Few-anyon systems in a parabolic dot

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The energy levels of two and three anyons in a two-dimensional parabolic quantum dot and a perpendicular magnetic field are computed as power series in $1/|J|$, where J is the angular momentum. The particles interact repulsively through a Coulombic $(1/r)$ potential. In the two-anyon problem, the reached accuracy is better than one part in 10^5. For three anyons, we study the combined effects of anyon statistics and Coulomb repulsion in the ''linear'' anyonic states. [S0163-1829(98)04628-1]

I. INTRODUCTION

Recent developments in semiconductor technology (e.g., molecular-beam epitaxy and electron lithography) opened the possibility to create totally confined electron systems, the so-called artificial atoms or quantum dots.[1,2] This is one of the examples of present-day-physics' interest in confined finite systems, among which one can mention also the atomic and electronic traps,[3] and the condensation of confined bosons.[4]

Quantum dots exhibit very interesting properties like the possibility of varying their parameters (number of electrons, applied fields, dot's geometry, temperature) over a wide range, the observation of conductance oscillations,[5] and Kohn's theorem,[6] etc.

Theoretically, these systems have been studied mainly with the help of numerical methods. However, analytic approaches have proven to work extremely well providing, at the same time, a qualitative understanding of the quantum dynamics. Among these approaches one can mention the semiclassical quantization,[7] regularized perturbation theory,[8,9] Padé approximant techniques,[10,11] and the $1/N$ expansion.[12–14]

In the present paper, we continue the analytic-qualitative line of research and apply the $1/N$ expansion (N is the absolute value of the angular momentum) to compute the energy levels of two and three anyons in a model parabolic dot. The particles interact through a Coulombic $(1/r)$ repulsive potential. A magnetic field is applied perpendicularly to the plane of motion.

Numerical results for the two-anyon system were obtained by Myrheim *et al.*[15] Exact analytic solutions at particular values of the coupling constants were found in Ref. 16. Bohr-Sommerfeld quantization was applied to this system at low magnetic fields.[17] We shall see that our method provides extremely accurate solutions for states with angular momentum $|J|\geqslant 2$. A picture for the ''geometry'' of the states (the spatial distribution of probability) is obtained also. In the three-anyon problem, however, to our knowledge, there are no numerical or approximate calculations.

We start from the Hamiltonian of N_a anyons moving in a two-dimensional quantum dot in the presence of a perpendicular homogenous magnetic field. In the bosonic gauge, it is given by the expression[18]

$$H=\frac{1}{2m}\sum_{i=1}^{N_a}\left|\mathbf{p}_i+\frac{e}{c}\,\mathbf{A}_i-\hbar\,\nu\mathbf{a}_i\right|^2+\frac{m}{2}\,\omega_0^2\sum_{i=1}^{N_a}r_i^2+\sum_{i>j}\frac{e^2}{\kappa|\mathbf{r}_i-\mathbf{r}_j|},\tag{1}$$

in which the vector potential is taken in the symmetric gauge,

$$\mathbf{A}_i=\frac{1}{2}\,\mathbf{B}\times\mathbf{r}_i\,,\tag{2}$$

$\mathbf{a}_i$ is the statistical vector potential,

$$\mathbf{a}_i=\sum_{j\neq i}\frac{\mathbf{n}\times(\mathbf{r}_j-\mathbf{r}_i)}{|\mathbf{r}_j-\mathbf{r}_i|^2},\tag{3}$$

$\mathbf{n}$ is the unit vector perpendicular to the plane of motion of the anyons, e is the anyon charge, ν is the anyonic parameter, ω_0 is the frequency of the parabolic potential needed to confine the anyons in the dot, and κ is the dielectric constant of the medium. A dimensionless Hamiltonian is obtained by means of the change of variables $\mathbf{r}_i\rightarrow\sqrt{\hbar/(m\Omega)}\mathbf{r}_i$,

$$\begin{aligned}\frac{H}{\hbar\Omega}=&\frac{1}{2}\sum_{i=1}^{N_a}p_i^2+\frac{\omega_c}{2\Omega}\,\mathbf{n}\cdot\sum_{i=1}^{N_a}\mathbf{r}_i\times\mathbf{p}_i\\&+\nu\mathbf{n}\cdot\sum_{i>j}\frac{(\mathbf{r}_i-\mathbf{r}_j)\times(\mathbf{p}_i-\mathbf{p}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2}\\&+\frac{1}{2}\sum_{i=1}^{N_a}r_i^2+\frac{\omega_c}{4\Omega}\,\nu N_a(N_a-1)\\&+\frac{\nu^2}{2}\sum_{i\neq j,k}\frac{(\mathbf{r}_i-\mathbf{r}_j)\cdot(\mathbf{r}_i-\mathbf{r}_k)}{|\mathbf{r}_i-\mathbf{r}_j|^2|\mathbf{r}_i-\mathbf{r}_k|^2}+\beta^3\sum_{i>j}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|},\end{aligned}\tag{4}$$

where $\omega_c = eB/(mc)$ is the cyclotronic frequency, $\Omega^2 = \omega_c^2/4 + \omega_0^2$, and $\beta^3 = \sqrt{me^4/(\kappa^2\Omega\hbar^3)}$ is the square root of the ratio between the Coulombic and oscillator characteristic energies. The problem has one exactly solvable limit: a low-density limit, which we call the Wigner limit, reached when $\beta\to\infty$. In the $\beta\to 0$ (oscillator) limit, the two-anyon problem is exactly solvable,[19] whereas the three-anyon system has an infinite family of exact linear states.[20] In real semiconductors, $\beta\sim 1$.

