# SO(5)-symmetric description of the low-energy sector of a ladder system

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We study a system of two Tomonaga-Luttinger models coupled by a small transverse hopping (a two-chain ladder). We use Abelian and non-Abelian bosonization to show that the strong coupling regime at low energies can be described by an $SO(5)_1$ Wess-Zumino-Witten model (or equivalently five massless Majorana fermions) deformed by symmetry-breaking terms that nonetheless leave the theory critical at $T=0$. The SO(5) currents of the theory comprise the charge and spin currents and linear combinations of the so-called pi operators [S.C. Zhang, Science **275**, 1089 (1997)], which are local in terms both of the original fermions and those of the effective theory. Using bosonization we obtain the asymptotic behavior of all correlation functions. We find that the five-component ''superspin'' vector has power-law correlations at $T=0$; other fermion bilinears have exponentially decaying correlations and the corresponding tendencies are suppressed. Conformal field theory also allows us to obtain the energies, quantum numbers, and degeneracies of the low-lying states and fit them into deformed SO(5) multiplets. [S0163-1829(98)00935-7]

## I. INTRODUCTION

One of the most characteristic features of the high-$T_c$ cuprates is the proximity of antiferromagnetic (AF) and superconducting (SC) phases as a function of doping. As a result, much of the theoretical effort has focused on trying to consistently treat the insulating–underdoped–optimally doped region of the phase diagram, in which AF and SC tendencies compete and may have strong fluctuations.

An interesting recent proposal is that of Zhang.[1] He suggests that the simplest way of unifying AF and SC in the cuprates is to introduce a new five-component vector order parameter consisting of the three-component staggered magnetization, and two components associated with the real and imaginary parts of the $d$-wave SC order parameter. Clearly this new concept is only useful if there exists some kind of symmetry [higher than the known $SO(3)\otimes U(1)$] which relates the AF and SC sectors. His suggestion is that an approximate SO(5) symmetry emerges in the low-energy sector [SO(5) because the composite order parameter has five components and transforms like a vector]. If true, this would allow the construction of an SO(5) quantum nonlinear $\sigma$ model to explain the low-energy dynamics of the high-$T_c$ materials. This could explain the form of the phase diagram, and the so-called $\pi$ mode.[2]

However, there have been several criticisms of this theory. Some[3] have focused more on the details of microscopic calculations in the framework of the $t$-$J$ or Hubbard models. Others have added several physical objections.[4] One response to these criticisms has been to attempt to construct concrete examples of extended microscopic Hamiltonians that manifestly have an SO(5) symmetry.[5] But knowing the Hamiltonian does not necessarily tell us much about the low-energy behavior.

In this paper we study a two-chain ladder Hamiltonian that is related to popular two-dimensional models of the cuprates. One of the reasons that ladder systems have attracted such attention is that many experimental realizations of these systems are very closely related to the high-$T_c$ materials,[6] and some have even exhibited superconductivity.[7,8] From a theoretical point of view, powerful nonperturbative techniques such as bosonization and conformal field theory (CFT) exist in one dimension. This offers hope of starting with a microscopic Hamiltonian and ending up with a tractable effective-field theory. In this paper it is not our purpose to comment on the general validity of the SO(5) idea but to explicitly study a simplified and more tractable model.

There is a large body of literature on two-chain and ladder systems[9–17] (for a review see Ref. 6). Using a combination of weak-coupling renormalization group (RG) and bosonization, the phase diagram has been intensively investigated. These analyses reveal that for small interchain hopping there are interesting strong-coupling phases. However, while Abelian bosonization and weak-coupling RG are good for determining the phase diagram, they do not explicitly respect the symmetries of the system, nor do they provide detailed information about the correlations. In this paper we explore in more detail the strong-coupling region of a two-chain ladder system, taking care to preserve the full non-Abelian symmetries and obtain the correlations.

