Property of Fibonacci numbers and the periodiclike perfectly transparent electronic states in Fibonacci chains

Xiuqing Huang
Department of Physics, Nanjing University, Nanjing 210093, China

Changde Gong
Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China and National Laboratory of Solid State Microstructure, Nanjing University, Nanjing 210093, China

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We study the properties of Fibonacci numbers and the transparency of clusters for electrons at some values of the energy. For the mth Fibonacci number F_m, a set of divisors are obtained by $F_m/k=\lfloor F_m/k \rfloor$, $1<k\leqslant F_m$. Interestingly, the numerical and analytical results show that any new divisors of the mth Fibonacci sequence will appear periodically in the following Fibonacci sequence. Furthermore, in the mixing Fibonacci system, we perform computer simulations and analytical calculations to study the transparent properties and spatial distributions of electronic states with the energies determined by the divisors of Fibonacci systems. The results show that the transmission coefficients are unity and the corresponding wave functions have periodic-like features. We report that an infinite number of one-dimensional disordered lattices, which are composed of some specific Fibonacci clusters, exhibit an absence of localization. [S0163-1829(98)04325-2]

I. INTRODUCTION

The medieval mathematician Leonardo Fibonacci of Pisa considered the idealized generation of rabbits and introduced the so-called Fibonacci sequence. Since then, much research had discovered the link between this pure mathematical model and real natural phenomena.[1–4] However, the study of one-dimensional Fibonacci systems has become particularly relevant since the remarkable discovery of Schechtman *et al.*,[5] i.e., the discovery of one-dimensional Fibonacci quasicrystals in rapidly solidified alloys,[6] together with the realization of a quasiperiodic superlattice.[7] It has been found that both electronic and phonon spectra of Fibonacci chains or Fibonacci multilayers are of Cantor-set structures and the corresponding eigenstates may be localized, extended, or critical.[8–17] The milestone work of Anderson showed that the tight-binding models with a site-energy disorder cause the vanishing of the diffusion in one- and two-dimensional systems.[18] However, Dunlap, Wu, and Phillips (DWP) (Refs. 19–22) explicitly demonstrated that diffusion occurs in models involving a specific type of correlation of the random site energies known as random dimer models (RDM), in which two kinds of site energy ε_a and ε_b are assigned at random to pairs of lattice sites. The basic reason for the appearance of extended states in such systems has been traced to the existence of short-range spatial correlations (clustering effect). It is worth mentioning that the idea of a clustering effect, first pointed out by DWP, has been the subject of many papers.[23–26] Moreover, some types of correlations have been shown to be responsible for the appearance of extended electronic states in one-dimensional quasiperiodic lattices, such as the copper mean chain and the Thue-Morse lattice.[27,28] In the recent work on the aperiodic structure, several groups have provided evidence of the possibility of Fibonacci and Thue-Morse superlattices as selective electronic filters.[29,30]

As is well known, the Fibonacci numbers are rather ubiquitous, emerging in different problems of diverse disciplines. For instance, the arrangement of botanical elements according to the Fibonacci sequence has been well known for more than a century.[1] In fact, many distinctive physical properties of low-dimensional Fibonacci lattices are also related to the Fibonacci number.[31,32] In the next section we will present an interesting kind of periodicity of divisors of Fibonacci numbers.

In recent works, the transparent properties in the mixing Fibonacci (MF) systems have been reported.[17,33] One of the aims of this paper is to also investigate the transparency of electrons in MF systems with electronic energies related to the divisors of the Fibonacci number. These discussions are given in Sec. III.

Then, in Sec. IV, we report that periodiclike wave functions with perfect transmission coefficients can be found in MF systems. In this section, we present the interesting numerical results and theoretically explain the numerical finding of the wave functions.

Finally, in Sec. V, we consider the problem of the relationship between the random system and the Fibonacci chain that was previously discussed by Phillips and Wu.[21] In view of the above-mentioned works, the research has been restricted to two cases: the RDM-like systems with internal symmetrical defects, and aperiodic systems with long-range correlation clusters. In this section, we construct some random systems by using some quasiperiodic clusters. The analytical studies show that these random systems permit the perfect transmission of some special electronic states and the corresponding wave functions are found to be extended.