Introducing Jacobi coordinates,

$$\boldsymbol{\rho}_k = \sqrt{\frac{2k}{k+1}}\left\{\frac{1}{k}\sum_{i=1}^{k}\mathbf{r}_i - \mathbf{r}_{k+1}\right\}, \quad 1\leqslant k\leqslant N_a-1, \tag{5}$$

$$\boldsymbol{\rho}_{N_a} = \frac{1}{\sqrt{N_a}}\sum_{i=1}^{N_a}\mathbf{r}_i, \tag{6}$$

the center-of-mass and relative motions are separated

$$\frac{H}{\hbar\Omega} = \frac{H_{CM}}{\hbar\Omega} + \frac{H_{rel}}{\hbar\Omega}, \tag{7}$$

where

$$\frac{H_{CM}}{\hbar\Omega} = \frac{1}{2}p_{N_a}^2 + \frac{\omega_c}{2\Omega}\,\mathbf{n}\cdot(\boldsymbol{\rho}_{N_a}\times\mathbf{p}_{N_a}) + \frac{1}{2}\rho_{N_a}^2, \tag{8}$$

is the center-of-mass Hamiltonian and

$$\begin{aligned}\frac{H_{rel}}{\hbar\Omega} &= \sum_{i=1}^{N_a-1} p_i^2 + \frac{\omega_c}{2\Omega}\,\mathbf{n}\cdot\sum_{i=1}^{N_a-1}\boldsymbol{\rho}_i\times\mathbf{p}_i \\ &\quad + \nu\mathbf{n}\cdot\sum_{i>j}\frac{\mathbf{r}_{ij}\times\mathbf{p}_{ij}}{r_{ij}^2} + \frac{1}{4}\sum_{i=1}^{N_a-1}\rho_i^2 + \frac{\omega_c}{4\Omega}\nu N_a(N_a-1) \\ &\quad + \frac{\nu^2}{2}\sum_{i\neq j,k}\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2} + \beta^3\sum_{i>j}\frac{1}{r_{ij}}\end{aligned} \tag{9}$$

is the Hamiltonian of the relative motion. We introduced the following notation: $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j$.

We will obtain approximate expressions for the energy eigenvalues of $H_{rel}/(\hbar\Omega)$ for $N_a = 2$ and $N_a = 3$ as a function of β by means of the $1/|J|$ expansion.

II. THE TWO-ANYON SYSTEM

In the two-anyon problem, we have only one Jacobi coordinate, $\boldsymbol{\rho}_1$, and the Hamiltonian of the internal motion reads

$$\begin{aligned}\frac{H_{rel}}{\hbar\Omega} &= p_1^2 + \frac{\omega_c}{2\Omega}\,\mathbf{n}\cdot(\boldsymbol{\rho}_1\times\mathbf{p}_1) + 2\nu\mathbf{n}\cdot\frac{\boldsymbol{\rho}_1\times\mathbf{p}_1}{\rho_1^2} + \frac{1}{4}\rho_1^2 \\ &\quad + \frac{\nu^2}{\rho_1^2} + \frac{\beta^3}{\rho_1} + \frac{\omega_c\nu}{2\Omega}.\end{aligned} \tag{10}$$

Notice that $\mathbf{n}\cdot(\boldsymbol{\rho}_1\times\mathbf{p}_1) = J$. After the scaling transformation $\rho_1^2\to|J|\rho_1^2$, we get

$$\begin{aligned}h &= \frac{1}{|J|}\left[\frac{H_{rel}}{\hbar\Omega} - \frac{\omega_c(J+\nu)}{2\Omega}\right] \\ &= \frac{(\tilde{\nu}+1)^2}{\rho_1^2} + \frac{1}{4}\rho_1^2 + \frac{\tilde{\beta}^3}{\rho_1} - \frac{1}{J^2}\left(\frac{\partial^2}{\partial\rho_1^2} + \frac{1}{\rho_1}\frac{\partial}{\partial\rho_1}\right),\end{aligned} \tag{11}$$

where $\rho_1 = |\boldsymbol{\rho}_1|$. We have "renormalized" the coupling constant, $\tilde{\beta}^3 = \beta^3/|J|^{3/2}$, and the statistical parameter, $\tilde{\nu} = \nu/J$, in order to take account of the Coulomb repulsion and the statistical interaction in a nonperturbative way when taking the formal limit $|J|\to\infty$. We shall look for symmetric eigenfunctions of h, i.e., $|J|$ shall be even.

In the $|J|\to\infty$ limit, the only term surviving in the Hamiltonian is the effective (classical) potential energy

$$U_{eff} = \frac{(\tilde{\nu}+1)^2}{\rho_1^2} + \frac{1}{4}\rho_1^2 + \frac{\tilde{\beta}^3}{\rho_1}. \tag{12}$$

Minimizing U_{eff}, we obtain the leading contribution to the energy, $\epsilon_0 = U_{eff}(\rho_{01})$, where the radius of the "Bohr orbit" is obtained from

$$\frac{1}{2}\rho_{01}^4 - \tilde{\beta}^3\rho_{01} = 2(\tilde{\nu}+1)^2. \tag{13}$$