It is well known that many two-chain ladder systems are spin liquids; that is, they exhibit a spin gap for a wide range of different fillings and couplings. This is because the Luttinger liquid is a quantum critical system, and as such, highly unstable to perturbations such as interchain coupling. In general there are a number of relevant couplings that can drive the system into a spin gap phase (an explicit example is discussed in Sec. VI). However, in this paper we study a simplified system in which there is no backscattering and as a result, no spin gap. This model is of interest because it displays remarkable similarities to some aspects of the Zhang proposal in two dimensions.[1]

The model we consider is a system of two spinful Tomonaga-Luttinger (TL) models in the repulsive regime, coupled by a small interchain hopping. This corresponds to the case of no backscattering and was studied in Refs. 11, 14, and 17. We demonstrate that the hopping only generates couplings in a certain sector of the theory (which we call ''flavor''), freezing it out of the effective action at energy scales below $t_\perp$. In agreement with the above references, we find that this leaves a critical (at $T=0$) spin and charge sector with conformal charge 5/2. However, we go on to show that this can be represented as a system of five massless Majorana fermions, or equivalently, an $SO(5)_1$ Wess-Zumino-Witten (WZW) model, deformed away from the symmetric point by marginal current-current interactions. These SO(5) breaking terms are associated with spin charge separation (spin and charge velocities not equal $v_s \neq v_c$) and the anomalous charge exponent ($K_c \neq 1$), which distinguish the spin and charge sectors. Thus the system is never exactly SO(5) symmetric except in the trivial noninteracting case. Nonetheless, this representation does have strong analogies with the Zhang proposal in two dimensions; the physics can be understood using an SO(5) symmetric $\sigma$ model with symmetry-breaking terms. In this way we obtain the asymptotic behavior of all correlation functions; the correlations of the five-component ''superspin'' are enhanced (power law at $T=0$); we obtain their scaling dimensions. Other fermion bilinears die away exponentially fast.

Sections II–V are concerned with an analysis of this model, including its detailed symmetric description, the relevant currents, the $\pi$ operators, its correlations, and low-lying multiplets in the excitation spectrum. One important way in which the system we are studying differs from that considered by Zhang is that we are away from half-filling, which is a very special point in one dimension. Exactly at half-filling it is necessary to consider the Umklapp term, which causes a Mott gap in the charge sector.[18] Then the low-energy effective Hamiltonian is simply a pure spin Heisenberg model (with exchange $J \sim 4t^2/U$ in the case of the repulsive Hubbard model at strong $U$). We comment further on this difference in Sec. IV.

In Sec. VI we finally consider the case of two coupled Luttinger liquids, which differs from the previous model in that it includes marginal backscattering terms. An example of this is provided by some regions of the phase diagram of a system of two Hubbard chains coupled by single-particle hopping. In this more physical case, we show in detail how the additional marginal terms cause a spin gap to appear in agreement with Refs. 9–17, and numerical work such as Ref. 19. Then the spectrum and correlations are as in Ref. 20; there is a spin gap but the charge sector remains gapless.

Finally, we conclude. There is also an appendix that sketches out a bosonization prescription that enables us to calculate the correlation functions of fermion bilinears.

## II. A SIMPLE MODEL

Many systems of interacting one-dimensional fermions away from half-filling fall into the Luttinger liquid universality class. That is, they exhibit spin-charge separation, gapless excitations, anomalous power law correlations and the absence of a quasiparticle pole (see Ref. 21 for a recent review, and references therein). For example, the one-dimensional repulsive Hubbard model away from half-filling is known from its exact solution to be a Luttinger liquid all the way from $U=0$ to $U=\infty$, as is the $t$-$J$ model for small enough $J/t$.

One of the simplest two-chain models of this type that can be written down consists of two Tomonaga-Luttinger (TL) models (labeled by a chain index $i=1,2$) coupled by a small transverse hopping $t_\perp \ll t$:

$$H=H_{\mathrm{TL}}(1)+H_{\mathrm{TL}}(2)+H_\perp \tag{1}$$

where the TL Hamiltonian is a sum of three pieces ($H_0+H_2+H_4$):

$$H_0(i)=iv_F\sum_\alpha \int dx(R^\dagger_{\alpha,i}\partial_x R_{\alpha,i}-L^\dagger_{\alpha,i}\partial_x L_{\alpha,i}),$$

$$H_2(i)=g_2\sum_{\alpha,b'}\int dx j^R_{\alpha,i}(x)j^L_{\beta,i}(x),$$

$$H_4(i)=g_4\sum_{\alpha,\beta}\int dx(j^R_{\alpha,i}(x)j^R_{\beta,i}(x)+j^L_{\alpha,i}(x)j^L_{\beta,i}(x)). \tag{2}$$