II. THE PERIODICITY OF DIVISORS OF THE FIBONACCI NUMBER

The Fibonacci sequence can be generated by the substitution rules: $B\rightarrow A$ and $A\rightarrow AB$, where A and B correspond to

TABLE I. The relationship among the Fibonacci generation (m), the Fibonacci number F_m, and corresponding divisors k. The divisors k exhibit different kinds of periodicity.

m	F_m	k
1	1	
2	1	
3	2	2
4	3	3
5	5	5
6	8	2, 4, 8
7	13	13
8	21	3, 7, 21
9	34	2, 17, 34
10	55	5, 11, 55
11	89	89
12	144	2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144
13	233	233
14	377	13, 29, 377
15	610	2, 5, 10, 61, 122, 305, 610
16	987	3, 7, 21, 47, 141, 329, 987
17	1597	1597
18	2584	2, 4, 8, 17, 19, 34, 38, 68, 76, 136, 152, 323, 646, 1292, 2584
19	4181	37, 113, 4181
20	6765	3, 5, 11, 15, 33, 41, 55, 123, 165, 205, 451, 615, 1353, 2255, 6765
21	10946	2, 13, 26, 421, 842, 5473, 10946
22	17711	89, 199, 17711
23	28657	28657
46	1836311903	139, 461, 28657, 64079, . . .
69	117669030460994	2, 137, 274, 829, 1658, 18077, 28657, 36154, 57314, 113573, 227146, . . .

two kinds of atoms. Let F_m denote the Fibonacci number of the mth sequence, which obeys the recursion relation $F_{m+2}=F_{m+1}+F_m$, $m\geqslant 1$, with initial conditions $F_1=1$ and $F_2=1$. The first ten Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, and the ''golden mean'' $\tau=\lim_{m\to\infty}F_m/F_{m+1}=(\sqrt{5}-1)/2$. Let us consider the mth Fibonacci sequence with the Fibonacci number F_m, for which a set of divisors k can be obtained. These divisors are defined as

$$\frac{F_m}{k}=\left\lfloor\frac{F_m}{k}\right\rfloor,\quad 1<k\leqslant F_m, \tag{1}$$

where ''$\lfloor\ \rfloor$'' denotes the greatest integer function.

From the definition (1), we can easily obtain the divisors k to an arbitrary high generation of the Fibonacci sequence. Table I shows the divisors k of F_m up to $F_{23}=28657$ and a part of divisors k of the $F_{46}=1836311903$ and $F_{69}=117669030460994$. On the basis of the above results of Table I, we can conclude that for any new divisors k of mth Fibonacci sequence, they will appear periodically in the following sequence, the corresponding period $P(k)$ is given by the following expression:

$$P(k)=m,\quad m\geqslant 3. \tag{2}$$

Before proving the above formula (2), it is helpful to write down the following recursion relation concerned with the golden mean τ:

$$\tau^m+\tau^{m-1}=\tau^{m-2}, \tag{3}$$

where the formula $\tau^2+\tau-1=0$ is applied.

Using the recursion relation of the Fibonacci number, one can easily obtain the recursion relation between the golden mean τ and the Fibonacci number F_m as follows:[31]

$$\tau^m=(-1)^m(F_{m-1}-\tau F_m). \tag{4}$$

Because τ is an irrational number, from Eqs. (3) and (4) for any positive integer l we have

$$F_{lm}=LF_m, \tag{5}$$

where L is the sum of the Fibonacci number; therefore it is an integer.

The definition of Eq. (1) and the conclusion of Eq. (5) lead to the obvious result that the divisors k of F_m must also be the divisors of F_{lm}. The result is in good agreement with the numerical simulation of Table I.