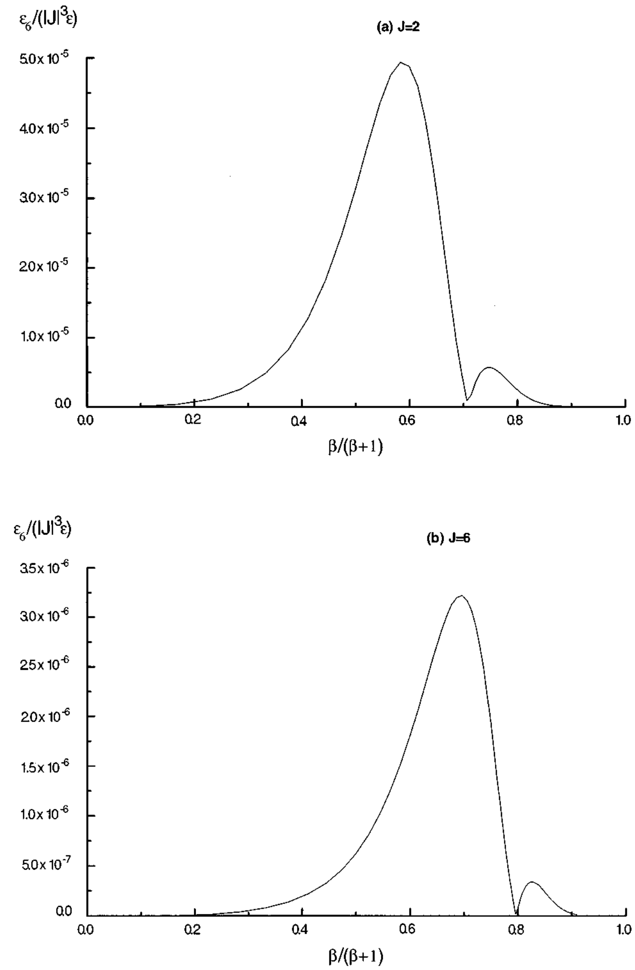

FIG. 1. Relative weight of ϵ_6 in ϵ. Two anyons in states with $n=0$ and $\nu=1/2$ are studied. (a) $J=2$, (b) $J=6$.

Substituting $\rho_1=\rho_{01}+y_1/|J|^{1/2}$ in the right-hand side (rhs) of Eq. (11) and expanding, we get

$$h=\sum_{i=0}^{\infty}\frac{h_i}{|J|^{i/2}}, \tag{14}$$

where the operator coefficients are given by

$$h_0=\frac{3}{4}\rho_{01}^2-\frac{(\tilde{\nu}+1)^2}{\rho_{01}^2}, \tag{15}$$

$$h_1=0, \tag{16}$$

$$h_2=-\frac{\partial^2}{\partial y_1^2}+\frac{1}{4}\left(3+\frac{4(\tilde{\nu}+1)^2}{\rho_{01}^4}\right)y_1^2, \tag{17}$$

$$h_i=(-1)^i\left\{\left(\frac{1}{2\rho_{01}^{i-2}}+\frac{(i-1)(\tilde{\nu}+1)^2}{\rho_{01}^{i+2}}\right)y_1^i+\frac{1}{\rho_{01}^{i-2}}y_1^{i-3}\frac{\partial}{\partial y_1}\right\},\quad i\geqslant 3. \tag{18}$$

Similar series are written for the wave function and the scaled energy, that is,

$$\psi=\sum_{i=0}^{\infty}\frac{\psi_i}{|J|^{i/2}}, \tag{19}$$

$$\epsilon=\frac{1}{|J|}\left[\frac{E_{rel}}{\hbar\Omega}-\frac{\omega_c(J+\nu)}{2\Omega}\right]=\sum_{i=0}^{\infty}\frac{\epsilon_i}{|J|^{i/2}}. \tag{20}$$

Inserting Eq. (19), Eq. (20), and the series expansion for the Hamiltonian into the Schödinger equation, we may compute the coefficients ψ_i and ϵ_i in a systematic way. Up to second order, for example, the system is described in terms of small oscillations around the equilibrium orbit, i.e., the wave function is

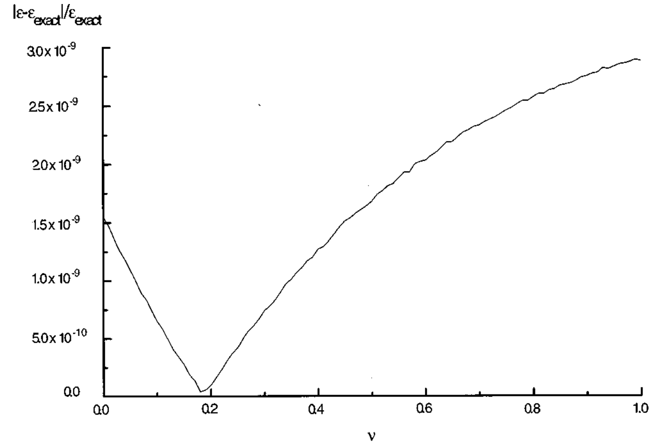

FIG. 2. Comparison between the $1/|J|$ estimate and the exact solution found in Ref. 16 for two anyons with $J=6$, $n=0$.

$$\Psi_0=e^{iJ\theta}|n\rangle, \tag{21}$$

where θ is the angle associated to the vector $\boldsymbol{\rho}_1$, the $|n\rangle$ are two-dimensional harmonic oscillator radial states with frequency $\omega_1=\sqrt{3+4(\tilde{\nu}+1)^2/\rho_{01}^4}$, and the first two coefficients for the energy are

$$\epsilon_0=\frac{3}{4}\rho_{01}^2-\frac{(\tilde{\nu}+1)^2}{\rho_{01}^2}, \tag{22}$$

$$\epsilon_2=\omega_1\left(n+\frac{1}{2}\right). \tag{23}$$

Afterward, we may take account of anharmonicities. The results for the next two coefficients are the following:

$$\epsilon_4=-\frac{1}{4\rho_{01}^2}+\frac{3(2n^2+2n+1)}{2\omega_1^2\rho_{01}^2}\left(1+\frac{6(\tilde{\nu}+1)^2}{\rho_{01}^4}\right)-\frac{(30n^2+30n+11)}{4\omega_1^4\rho_{01}^2}\left(1+\frac{4(\tilde{\nu}+1)^2}{\rho_{01}^4}\right)^2, \tag{24}$$

$$\begin{aligned}\epsilon_6=&-\frac{3(2n+1)}{4\omega_1\rho_{01}^4}+\frac{(2n+1)}{\omega_1^3\rho_{01}^4}\left[5n^2+5n+9+\frac{(\tilde{\nu}+1)^2(50n^2+50n+81)}{\rho_{01}^4}\right]-\frac{(2n+1)}{2\omega_1^5\rho_{01}^4}\left[87n^2\right.\\&\left.+87n+86+\frac{12(\tilde{\nu}+1)^2(87n^2+87n+86)}{\rho_{01}^4}+\frac{4(\tilde{\nu}+1)^4(713n^2+713n+709)}{\rho_{01}^8}\right]+\frac{9(2n+1)}{2\omega_1^7\rho_{01}^4}\\&\times(25n^2+25n+19)\left(1+\frac{6(\tilde{\nu}+1)^2}{\rho_{01}^4}\right)\left(1+\frac{4(\tilde{\nu}+1)^2}{\rho_{01}^4}\right)^2-\frac{15(2n+1)}{8\omega_1^9\rho_{01}^4}(47n^2+47n+31)\left(1+\frac{4(\tilde{\nu}+1)^2}{\rho_{01}^4}\right)^4.\end{aligned} \tag{25}$$

Notice that in both Wigner ($\beta\to\infty$) and oscillator ($\beta\to 0$) limits the corrections ϵ_4 and ϵ_6 go to zero. The expressions found in Ref. 14 are reproduced if we take $\nu=0$.