The current (or density) is simply defined as

$$j^R_{\alpha,i}=R^\dagger_{\alpha,i}R_{\alpha,i} \quad j^L_{\alpha,i}=L^\dagger_{\alpha,i}L_{\alpha,i} \tag{3}$$

and the electrons fields $R_{\alpha,i}$ and $L_{\alpha,i}$ are slowly varying on an atomic scale: the electron annihilation operator at site $x$, chain $i$, and spin $\alpha$ may be expressed as

$$c_{\alpha,i}(x)=R_{\alpha,i}(x)e^{ik_Fx}+L_{\alpha,i}(x)e^{-ik_Fx}. \tag{4}$$

In terms of these fields, the simple interchain hopping term becomes

$$H_\perp=t_\perp\int dx\sum_\alpha (R^\dagger_{\alpha,1}(x)R_{\alpha,2}(x)+L^\dagger_{\alpha,1}(x)L_{\alpha,2}(x)+\mathrm{H.c.}). \tag{5}$$

For simplicity, we have assumed that the Hamiltonian is invariant under spin rotation, and so the coupling constants $g_2$ and $g_4$ are the same for parallel and antiparallel spin configurations. Normal ordering is assumed throughout in products of local fields (definition of currents, Hamiltonians, etc.).

It is worth making a quick observation about the difference between the terms TL *liquid* and TL *model*: The TL model is an idealized and specific Hamiltonian, written down in Eq. (2). It has a perfectly linear dispersion, an infinitely deep Fermi sea, has only density-density interactions, and is exactly solvable for all values of the coupling constants (the model is unstable beyond a critical value of $g_2$).[21] The TL liquid (which is the generic state corresponding to many realistic Hamiltonians like the Hubbard model away from half-filling) differs in that the dispersion is no longer exactly linear, and the Fermi sea no longer infinitely deep. But from our point of view the most important difference in the low-energy sector is the presence of marginally irrelevant couplings (backscattering). In a single chain system these are not very important when repulsive—they simply give logarithmic corrections to the correlation functions. In Sec. VI we will study the effect of these additional terms in the two chain system, in order to establish the behavior of the more realistic coupled TL *liquids*, but for the moment, we will restrict our attention to the simpler case of coupled TL *models*.

The model (1), even though it is made of TL models, is not exactly solvable because of the interchain hopping. However, we will argue presently that the model segregates into three different *sectors*, respectively associated with charge, spin, and flavor, and that the combined effect of interchain hopping and interactions is to make the flavor sector massive, leaving only the charge and spin sectors critical (i.e., gapless). To each sector one may associate current operators, expressed as bilinears of the electron fields:

$$\text{charge,}\quad J_R(x)=\sum_{\alpha,i} R^\dagger_{\alpha,i}(x)R_{\alpha,i}(x);$$

$$\text{spin,}\quad \mathbf{J}_R(x)=\frac{1}{2}\sum_{i,\alpha,\beta} R^\dagger_{\alpha,i}(x)\,\boldsymbol{\sigma}_{\alpha\beta}R_{\beta,i}(x);$$

$$\text{flavor,}\quad \mathbf{I}_R(x)=\frac{1}{2}\sum_{i,j,\alpha} R^\dagger_{\alpha,i}(x)\,\boldsymbol{\sigma}_{ij}R_{\alpha,j}(x); \tag{6}$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices (left-moving currents are defined similarly). These currents have the following commutation relations (they may be derived from Wick's theorem):

$$[J_R(x),J_R(y)]=-\frac{2i}{\pi}\,\delta'(x-y),$$

$$[J^a_R(x),J^b_R(y)]=-\frac{i}{2\pi}\,\delta^{ab}\,\delta'(x-y)+i\varepsilon^{abc}J^c_R(y)\,\delta(x-y),$$

$$[I^i_R(x),I^j_R(y)]=-\frac{i}{2\pi}\,\delta^{ij}\,\delta'(x-y)+i\varepsilon_{ijk}I^k_R(y)\,\delta(x-y), \tag{7}$$