III. THE DIVISORS AND TRANSMISSION COEFFICIENT

Making use of the properties of the Fibonacci number above, we now study the electronic properties of the MF system, that have been studied by several groups.[16,17,33] The Schrödinger equation for such a one-dimensional lattice with nearest-neighbor interactions can be written in the transfer matrix form as follows:

$$\Psi_n=M_n\Psi_{n-1},$$
$$M_n=\begin{bmatrix}\dfrac{E-\varepsilon_n}{t_{n,n+1}} & -\dfrac{t_{n,n-1}}{t_{n,n+1}}\\ 1 & 0\end{bmatrix},\quad \Psi_n=\begin{bmatrix}\varphi_{n+1}\\ \varphi_n\end{bmatrix}. \tag{6}$$

Here, φ_n denotes the amplitude of the wave function in the Wannier representation, $t_{n,n\pm1}$ is the hopping matrix element from the Wannier state $|n\rangle$ to $|n\pm1\rangle$, ε_n is the site energy, and M_n is a 2×2 transfer matrix. In the MF lattice, if one chooses the site energies and hopping integers in such a way that $\varepsilon_A=-\varepsilon_B=\varepsilon$, $t_{AB}\equiv1$, and $t_{AA}\equiv t$, then the string of transfer matrices of mth MF can be described as[17]

$$M_m=M(N)=\cdots M_BM_AM_AM_BM_A,$$
$$M_B=\begin{bmatrix}E-\varepsilon & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}E+\varepsilon & -1\\ 1 & 0\end{bmatrix}, \tag{7}$$
$$M_A=\begin{bmatrix}E-\varepsilon & -t\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dfrac{E-\varepsilon}{t} & -\dfrac{1}{t}\\ 1 & 0\end{bmatrix}\begin{bmatrix}E+\varepsilon & -1\\ 1 & 0\end{bmatrix},$$

where $N=F_m$ is the Fibonacci number, E is the electron energy, and M_A and M_B are unimodular.

Defining $x_m=\frac{1}{2}\mathrm{Tr}M_m$, Kumar found that there exists a constant[16]

$$\begin{aligned} J &= X_{m+1}^2 + X_m^2 + X_{m-1}^2 - 2X_{m+1}X_mX_{m-1} - 1 \\ &= \tfrac{1}{8}\mathrm{Tr}[M_A, M_B] \\ &= \frac{1}{4t^2}[(1+t^2)\varepsilon - (1-t^2)E]^2. \end{aligned} \quad (8)$$

When $J=0$, or $[M_A, M_B]=0$, then $E=\varepsilon(1+t^2)/(1-t^2)$ is obtained.[16,17] Consequently, the transfer matrix M_A and M_B can be decomposed as[33]

$$M_B = M_0^2, \quad \text{and} \quad M_A = M_0^3, \qquad (9)$$
$$M_0 = \begin{bmatrix} \dfrac{2\varepsilon t}{1-t^2} & -t \\ \dfrac{1}{t} & 0 \end{bmatrix}.$$

At the same time, $M(N)$ becomes the simple form $M(N)=M_0^N$, which can be explicitly evaluated in terms of the Chebyshev polynomial of the second kind. If $|\varepsilon t/(1-t^2)| \leqslant 1$, expression (4) of Ref. 17 can be readily obtained. Whereafter the transmission coefficient of the studied system can be determined by

$$T(N,E) = \frac{1}{1 + [(1-t^2)^2/(4-E^2)t^2]\sin^2(N\phi)}, \qquad (10)$$

where $2\cos\phi = \sqrt{E^2-\varepsilon^2}$. From the expression (10), we can see that some transparent electronic states $[T(N,E)=1]$ can be found in finite Fibonacci systems with $N\phi = K\pi$. On account of the restriction of $[M_A, M_B]=0$, the integer K will influence the choice of the model parameters ε and t. It seems that the authors of Ref. 17 had ignored this point.