We show in Fig. 1 the relative weight of ϵ_6 in ϵ for the first states with $J=2$ and 6. The parameter ν was fixed to 1/2 (semions). The relative contribution of ϵ_6 is never greater than 5×10^{-5} or 3×10^{-6} for $J=2$ or 6, respectively. This shows that the $1/|J|$ series is extremely well behaved.

A comparison with the exact solutions found in Ref. 16 is carried on in Fig. 2, where the relative difference $|\epsilon-\epsilon_{exact}|/\epsilon_{exact}$ is plotted against ν. The state with $J=6$, $n=0$ is shown. It may be easily verified that $\psi_{exact}=\rho_1^{|J+\nu|}(1+\rho_1/\sqrt{2|J+\nu|+1})e^{-\rho_1^2}$, $\epsilon_{exact}=|J+\nu|+2$, are exact solutions of the two-anyon problem at $\beta^3=\sqrt{2|J+\nu|+1}$. The comparison shows that the relative error of our estimate is not greater than 10^{-8}.

III. THE THREE-ANYON SYSTEM

The internal Hamiltonian of the system of three anyons in Jacobi coordinates $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ is written as

$$\begin{aligned}\frac{H_{rel}}{\hbar\Omega}=&\sum_{i=1}^{2}p_i^2+\frac{\omega_c}{2\Omega}\,\mathbf{n}\cdot\sum_{i=1}^{2}\boldsymbol{\rho}_i\times\mathbf{p}_i+\frac{3\omega_c\nu}{2\Omega}\\&+\nu\mathbf{n}\cdot\left[2\frac{\boldsymbol{\rho}_1\times\mathbf{p}_1}{\rho_1^2}+2\frac{(\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2)\times(\mathbf{p}_1+\sqrt{3}\mathbf{p}_2)}{|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|^2}+2\frac{(\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2)\times(\mathbf{p}_1-\sqrt{3}\mathbf{p}_2)}{|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|^2}\right]\\&+\frac{1}{4}\sum_{i=1}^{2}\rho_i^2+9\nu^2\frac{(\rho_1^2+\rho_2^2)^2+4[\rho_1^2\rho_2^2-(\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)^2]}{\rho_1^2|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|^2|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|^2}+\beta^3\left[\frac{1}{\rho_1}+\frac{2}{|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|}+\frac{2}{|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|}\right].\end{aligned}\tag{26}$$

Doing the same scaling transformation $\rho_i^2\to|J|\rho_i^2$, and making explicit the dependence on $|J|$, we get

$$\begin{aligned}\frac{1}{|J|}\left[\frac{H_{rel}}{\hbar\Omega}-\frac{\omega_c(J+3\nu)}{2\Omega}\right]=&\frac{1}{4}\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}\right)+\frac{1}{4}(\rho_1^2+\rho_2^2)+\tilde{\beta}^3\left[\frac{1}{\rho_1}+\frac{2}{|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|}+\frac{2}{|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|}\right]\\&+4\tilde{\nu}\frac{\rho_1^2(2-3\cos^2\theta)+3\rho_2^2(2-\cos^2\theta)}{|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|^2|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|^2}+\frac{\tilde{\nu}}{\rho_1^2}\\&+9\tilde{\nu}^2\frac{(\rho_1^2+\rho_2^2)^2+4\rho_1^2\rho_2^2\sin^2\theta}{\rho_1^2|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|^2|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|^2}+\frac{1}{J}\left[\imath\left(\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right)\frac{\partial}{\partial\theta}\right.\\&+\tilde{\nu}\left\{12\imath\frac{\rho_1\rho_2\sin 2\theta}{|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|^2|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|^2}\left(\rho_1\frac{\partial}{\partial\rho_2}-\rho_2\frac{\partial}{\partial\rho_1}\right)\right.\\&\left.\left.+\imath\frac{2}{\rho_1^2}\frac{\partial}{\partial\theta}-8\imath\frac{\rho_1^2(1-3\cos^2\theta)+3\rho_2^2(1+\cos^2\theta)}{|\boldsymbol{\rho}_1+\sqrt{3}\boldsymbol{\rho}_2|^2|\boldsymbol{\rho}_1-\sqrt{3}\boldsymbol{\rho}_2|^2}\frac{\partial}{\partial\theta}\right\}\right]\\&+\frac{1}{J^2}\left[-\left\{\frac{\partial^2}{\partial\rho_1^2}+\frac{1}{\rho_1}\frac{\partial}{\partial\rho_1}+\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}+\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}\right)\frac{\partial^2}{\partial\theta^2}\right\}\right],\end{aligned}\tag{27}$$

where $\rho_1=|\boldsymbol{\rho}_1|$, $\rho_2=|\boldsymbol{\rho}_2|$, and $\cos\theta=\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2/(\rho_1\rho_2)$. The ''renormalized'' $\tilde{\beta}^3=\beta^3/|J|^{3/2}$ and $\tilde{\nu}=\nu/J$ were introduced.