and currents of different types (i.e., charge, spin, and flavor) commute. Thus in the language of non-Abelian bosonization, the charge current obeys a U(1) Kac-Moody algebra, and the spin and flavor currents obey $SU(2)_2\equiv SO(3)_1$ algebras.[22,23] It is simple to show that the Hamiltonian (1) may be expressed as $H=H_0+V_c+V_f$, where only the above currents appear. This is just a matter of taking careful account of point-splitting and normal ordering:[24]

$$H_0=\frac{\pi v_F}{2}\int dx(J_R^2+\mathbf{J}_R^2+\mathbf{I}_R^2+[R\to L]),$$

$$V_c=\frac{1}{2}\int dx\{g_2J_RJ_L+g_4(J_R^2+J_L^2)\},$$

$$V_f=2\int dx\{g_2I^z_RI^z_L+g_4[(I^z_R)^2+(I^z_L)^2]+t_\perp(I^x_R+I^x_L)\}. \tag{8}$$

Therefore the model (1) decouples into three independent sectors (charge, spin, flavor). The important point is that the hopping term only involves the flavor sector, which is decoupled from the other two. The effect of interactions ($g_2$ and $g_4$) on the charge sector will be a velocity renormalization and anomalous scaling exponents ($K_c\neq 1$). The combined effect of interactions and transverse hopping on the flavor sector is more dramatic. The RG analysis of Ref. 17 shows unambiguously that in the repulsive regime ($K_c<1$), the system scales to strong coupling at energies $<t_\perp$ (in the notation of Ref. 17 our model corresponds to initial conditions of $g_i^{(1)}=0$, $g_i^{(2)}=-g_i^{||}=g_0$ for $i=0,\pi,f,t,b$). The combination of the small hopping term $t_\perp$ and the interaction terms leads to the generation of important couplings in the RG process, giving a gap in some channels. What our analysis tells us is that all of this physics is only happening in the flavor sector, and thus it is this sector that becomes gapped, while the total spin and total charge sectors remain untouched and critical. So at low enough energies the flavor sector is frozen out of the effective theory, and our task is simply to understand the remaining charge and spin degrees of freedom.

### III. SPINOR AND VECTOR DESCRIPTIONS

Each electron field $R_{\alpha,i}$ or $L_{\alpha,i}$ carries charge, spin, and flavor. The separation of the model into charge, spin and flavor sectors is therefore difficult to describe in terms of these operators. However, one may introduce a different set of Fermi fields in terms of which this separation is much more natural. To this end, we must use some representation theory of Lie groups.

Let us first consider the model (1), but without interactions or interchain hopping (i.e., two free, decoupled chains). This model has SO(8) symmetry, and this may be shown has follows. Each complex field $R,L$ may be written in terms of its real and imaginary parts: $R_{\alpha,i}=R_{1,\alpha,i}+iR_{2,\alpha,i}$ and then, except for a total derivative, the Hamiltonian $H_0$ takes the form

$$H_0=iv_F\sum_\mu\int dx(R_\mu\partial_xR_\mu-L_\mu\partial_xL_\mu), \tag{9}$$

where the composite index $\mu$, running from 1 to 8, stands for spin, chain, and real/imaginary part. The eight Fermi fields $R_\mu$ (or $L_\mu$) can undergo an internal SO(8) rotation that leaves $H_0$ invariant. Hence the model has a chiral SO(8) symmetry. It is well known that a collection of $N$ real free fermions like this is equivalent to a special kind of conformal field theory: a level-1 SO($N$) WZW model.[25,26] Chiral SO(8) currents may be defined in terms of those real fermions as follows:

$$J^A_R=\frac{1}{2}\sum_{\mu,\nu=1}^{8} R_\mu S^A_{\mu\nu}R_\nu, \tag{10}$$

where $S^A_{\mu\nu}$ is a matrix representation of the generators of SO(8) [$A$ runs from 1 to $\frac{1}{2}N(N-1)=28$, the number of generators]. Left-moving currents are defined similarly. The charge, spin and flavor currents (6) are special cases of the above and correspond to specific values of the index $A$ if the generators $S^A_{ij}$ are chosen judiciously.