In fact, for a given MF system with size N, the model parameter t and an integer K, from the transparency condition $N\phi = K\pi$ of Eq. (10) along with $[M_A, M_B]=0$, we find another parameter ε and the corresponding allowed transparent state E as follows:

$$\begin{aligned} \varepsilon\left(\frac{N}{K}\right) &= \frac{t^2-1}{t}\cos\left(\frac{K\pi}{N}\right), \\ E\left(\frac{N}{K}\right) &= -\frac{1+t^2}{t}\cos\left(\frac{K\pi}{N}\right). \end{aligned} \qquad (11)$$

In the following, we consider only some special cases of Eq. (11). These cases are confined by $k=N/K$, where the values of k are given by Eq. (1). An example of these divisors is shown in Table I. According to the expression (11), when $k=2$, $t\neq 1$, then $\varepsilon(2)=0$, $E(2)=0$. The corresponding electronic state exists in the center of the energy spectrum of the transfer model, which can perfectly transmit from the Fibonacci chain with $N=F_{3+3i}$ $(i=1,2,\ldots)$. When $k=3$, $t=2$, the expression (11) shows that $\varepsilon(3)=0.75$, $E(3)=-1.25$. The corresponding electronic state was studied in Ref. 17 in the MF chain with $N=F_{17}=1597$. The authors found $T[F_{17},E(3)]=0.5909\ldots \neq 1$. This can be well explained from the given divisors k of Table I. As we can see, $k=3$ is the forbidden divisor in the 17th Fibonacci chain. In fact, it is possible to find many Fibonacci systems where the electronic state $E(3)=-1.25$ satisfies the transparency condition. These Fibonacci systems are $N=F_{4+4i}$ $(i=1,2,\ldots)$. In general, for a new divisor k of the mth Fibonacci chain, the corresponding electronic state $E(k)$ will propagate through the corresponding Fibonacci chain ballistically. Consider, for example, the situation $k=4$ and $t=2$. It follows from Eq. (11) that the model parameter $\varepsilon(4)=3\sqrt{2}/4$ and energy $E(4)=-5\sqrt{2}/4$. Accordingly, the transmission coefficients for this state with several lattice lengths can be obtained from Eq. (10). They are $T[F_{15},E(4)]=0.2799\ldots$, $T[F_{16},E(4)]=0.4375\ldots$, $T[F_{17},E(4)]=0.4376\ldots$, and $T[F_{18},E(4)]=1$. As can be seen from Table I, $k=4$ is the allowed divisor of the 18th MF lattice and the related state can move through it without decaying, while $k=4$ is not allowed in the 15th, 16th, and 17th MF systems. Therefore, the corresponding transmission coefficients are below unity. Of course, the above discussion can be extended to the higher k.

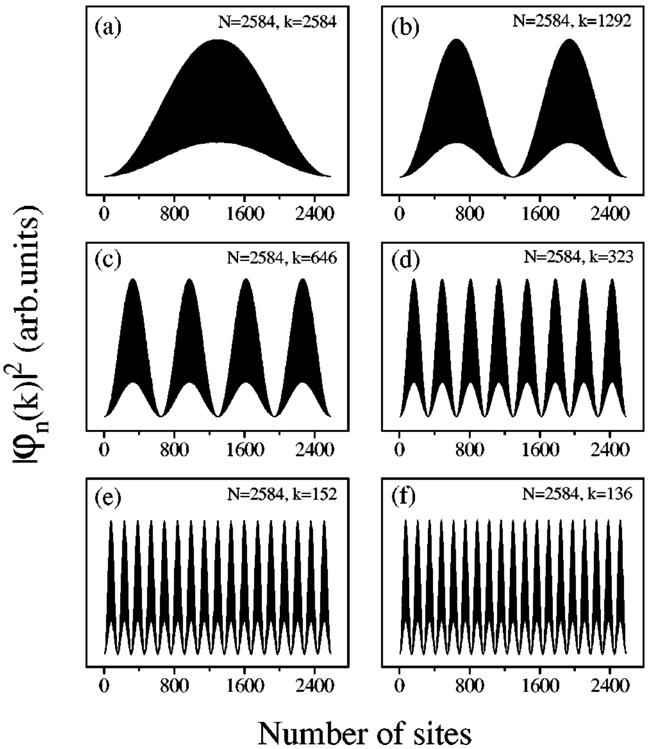

FIG. 1. The wave functions versus site number for the 18th MF lattice with 2584 sites. Here the energies $E(k)=-2.5\cos(\pi/k)$. (a), (b), (c), (d), (e), and (f) correspond, respectively, to $k=2584$, 1292, 646, 323, 152, and 136.