The minimum of the classical potential entering Eq. (27) is reached in the configuration of an equilateral triangle ($\rho_{01}=\rho_{02}$, $\theta=\pm\pi/2$). We choose, for example, $\rho_{01}=\rho_{02}$, $\theta=\pi/2$. ρ_{01} is obtained as the solution of the equation

$$\rho_{01}^4-3\rho_{01}\tilde{\beta}^3=(3\tilde{\nu}+1)^2.\tag{28}$$

Then, introducing $\rho_1=\rho_{01}+y_1/\sqrt{|J|}$, $\rho_2=\rho_{01}+y_2/\sqrt{|J|}$, and $\theta=\pi/2+z/\sqrt{|J|}$ in the rhs of Eq. (27), we obtain for the Hamiltonian h a series like Eq. (14). The first operator coefficients are given by

$$h_0=\frac{3}{2}\rho_{01}^2-\frac{(3\tilde{\nu}+1)^2}{2\rho_{01}^2},\tag{29}$$

$$h_1=0,\tag{30}$$

$$h_2=-\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+\frac{2}{\rho_{01}^2}\frac{\partial^2}{\partial z^2}\right)-\frac{2\imath}{\rho_{01}^3}\,\mathrm{sgn}(J)(y_1-y_2)\frac{\partial}{\partial z}+\frac{1}{4}\left(\frac{3}{\rho_{01}^4}+1\right)(y_1^2+y_2^2)$$
$$+\frac{1}{16}\left(1-\frac{1}{\rho_{01}^4}\right)(5y_1^2+6y_1y_2+5y_2^2+3\rho_{01}^2z^2)+\frac{9\tilde{\nu}^2}{16\rho_{01}^4}(y_1^2+y_2^2+6y_1y_2-\rho_{01}^2z^2)$$
$$+\tilde{\nu}\left[\frac{3}{8\rho_{01}^4}(3y_1^2+3y_2^2+2y_1y_2)-\frac{9z^2}{8\rho_{01}^2}+\frac{3\imath\,\mathrm{sgn}(J)z}{2\rho_{01}}\left\{\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}\right\}-\frac{3\imath\,\mathrm{sgn}(J)}{\rho_{01}^3}(y_1-y_2)\frac{\partial}{\partial z}\right], \tag{31}$$

$$h_3=-\frac{1}{\rho_{01}}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)+\frac{2}{\rho_{01}^3}(y_1+y_2)+\frac{3\imath\,\mathrm{sgn}(J)}{\rho_{01}^4}(y_1^2-y_2^2)\frac{\partial}{\partial z}-\frac{1}{\rho_{01}^5}(y_1^3+y_2^3)$$
$$-\frac{1}{64\rho_{01}}\left(1-\frac{1}{\rho_{01}^4}\right)(19y_1^3+3y_1^2y_2+33y_1y_2^2+9y_2^3-9\rho_{01}^2y_1z^2+21\rho_{01}^2y_2z^2)$$
$$-\frac{9\tilde{\nu}^2}{64\rho_{01}^5}(y_2^3-5y_1^3-21y_1^2y_2-39y_1y_2^2+7\rho_{01}^2y_1z^2-11\rho_{01}^2y_2z^2)+\tilde{\nu}\left[\frac{3\imath\,\mathrm{sgn}(J)}{2\rho_{01}^4}\left\{(4y_1^2-2y_2^2-2y_1y_2-\rho_{01}^2z^2)\frac{\partial}{\partial z}\right.\right.$$
$$\left.\left.-\rho_{01}^2z\left(y_1\frac{\partial}{\partial y_2}+y_2\frac{\partial}{\partial y_1}\right)+2\rho_{01}^2y_2z\frac{\partial}{\partial y_2}\right\}-\frac{3}{32\rho_{01}^5}(21y_1^3+15y_2^3+5y_1^2y_2+23y_1y_2^2+\rho_{01}^2y_1z^2-13\rho_{01}^2y_2z^2)\right], \tag{32}$$

$$h_4=\frac{1}{\rho_{01}^2}\left(y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}\right)-\frac{3}{\rho_{01}^4}(y_1^2+y_2^2)\frac{\partial^2}{\partial z^2}-\frac{4\imath\,\mathrm{sgn}(J)}{\rho_{01}^5}(y_1^3-y_2^3)\frac{\partial}{\partial z}+\frac{5}{4\rho_{01}^6}(y_1^4+y_2^4)$$
$$+\frac{1}{256\rho_{01}^2}\left(1-\frac{1}{\rho_{01}^4}\right)\left(\frac{329}{4}y_1^4-25y_1^3y_2+\frac{123}{2}y_1^2y_2^2+135y_1y_2^3+\frac{9}{4}y_2^4-\frac{99}{2}\rho_{01}^2y_1^2z^2\right.$$
$$\left.+27\rho_{01}^2y_1y_2z^2+\frac{141}{2}\rho_{01}^2y_2^2z^2+\frac{41}{4}\rho_{01}^4z^4\right)+\tilde{\nu}\left[-\frac{\imath\,\mathrm{sgn}(J)}{8\rho_{01}^5}\left\{6(11y_1^3-y_1^2y_2-7y_1y_2^2-3y_2^3\right.\right.$$
$$\left.-3\rho_{01}^2y_1z^2-\rho_{01}^2y_2z^2)\frac{\partial}{\partial z}+\rho_{01}^2z(9y_1^2-3y_2^2-18y_1y_2-\rho_{01}^2z^2)\frac{\partial}{\partial y_1}-\rho_{01}^2z(9y_1^2-27y_2^2+6y_1y_2-\rho_{01}^2z^2)\frac{\partial}{\partial y_2}\right\}$$
$$\left.+\frac{3}{512\rho_{01}^6}(503y_1^4-28y_1^3y_2+266y_1^2y_2^2+612y_1y_2^3+183y_2^4-58\rho_{01}^2y_1^2z^2+20\rho_{01}^2y_1y_2z^2-154\rho_{01}^2y_2^2z^2-41\rho_{01}^4z^4)\right]$$
$$+\frac{3\tilde{\nu}^2}{1024\rho_{01}^6}(549y_1^4-411y_2^4-84y_1^3y_2+2718y_1^2y_2^2+1836y_1y_2^3-750\rho_{01}^2y_1^2z^2+1266\rho_{01}^2y_2^2z^2+60\rho_{01}^2y_1y_2z^2+37\rho_{01}^4z^4). \tag{33}$$