The currents (10) are bilinears in the electrons fields $R_\mu$ ($\alpha=1,\ldots,8$). However, the $SO(8)_1$ WZW model contains other fields, belonging to a different representation of SO(8), in terms of which these currents are also bilinears.

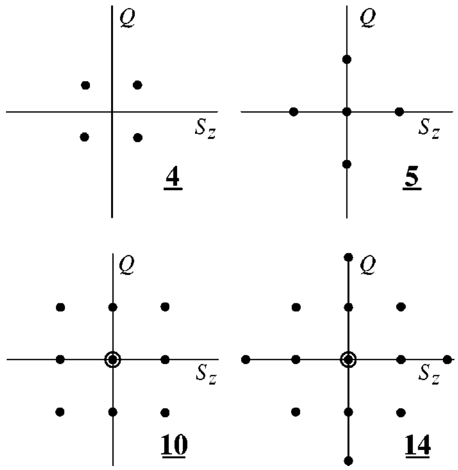


FIG. 1. The $S_z-Q$ diagrams associated with the lowest nontrivial SO(5) multiplets.

Among all SO($N$) groups, SO(8) is peculiar in that its vector representation, of dimension 8, has properties identical to its spinor and conjugate spinor representations (also of dimension 8). Indeed, which one is called ''vector'' is a matter of convention, dictated by the way the SO(8) symmetry breaks down to smaller SO($N$) components. In order to decide to which SO(8) representation the electron fields belong, one must study in detail how each representation breaks down when the symmetry is reduced. Let us consider a two-stage symmetry breaking, in which the flavor sector, with its SU(2), is first segregated, and then the charge U(1) and spin SU(2) (note that U(1)$\sim$ SO(2) and SU(2) $\sim$ SO(3)]:

$$SO(8)\rightarrow SO(5)\otimes SO(3)^{\mathrm{fl.}}\rightarrow SO(2)^{\mathrm{c}}\otimes SO(3)^{\mathrm{sp.}}\otimes SO(3)^{\mathrm{fl.}}. \tag{11}$$

We stress that the goal of the present analysis is to fit the fields and states of the model into symmetry multiplets, without demanding the symmetry to be exact. In the first stage of this breakdown, the vector and spinor representations of SO(8) are decomposed as follows [irreducible representations will be commonly denoted by bold numbers giving their dimensions, with an occasional superscript distinguishing between vectors ($v$) and spinors ($s$)]:

$$\mathbf{8}^v\rightarrow(\mathbf{5},\mathbf{1})\oplus(\mathbf{1},\mathbf{3}),$$

$$\mathbf{8}^s\rightarrow(\mathbf{4},\mathbf{2}) \tag{12}$$

[here, for instance, the notation $(\mathbf{4},\mathbf{2})$ stands for a tensor product of the four-dimensional representation of SO(5) with a doublet of SU$(2)^{\mathrm{fl.}}$]. Since SO(5) representations are not all that familiar, we provide a pictorial view of the lowest nontrivial ones on Fig. 1. The multiplet **4** is the spinor representation of SO(5), while **5** is the vector representation and **10** the adjoint representation, i.e., the representation of the SO(5) symmetry currents or generators. The decomposition of these SO(5) representations in terms of spin multiplets and charge quantum numbers is best appreciated on Fig. 1. For instance, the SO(5) spinor **4** breaks down into two spin-$\frac{1}{2}$ doublets, one with charge $+1$ and the other with charge $-1$. On the other hand, the vector representation breaks down into a spin-1 triplet of charge zero and two singlets of charges $\pm 2$.

We may now ascertain that the electron fields $R_\mu$ belong to the *spinor* representation of SO(8). Indeed, the lowest excited states of $H_0$, obtained by acting on the vacuum with the lowest electron creation operators, form a multiplet of four states of charge $+1$ and four states of charge $-1$. This is precisely the charge content of the spinor multiplet $\mathbf{8}^s$, since the spinor **4** of SO(5) contains two states of charge $+1$ and two of charge $-1$, and appears twice in the decomposition (12), because of the flavor doublet.