IV. PERIODICLIKE WAVE FUNCTIONS

In this section, we will discuss the behavior of the wave function for the states with the electronic energies determined by k of Eq. (1). We study the spatial distributions for two cases with the Fibonacci number $F_{18}=2584$ and $F_{16}=987$, and some very interesting wave-function properties are numerically found. Examples are presented in Figs. 1 and 2, and the corresponding divisors k are indicated in the figures. From the figures we obtain the following conclusions: although the arrangement of atoms is quasiperiodic, most of the global structures of the wave functions for allowed per-

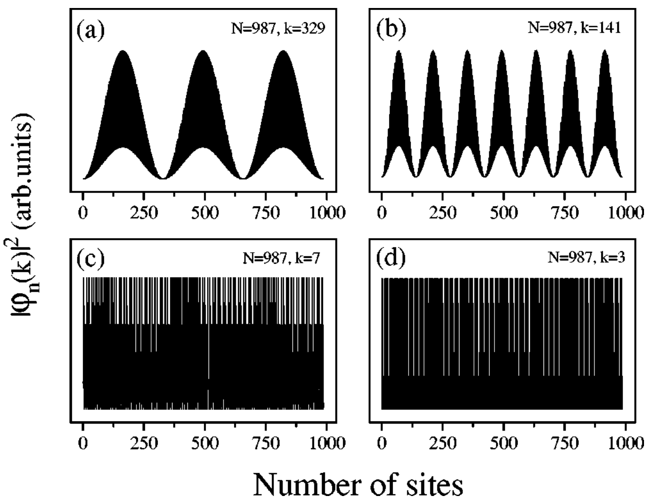

FIG. 2. The wave functions versus site number for the 16th MF lattice with 987 sites. Here the energies $E(k)=-2.5\cos(\pi/k)$. (a), (b), (c), and (d) correspond, respectively, to $k=329$, 141, 7, and 3.

fect transmitted electronic states are periodiclike. For a given system size N, the smaller k is, the more periods the envelope of the wave function has, and the period of the envelope is exactly equal to N/k. In Figs. 2(c) and 2(d) we notice that when k becomes rather small (compared with N), the corresponding wave functions will change from periodiclike to homogeneous.

Now, we proceed to explain the particular periodicity of the envelopes of the wave functions in the MF systems. When ε and E are given by Eq. (11), we have $M_0^k=(-1)^{k-1}I$, where k is a divisor of F_m and I is a 2×2 unit matrix, which indicates that the corresponding envelopes of the wave functions must be periodic. As one can see, $|\varphi_{i+m'k}|=|M_{11}(i+m'k)\varphi_1|=|M_{11}(i)\varphi_1|=|\varphi_i|$, where i ($i<k$) and m' ($m'\leqslant F_m/k$) are integers. To gain further insight into the behavior of the wave function, we next analyze the detailed structure of the wave-function amplitudes from sites 1 to k. In this paper, we have chosen the initial wave functions $\varphi_0=0$ and $\varphi_1=1$. At this point the overall behavior of the wave-function amplitudes can be obtained exactly by the matrix element $M_{11}(i)$. From Eq. (9), for a big k, the transfer matrix $M(i)$ can be approximately expressed as $M(i)=M_0^i$, and the corresponding matrix element $M_{11}(i)$ is given by

$$M_{11}(i)=\begin{cases} 1+2\sum\limits_{j=2}^{i}\cos\left(\dfrac{j\pi}{k}\right) & \text{for } i \text{ even} \\ -2\sum\limits_{j=1}^{i}\cos\left(\dfrac{j\pi}{k}\right) & \text{for } i \text{ odd}, \end{cases} \tag{12}$$

where $1\leqslant i\leqslant k$. Then, the wave-function amplitudes are as follows:

$$|\varphi_i(k)|=|M_{11}(i)\varphi_1| \approx\left|2\sum_{j=1}^{i}\cos\left(\frac{j\pi}{k}\right)\right|, \quad 1\leqslant i\leqslant k. \tag{13}$$

