq 5.75(5)$, where the proportion of spins $S=1$ among the effective spins remains stationary at the intermediate value 0.315(5).

as in the random-singlet solution of Ma-Dasgupta. It is then easy to obtain the asymptotic behavior of the total number $N(\Gamma)$ of spins (4.12),

$$N(\Gamma) \underset{\Gamma\to\infty}{\propto} \frac{1}{\Gamma^2}, \tag{4.28}$$

and the asymptotic behavior of the proportion $\epsilon(\Gamma)$ [Eq. (4.3)] of defects from Eq. (4.26):

$$\epsilon(\Gamma) = \frac{N_2(\Gamma)}{N(\Gamma)} \underset{\Gamma\to\infty}{\propto} \frac{\ln 3}{\Gamma}. \tag{4.29}$$

## V. CRITICAL REGIME

In Fig. 9, we plot the proportion $[N_{(S=1)}(\Gamma)]/N(\Gamma)$ of spins $S=1$ among the effective spins for various values of the disorder; this proportion flows toward 0 in the weak-disorder phase and toward 1 in the strong-disorder phase. Between these two attractive values, there is an unstable fixed point at $d_c \simeq 5.75(5)$, where the proportion of spins $S=1$ among the effective spins remains stationary at the intermediate value 0.315(5). The proportions $\rho_i(\Gamma)$ of the four types of bonds reach a stationary state characterized by (see Fig. 10)

$$\rho_1(\Gamma) \sim 0.17, \quad \rho_2(\Gamma) \sim 0.35, \quad \rho_3(\Gamma) \sim 0.33,$$

$$\rho_4(\Gamma) \sim 0.15. \tag{5.1}$$

We find, of course, that the four probability distributions $\mathcal{P}_i(x,\Gamma)$ for $i=1$, 2, 3, and 4 coincide up to statistical fluctuations [otherwise, the proportions $\rho_i(\Gamma)$ would not remain stationary] and follow the exponential form (see Fig. 11)

$$\mathcal{P}_i(x,\Gamma) \simeq \alpha_c(\Gamma) e^{-\alpha_c(\Gamma) x}, \tag{5.2}$$

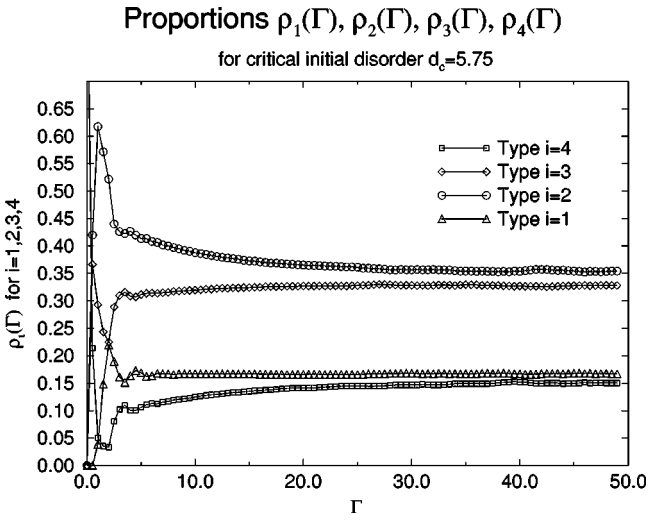



FIG. 10. The proportions $\rho_i(\Gamma)$ of the four types $i=1$, 2, 3, and 4 of bonds at scale $\Gamma$, for the critical initial disorder $d_c=5.75$ [Eq. (5.1)].

where the parameter $1/\alpha_c(\Gamma)$ (see Fig. 12) follows the behavior of the effective model of Hyman and Yang:[9]

$$\frac{1}{\alpha_c(\Gamma)} \simeq \frac{\Gamma}{2} + \text{Cst.} \tag{5.3}$$

The magnetic susceptibility is given by the effective number of free spins

$$\chi \propto \frac{1}{T \ln^3 T}. \tag{5.4}$$

## VI. CONCLUSION

We have introduced a real-space renormalization scheme that allows a study of the spin-1 chain. Within this scheme we obtained a complete characterization of the weak-coupling phase, the critical regime, and the strong-disorder phase. In all phases we were able to follow the spin populations and to obtain the probability distributions of the different types of bonds that appear under renormalization. It is only in the weak- and strong-coupling limits that we were able to obtain approximate analytical flow equations.