A different set of real fermions, denoted $\xi_i$ ($i=1,\ldots,8$), belongs to the *vector* representation of SO(8). These new fermions are related in a complicated, nonlocal way to the original fermions. The transformation relating them may be explicitly obtained via Abelian bosonization, if one takes care to preserve the anticommutation factors, but this is not a particularly illuminating procedure. The important point is that they are just a different basis or representation for the same system. These fermions obey the usual anticommutation relations $\{\xi_i(x),\xi_j(y)\}=\delta_{ij}\delta(x-y)$. The SO(8) currents (10) may also be expressed as bilinears of these fermions, albeit with the help of a different set of SO(8) matrices:

$$J_R^A=\frac{1}{2}\sum_{ij}\xi_i T_{ij}^A\xi_j. \tag{13}$$

A characteristic feature of the vector representation is its particularly simple decomposition into charge, spin, and flavor components: in the first stage of the breakdown (11), the vector representation decomposes as $\mathbf{8}^v\rightarrow(\mathbf{5},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})$. In the second stage, the SO(5) vector decomposes as $\mathbf{5}\rightarrow(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})$ (this time, doublets on the right-hand side correspond to spin and charge multiplets, respectively). We may thus distinguish three Majorana fermions ($\xi_i^s$, $i=1,2,3$) for spin, three others ($\xi_i^f$, $i=1,2,3$) for flavor, and the remaining two ($\xi_i^c$, $i=1,2$) for charge. The spin, flavor, and charge current then have the following expressions:

$$J_R^i=-\frac{i}{2}\epsilon_{ijk}\xi_j^s\xi_k^s,$$

$$I_R^i=-\frac{i}{2}\epsilon_{ijk}\xi_j^f\xi_k^f, \tag{14}$$

$$J_R=-i\epsilon^{jk}\xi_j^c\xi_k^c=-2i\xi_1^c\xi_2^c. \tag{15}$$

It is interesting to note that these currents [and eventual SO(5) currents] are local in terms both of the electron fields *and* in terms of the above Majorana fermions, even though the fermion operators themselves are nonlocally related. The Majorana fermions are nonetheless legitimate operators of the theory. For instance, at half-filling, if all sectors became gapped, the three spin fermions $\xi_i^s$ would describe the triplet of spin excitations characteristic of a gapped spin-1 chain.[27]

### IV. THE SO(5) CURRENTS AND ZHANG'S Π OPERATORS

We have seen above that the spin and charge degrees of freedom, which make up the critical sector of the theory (1), may be described by the five Majorana fermions $\xi^c_{1,2}$ and $\xi^s_{1,2,3}$. Except for the interaction $V_c$ of Eq. (8), the low-energy sector is equivalent to a level-1 SO(5) WZW model with conformal charge $c=\frac{5}{2}$.[14] Indeed, the above Majorana fermions may be arranged in the following suggestive sequence:

$$\xi_1=\xi^c_2 \quad \xi_2=\xi^s_1 \quad \xi_3=\xi^s_2 \quad \xi_4=\xi^s_3 \quad \xi_5=\xi^c_1 \tag{16}$$

plus corresponding left-moving fields. That the noninteracting part of the spin-charge sector is equivalent to a level-1 SO(5) WZW model means that this part of the Hamiltonian may be simply expressed as[28]

$$H_0=iv_F\sum_{j=1}^{5}\int dx(\xi_j\partial_x\xi_j-\overline{\xi}_j\partial_x\overline{\xi}_j) \tag{17}$$

(here $\overline{\xi}_j$ denote the left-moving fields).

It is then useful and instructive to introduce the full ten SO(5) currents. Four of those currents are provided by the charge and spin currents of Eq. (6). The remaining six, corresponding to Zhang's $\pi$ operators,[1] may be expressed in the continuum limit either in terms of the Majorana fermions $\xi^c_{1,2}$ and $\xi^s_{1,2,3}$, or directely in terms of the electron fields.