Since k is big, it is not hard to transform Eq. (13) into

$$|\varphi_i(k)|\approx 2\left|\int_0^i\cos\left(\frac{y\pi}{k}\right)dy\right| =\left|\frac{2k}{\pi}\sin\left(\frac{i\pi}{k}\right)\right|, \quad 1\leqslant i\leqslant k. \tag{14}$$

With the aid of Eq. (14), we are able to explain the interesting numerical results of Figs. 1 and 2. Let us for simplicity consider Fig. 1(c), which is the charge distribution of the state for the 18th MF lattice with the chain length $N=2584$ and divisor $k=646$. For this case, the positions of both the minima and the maxima appearing in the figure can be determined by Eq. (14). They are 0, 646, 1292, 1938, and 2584 for the minima, while 323, 969, 1615, and 2261 are for the maxima, which is in good agreement with the numerical results of Fig. 1(c). In general, for a given MF system with atoms N and divisor k, the minima and the maxima positions of the charge distribution of the corresponding state are pk (where $p=0,1,2,\ldots,N/k$) and $(p-\frac{1}{2})k$ (where $p=1,2,\ldots,N/k$), respectively. Consequently, the global feature of the wave function for this state is periodiclike, and there are k atoms in each period. In addition, for small k, we have found some homogenous wave functions [Figs. 2(c) and 2(d)] that have unity transmission coefficients. In particular, by comparing Fig. 2(d) with Fig. 1(a) of Ref. 17, one may find immediately that both figures are almost identical. In the above discussion, we have pointed out that these two states have absolutely opposite transmission coefficients 1 for Fig. 2(d) of our paper and 0.5909 ... for Fig. 1(a) of Ref. 17.

V. RANDOM SYSTEMS

According to expression (2), for a given electronic energy $E(k)$, a great number of MF chains that permit the perfect transmission of this electron can be easily obtained. They are $N=F_{m+mi}$, for $m=3,4$, and $N=F_{mi}$, for $m\geqslant 5$, where $i=1,2,\ldots$. Consequently, we can see that there are infinite kinds of disorder systems where the electronic transmission

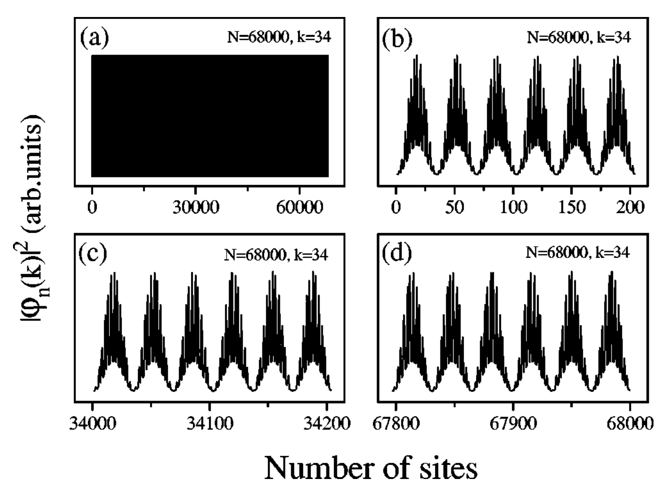

FIG. 3. The wave functions versus site number for the random system that is composed of 100 X-*type* and 34 Y-*type* MF clusters, where $X=F_9$ and $Y=F_{18}$, the total system size $N=68000$ and the energy $E(k)=-2.5\cos(\pi/34)$. (a) The global wave functions, (b), (c), and (d) are the enlarged figures of (a). They correspond to the left, center, and right part of (a), respectively.