The operator h_2 is diagonalized by changing variables $y_1=(y_s+y_m)/\sqrt{2}$, $y_2=(y_s-y_m)/\sqrt{2}$, $z=\sqrt{2}z_m/\rho_{01}$, and making the ‘‘gauge’’ transformation $h_2'=e^{\imath f}h_2e^{-\imath f}$, where $f=\mathrm{sgn}(J)y_mz_m/(2\rho_{01}^2)$, the result is

$$h_2'=h_s+h_m, \tag{34}$$

where h_s describes the motion of a harmonic oscillator in the coordinate y_s (the symmetric mode),

$$h_s=-\frac{\partial^2}{\partial y_s^2}+\frac{\omega_1^2}{4}y_s^2, \tag{35}$$

with $\omega_1=\sqrt{3+(3\tilde{\nu}+1)^2/\rho_{01}^4}$, and h_m accounts for two-dimensional motion in a ‘‘fictitious’’ magnetic field (the mixed mode)

$$h_m=-\frac{\partial^2}{\partial y_m^2}-\frac{\partial^2}{\partial z_m^2}+\frac{\omega_2^2}{4}(y_m^2+z_m^2)$$
$$-\frac{\imath\,\mathrm{sgn}(J)(3\tilde{\nu}+1)}{\rho_{01}^2}\left(y_m\frac{\partial}{\partial z_m}-z_m\frac{\partial}{\partial y_m}\right), \tag{36}$$

where $\omega_2=\sqrt{3/2-(3\tilde{\nu}+1)^2/(2\rho_{01}^4)}$.

The first two coefficients in the expansion for the energy are

$$\epsilon_0=\frac{3}{2}\rho_{01}^2-\frac{(3\tilde{\nu}+1)^2}{2\rho_{01}^2}, \tag{37}$$

$$\epsilon_2=\omega_1\left(n_s+\frac{1}{2}\right)+\omega_2(2n+|m|+1)+\omega_3\,\mathrm{sgn}(J)m, \tag{38}$$

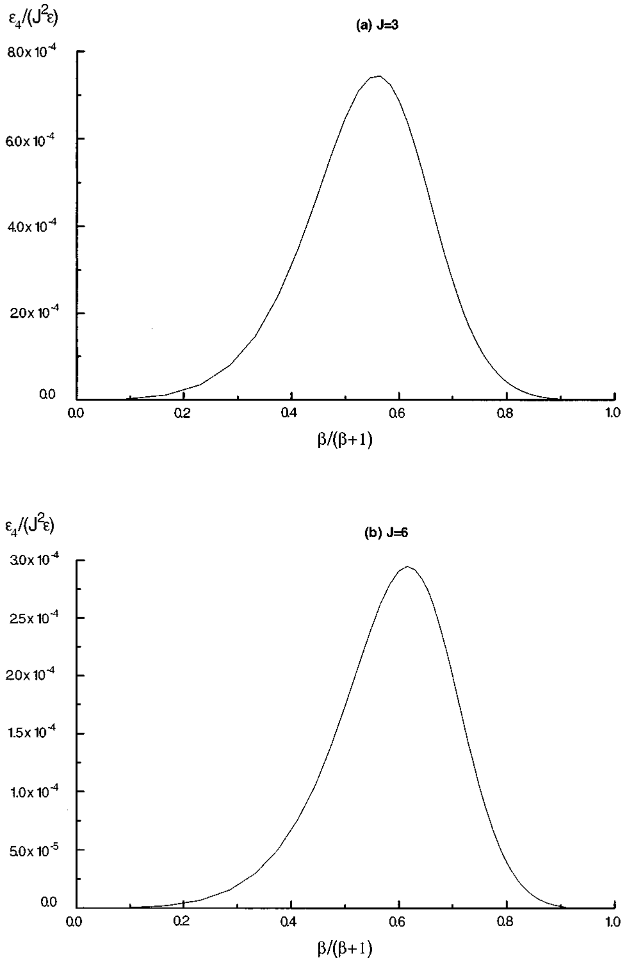

FIG. 3. Relative weight of ϵ_4 for three anyons in states with $\nu=1/2$. (a) $J=3$, (b) $J=6$.

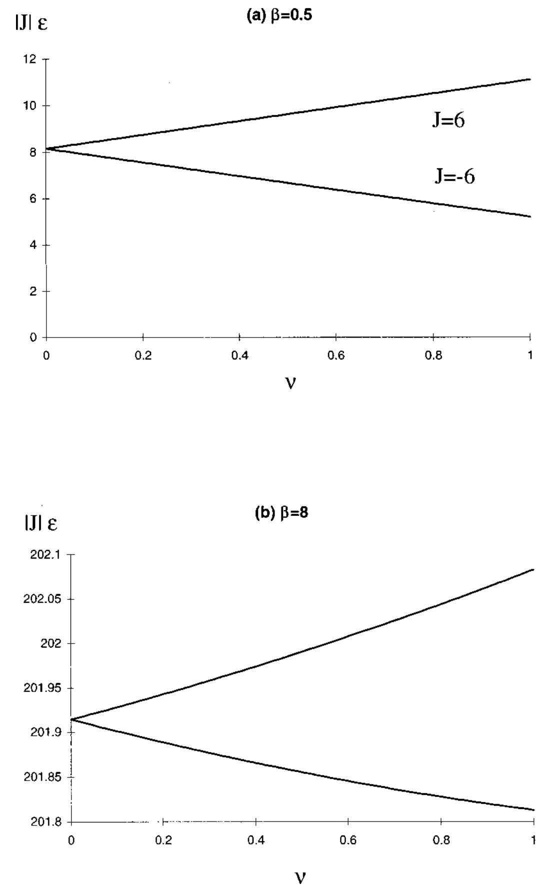

FIG. 4. $|J|\epsilon$ vs ν for three anyons in states with $J=\pm 6$. (a) $\beta=0.5$, (b) $\beta=8$.

where $\omega_3=(3\tilde{\nu}+1)/\rho_{01}^2$, and the quantum numbers n_s, n, and m may be used to approximately label the states.