The renormalization scheme that we used is an extension of the Ma-Dasgupta idea. These schemes have in common the fact that they are consistent for arbitrarily weak initial disorder. They do not create bonds stronger than the original decimated bond. In the spin-$\frac{1}{2}$ case, it is believed that this means that there is no critical disorder. In fact, this is suggested by bosonization: most bosonic forms of randomness give rise to relevant operators along the massless line of the pure system when the anisotropy is varied. The simplest assumption[2] is thus that the system flows immediately to the random-singlet phase (there is a region of stability of the spin liquid but this happens only for attractive enough interactions between the Jordan-Wigner fermions, i.e., for negative enough anisotropy).

However, this is not the case for the spin-1 chain. Here the Haldane gap is perturbatively insensitive to disorder, as naively expected. This is known from bosonization studies of

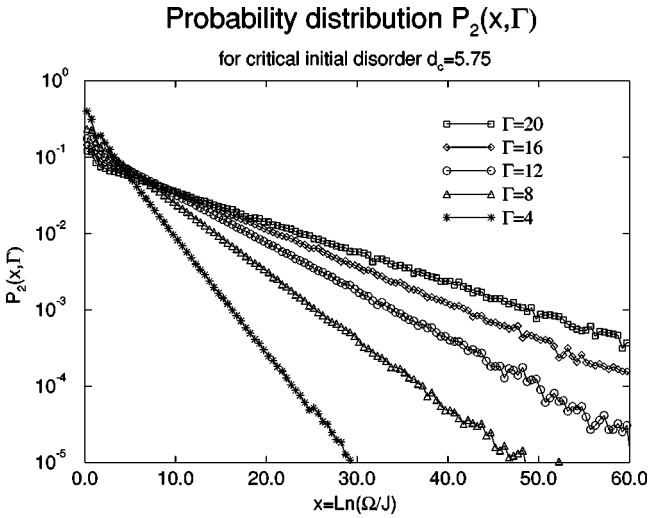



FIG. 11. Linear-log plot of the probability distribution $\mathcal{P}_2(x,\Gamma)$ for $\Gamma=4$, 8, 12, 16, and 20, for critical initial disorder $d_c=5.75$: $\mathcal{P}_2(x,\Gamma)$ takes the exponential form (5.2) with a parameter $\alpha_c(\Gamma)$ plotted in Fig. 12.

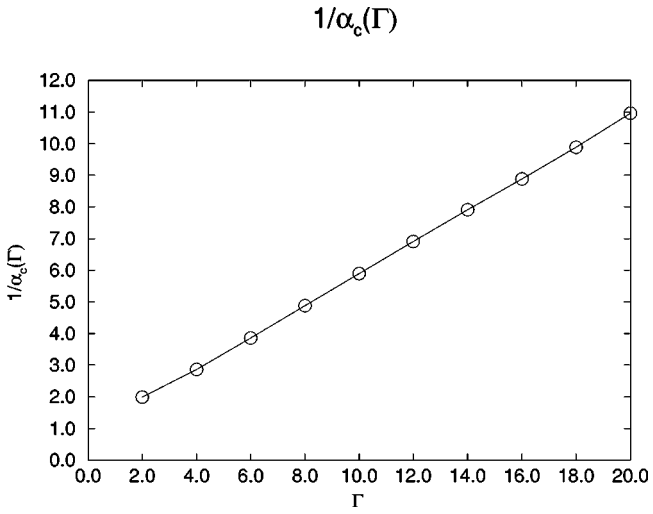



FIG. 12. Plot of the inverse of the parameter $\alpha_c(\Gamma)$ defined in Eq. (5.2) as a function of $\Gamma$: it follows the behavior (5.3).

the spin-$\frac{1}{2}$ two-leg ladder[12] as well as of the anisotropic spin-1 chain.[13] So we may be in a situation with a first critical disorder strength corresponding to the vanishing of Haldane gap, but which is unreachable by the real-space scheme. With increasing disorder there is then the second critical disorder strength for which the string order vanishes. This second transition is described by our renormalization scheme which is then asymptotically exact. Conversely, the bosonization methods are unable to follow the flow to strong coupling, and thus are unable to describe even the weak-disorder phase captured by the real-space scheme. It may be also that there is nothing like a critical value of the disorder for the vanishing of the Haldane gap, if for example there are states of arbitrarily small energies in the gap, as in the case of the Lifshitz tails in the localization problem. It remains to be seen if there is a single theoretical approach that is able to deal with all known limiting cases.

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