coefficient of $E(k)$ is exactly unity. These models can be readily constructed as

$$S_{\text{random}}=\begin{cases} F^{j_1}_{m+mi_1}F^{j_2}_{m+mi_2}F^{j_3}_{m+mi_3}\cdots, & m=3,4 \\ F^{j_1}_{mi_1}F^{j_2}_{mi_2}F^{j_3}_{mi_3}\cdots, & m\geqslant 5, \end{cases} \tag{15}$$

where $i_1, j_1, i_2, j_2, i_3, j_3, \ldots$, are arbitrary positive integers.

From the knowledge that F_{m+mi}, for $m=3,4$, and F_{mi}, for $m\geqslant 5$ $(i=1,2,\ldots)$ meets the perfect transmission condition for the considered energy value $E(k)$, we can prove that the disorder systems (15) will also allow the electronic state $E(k)$ to pass them with zero resistance. To this end, let us consider the global transfer matrices of the disorder systems (15) for the allowed energy $E(k)$. With the help of Eq. (7) and the condition $[M_A, M_B]=0$, the global transfer matrices have the particular form $M(N')=M_0^{N'}$, which can also be represented by the expression (4) of Ref. 17 with $N'=\Sigma_{s=1}j_sF_{m+mi_s}$, for $m=3,4$, or $N'=\Sigma_{s=1}j_sF_{mi_s}$ for $m\geqslant 5$. By applying the same procedure above, we have the transmission coefficient $T=T[N',E(k)]$. Thus, the perfect transmitted condition is $N'\phi=K'\pi$. Because N' is the linear combination of the Fibonacci chains with same divisor k, k is, consequently, the divisor of N', which, in turn, implies $T=T[N',E(k)]=1$.

To illustrate the random systems (15) more clearly, we now present the simplest random system of expression (15). This system is composed of the following two kinds of MF clusters:

$$\begin{aligned} X&=F_6=\{BAABABAA\}, \\ Y&=F_9=\{BAABABAABAABABAABABAABAABABAABAABA\}, \end{aligned} \tag{16}$$

where A and B are considered as two kinds of site energies or hopping integers. From Table I we note that $k=2$ is the common divisor of the 6th and 9th FM chains that permit the electronic state $E(2)=0$ to move through them ballistically. Interestingly, by these two MF clusters, a related random system can be constructed,

$$S_{\text{random}}=\{\cdots XXXYYXYXXXYYXY\cdots\}. \tag{17}$$

With the above background we are able to verify that the electronic state $E(2)=0$ can also pass the random systems (17) with unity transmission coefficient. Naturally, we also expect that the wave functions, like Figs. 1 and 2, can be observed in these random systems. To support this argument, we consider a random system that contains randomly placed 100 X-*type* and 34 Y-*type* MF clusters, where $X=F_9$ and $Y=F_{18}$. In Fig. 3 we plot the wave functions for the energy $E(34)=-2.5\cos(\pi/34)$. We also show the detailed structure of Fig. 3(a) and these functions are shown in Figs. 3(c) and 3(d), which are seen to have a period of 34 atoms.

VI. CONCLUSION

In this paper, we have shown that there exists an interesting kind of periodicity in Fibonacci numbers. We have studied the relationship among the divisor of Fibonacci numbers, the transmission, and the form of wave function in MF systems. On the one hand, for the mth MF system and the divisors k of F_m, we have obtained the periodiclike wave functions for the energies relative to the divisors k, and proved that the period of envelopes of wave functions are exactly equal to F_m/k. On the other hand, we have presented the analytical derivation that shows the transmission coefficients corresponding to these states are unity. Furthermore, we have found that some Fibonacci clusters with the same divisor can randomly combine together to form infinite kinds of disorder lattices that have the same filtering function as the single Fibonacci cluster.

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