Up to this order, the wave function is given by

$$\Psi_0=e^{iJ\Xi}|J,n_s,n,m\rangle, \tag{39}$$

where Ξ accounts for overall rotations of the system, and $|J,n_s,n,m\rangle$ are the eigenfunctions of h_2'. We notice that, when $\beta\to 0$ the energy becomes

$$E_0=|J+3\nu|+2+2n_s+2n+|m|+\mathrm{sgn}(J)m. \tag{40}$$

These are the ''linear'' three-anyon states. We stress that they are obtained as harmonic excitations against the equilateral triangle configuration, and are not necessarily related to a cigarlike shape of the wave function ($\rho_1 \gg \rho_2$) as described in Ref. 21.

The set of numbers $\{J,n_s,n,m\}$ compatible with the symmetry constraints (the wave function shall be symmetric) are obtained upon comparison with harmonic-oscillator wave functions at $\nu=0$, $\beta=0$. Details may be found in Refs. 22 and 14. An additional requirement is that the state at $\beta=0$ should be a linear state. For example, the lowest linear states are the following: $|0,0,0,0\rangle$ [the ground state, starting from $E_0=2$ at the bosonic end], $|0,1,0,0\rangle$, and $|2,0,0,-1\rangle$ (starting from $E_0=4$), $|3,0,0,0\rangle$ and $|1,0,0,1\rangle$ (starting from $E_0=5$), $|4,0,0,-2\rangle$, $|2,1,0,-1\rangle$, and $|0,1,1,0\rangle$ (starting from $E_0=6$), etc. The lowest state with $J<0$ is $|-6,0,0,0\rangle$, which starts at $E_0=8$. Of course, states with small values of $|J|$ cannot be described within our method.

In what follows, we restrict the analysis to levels with quantum numbers $n_s=n=m=0$. This leaves only the linear anyonic states with $J=3k$, where k is an integer.[22] The geometry of the state is an equilateral triangle. It can be seen from Eq. (29) that the side of the triangle increases with ν when $J>0$, and decreases when $J<0$. Thus, the Coulomb repulsion is much more stronger for $J<0$ states, and the ordering of levels may dramatically change as β is increased. On the other hand, for $\beta\to\infty$ the side grows like $\rho_{01}\sim 3^{1/3}\tilde{\beta}$ and becomes independent of the anyonic parameter, as one expects. A strong coupling expansion[10] shows that the leading contribution to the energy (potential energy) is $\sim\beta^2$, the next corrections (quantum fluctuations) are ~ 1, the angular momentum and the statistical parameter enter the second order corrections, which are $\sim 1/\beta^2$.

The first anharmonic corrections to the energy are given by

$$\epsilon_4=\langle J,n_s,n,m|h_4'|J,n_s,n,m\rangle + \sum_{n_s',n',m'} \frac{\langle J,n_s,n,m|h_3'|J,n_s',n',m'\rangle}{\epsilon_2^{Jn_snm}-\epsilon_2^{Jn_s'n'm'}} \times\langle J,n_s',n',m'|h_3'|J,n_s,n,m\rangle, \quad (41)$$

where h_3' and h_4' are obtained from h_3 and h_4 by means of a gauge transformation, in the same way as explained above for h_2.

For a state with quantum numbers $|J,0,0,0\rangle$, we get

$$\langle J,0,0,0|h_4'|J,0,0,0\rangle=-\frac{5}{8\rho_{01}^2}+\frac{3}{4}\frac{1}{\rho_{01}^2\omega_1^2}+\frac{9}{8}\frac{1}{\rho_{01}^6\omega_1^2} +\frac{9}{16}\frac{1}{\rho_{01}^2\omega_1\omega_2}\left(1+\frac{1}{\rho_{01}^4}\right) +\frac{3}{8}\frac{\omega_2}{\rho_{01}^2\omega_1}+\frac{9}{16}\frac{1}{\rho_{01}^2\omega_2^2}\left(1-\frac{1}{\rho_{01}^4}\right) +\frac{27\tilde{\nu}^2}{16\rho_{01}^6\omega_1^2\omega_2^2}(3\omega_1^2-\omega_1\omega_2+6\omega_2^2) -\frac{3\tilde{\nu}}{8\rho_{01}^6\omega_1^2\omega_2^2}(9\omega_1^2-\omega_1\omega_2-18\omega_2^2), \quad (42)$$

$$h_3'=A\frac{\partial}{\partial y_s}+By_s+Cy_s^3+D, \quad (43)$$

where

$$A=-\frac{\sqrt{2}}{\rho_{01}}-\frac{3\sqrt{2}\imath\ \mathrm{sgn}(J)\tilde{\nu}}{4\rho_{01}^3}\xi^2\sin 2\alpha, \quad (44)$$

$$B=\frac{\sqrt{2}}{\rho_{01}}\Bigg\{\sin^2\alpha\frac{\partial^2}{\partial\xi^2}+\frac{\cos^2\alpha}{\xi^2}\frac{\partial^2}{\partial\alpha^2} +\left(\frac{(3\tilde{\nu}+2)\imath\ \mathrm{sgn}(J)}{\rho_{01}^2}\cos^2\alpha-\frac{\sin 2\alpha}{\xi^2} +\frac{3\tilde{\nu}\imath\ \mathrm{sgn}(J)}{2\rho_{01}^2}\right)\frac{\partial}{\partial\alpha} +\left(\frac{(3\tilde{\nu}+2)\imath\ \mathrm{sgn}(J)\sin 2\alpha\xi}{2\rho_{01}^2}+\frac{\cos^2\alpha}{\xi}\right)\frac{\partial}{\partial\xi} +\frac{\sin 2\alpha}{\xi}\frac{\partial^2}{\partial\xi\partial\alpha}+\frac{9\tilde{\nu}^2-1}{4\rho_{01}^4}\xi^2\cos^2\alpha -\frac{3}{16}\left(1+\frac{3\tilde{\nu}^2-2\tilde{\nu}-1}{\rho_{01}^4}\right)\xi^2\Bigg\}, \quad (45)$$

$$C=-\frac{\sqrt{2}}{4\rho_{01}}\left[1+\frac{(3\tilde{\nu}+1)^2}{\rho_{01}^4}\right], \quad (46)$$

$$D=-\frac{\sqrt{2}}{32\rho_{01}}\left(5+\frac{27\tilde{\nu}^2-6\tilde{\nu}-5}{\rho_{01}^4}\right)\xi^3\cos\alpha(4\cos^2\alpha-3) +\frac{3\sqrt{2}\imath\tilde{\nu}\ \mathrm{sgn}(J)}{2\rho_{01}^3}\xi^2\sin\alpha(4\cos^2\alpha-1)\frac{\partial}{\partial\xi} +\frac{3\sqrt{2}\imath\tilde{\nu}\ \mathrm{sgn}(J)}{2\rho_{01}^3}\xi\cos\alpha(4\cos^2\alpha-3)\frac{\partial}{\partial\alpha}. \quad (47)$$

Polar coordinates have been introduced according to $\xi^2=y_m^2+z_m^2$, $\tan\alpha=z_m/y_m$. The only nonvanishing matrix elements entering the sum (41) are the following

$$\langle 0,0|A|0,0\rangle=-\frac{\sqrt{2}}{\rho_{01}}, \quad (48)$$

$$\langle 0,0|A|0,\pm 2\rangle=\pm\frac{3\ \mathrm{sgn}(J)\tilde{\nu}}{2\rho_{01}^3\omega_2}, \quad (49)$$

$$\langle 0,\pm 2|A|0,0\rangle=\mp\frac{3\ \mathrm{sgn}(J)\tilde{\nu}}{2\rho_{01}^3\omega_2}, \quad (50)$$

$$\langle 0,0|B|0,0\rangle=-\frac{\sqrt{2}\omega_2}{2\rho_{01}}, \quad (51)$$

$$\langle 0,\pm 2|B|0,0\rangle=\mp\mathrm{sgn}(J)\frac{3\tilde{\nu}+2}{2\rho_{01}^3}-\frac{1}{4}\frac{\omega_2}{\rho_{01}} +\frac{9\tilde{\nu}^2-1}{4}\frac{1}{\omega_2\rho_{01}^5}, \quad (52)$$

$$\langle 0,\mp 3|D|0,0\rangle=-\frac{\sqrt{6}}{16\rho_{01}^5\omega_2^{3/2}}(5\rho_{01}^4-5 \mp 24\rho_{01}^2\ \mathrm{sgn}(J)\tilde{\nu}\omega_2+27\tilde{\nu}^2-6\tilde{\nu}). \quad (53)$$

Collecting everything, we arrive at

$$\begin{aligned}\epsilon_4 = {} & \langle J,0,0,0|h_4'|J,0,0,0\rangle - \left\{ \frac{\langle 0,3|D|0,0\rangle^2}{3\omega_2+3\omega_3\ \mathrm{sgn}(J)} + \frac{\langle 0,-3|D|0,0\rangle^2}{3\omega_2-3\omega_3\ \mathrm{sgn}(J)} \right\} - \frac{11}{\omega_1^4}\langle 0,0|C|0,0\rangle^2 \\ & + \frac{\omega_1}{4}\left\{ \frac{\langle 0,0|A|0,0\rangle^2}{\omega_1} + \frac{\langle 0,0|A|0,2\rangle\langle 0,2|A|0,0\rangle}{\omega_1+2\omega_2+2\omega_3\ \mathrm{sgn}(J)} + \frac{\langle 0,0|A|0,-2\rangle\langle 0,-2|A|0,0\rangle}{\omega_1+2\omega_2-2\omega_3\ \mathrm{sgn}(J)} \right\} \\ & - \left\{ \frac{\langle 0,0|A|0,2\rangle\langle 0,2|B|0,0\rangle}{\omega_1+2\omega_2+2\omega_3\ \mathrm{sgn}(J)} + \frac{\langle 0,0|A|0,-2\rangle\langle 0,-2|B|0,0\rangle}{\omega_1+2\omega_2-2\omega_3\ \mathrm{sgn}(J)} \right\} \\ & - \frac{6}{\omega_1^3}\langle 0,0|B|0,0\rangle\langle 0,0|C|0,0\rangle - \frac{1}{\omega_1}\left\{ \frac{\langle 0,0|B|0,0\rangle^2}{\omega_1} + \frac{\langle 0,2|B|0,0\rangle^2}{\omega_1+2\omega_2+2\omega_3\ \mathrm{sgn}(J)} + \frac{\langle 0,-2|B|0,0\rangle^2}{\omega_1+2\omega_2-2\omega_3\ \mathrm{sgn}(J)} \right\}. \qquad (54)\end{aligned}$$

It may be checked that the corrections go to zero in both the Wigner ($\beta \to \infty$) and the oscillator ($\beta \to 0$) limits.

We show in Fig. 3 the relative weight of ϵ_4 in ϵ for three semions in states with $J=3$ and $J=6$. The numbers are similar to those appearing in the two-anyon problem. Thus, we expect a similar accuracy to this order, i.e., one part in 10^3 or better.

In Fig. 4, the levels with $J=\pm 6$ are drawn. β is increased from 0.5 to 8. Notice that the Coulomb effects are stronger for the state with negative J, and that the levels become flatter (as a function of ν) as β rises.

In conclusion, the energy levels of two and three anyons in a model parabolic dot were computed by means of the $1/|J|$ expansion. The qualitative picture emerging from the $1/|J|$ expansion is that of a rigid structure (an orbit in the two-anyon system, an equilateral triangle for three anyons) against which harmonic and anharmonic oscillations are developed. The Coulomb repulsion is much stronger for negative J states. Comparison with exact particular solutions for two anyons shows excellent agreement.

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