It is interesting at this point to go back to the lattice definition of the $\pi$ operators:[1]

$$\begin{aligned}\Pi^\dagger_a&=\sum_{\mathbf{k},\alpha,\beta} g(\mathbf{k})c^\dagger_\alpha(\mathbf{k}+\mathbf{Q})(\sigma_a\sigma_2)_{\alpha\beta}c^\dagger_\beta(-\mathbf{k})\\ &=\sum_{\mathbf{m},\mathbf{n},\alpha,\beta} g_{\mathbf{m},\mathbf{n}}e^{i\mathbf{Q}\cdot\mathbf{m}}c^\dagger_\alpha(\mathbf{m})(\sigma_a\sigma_2)_{\alpha\beta}c^\dagger_\beta(\mathbf{n}),\end{aligned} \tag{18}$$

where **m** and **n** are vectorial site indices (in-chain and chain index). On a square lattice at half-filling Zhang takes $\mathbf{Q}=(\pi,\pi)$. On a two-chain system, away from half-filling, there are two possibilities: $\mathbf{Q}=(2k_F,\pi)$ for right movers and $\mathbf{Q}=(-2k_F,\pi)$ for left movers.

The structure factor $g(\mathbf{k})=\cos k_x-\cos k_y$ has the local form:

$$g_{\mathbf{m},\mathbf{n}}=\begin{cases}+2 & \text{if } (\mathbf{m},\mathbf{n}) \text{ are NN on the same chain}\\ -2 & \text{if } (\mathbf{m},\mathbf{n}) \text{ are NN on opposite chains}\\ 0 & \text{otherwise.}\end{cases} \tag{19}$$

Defining a "staggered $\pi$ density,"

$$\Pi^\dagger_a=8\int dx\,\mathrm{e}^{2ik_Fx}\pi^\dagger_a, \tag{20}$$

we find, with the help of Eq. (4), the following expressions:

$$\begin{aligned}\pi^\dagger_x&=-\frac{i}{2}[R^\dagger_{\uparrow,1}R^\dagger_{\uparrow,2}-R^\dagger_{\downarrow,1}R^\dagger_{\downarrow,2}],\\ \pi^\dagger_y&=-\frac{1}{2}[R^\dagger_{\uparrow,1}R^\dagger_{\uparrow,2}+R^\dagger_{\downarrow,1}R^\dagger_{\downarrow,2}],\\ \pi^\dagger_z&=\frac{i}{2}[R^\dagger_{\uparrow,1}R^\dagger_{\downarrow,2}+R^\dagger_{\downarrow,1}R^\dagger_{\uparrow,2}].\end{aligned} \tag{21}$$

Interestingly, this continuum expression for the $\pi$ operators is quite robust and does not depend too closely on the microscopic definition (18). One might have alternatively chosen the Henley-Kohno form $g(\mathbf{k})=\mathrm{sgn}(\cos k_x-\cos k_y)$ and this would not have changed the results, apart from derivative terms that are irrelevant in the RG sense—essentially because in a two-chain ladder system there are only two $k_y$ values, 0 or $\pi$. Another seemingly different microscopic expression for the $\pi$ operators is used in Ref. 29, but again, we have verified that the same expression (21) is obtained in the continuum limit.

We define the matrix $l^{ab}(x)$, analogously to Zhang:

$$\begin{pmatrix} 0 & & & & \\ \pi^\dagger_x+\pi_x & 0 & & & \\ \pi^\dagger_y+\pi_y & -J^z_R & 0 & & \\ \pi^\dagger_z+\pi_z & J^y_R & -J^x_R & 0 & \\ J_R & -i(\pi^\dagger_x-\pi_x) & -i(\pi^\dagger_y-\pi_y) & -i(\pi^\dagger_z-\pi_z) & 0\end{pmatrix} \tag{22}$$

(the matrix is antisymmetric and so we only wrote down the lower diagonal). Using Wick's theorem for the electron fields $R_{\alpha,i}$ and $L_{\alpha,i}$, we find that the $l^{ab}$ obey an $\mathrm{SO}(5)_1$ Kac-Moody algebra, different from the standard SO(5) algebra by a quantum anomaly coming from the necessity for normal ordering with respect to the vacuum:

$$\begin{aligned}[l^{ab}(x),l^{cd}(y)]=\delta(x-y)(&\delta^{ac}l^{bd}(x)-\delta^{ad}l^{bc}(x)\\ &-\delta^{bc}l^{ad}(x)+\delta^{bd}l^{ac}(x))\\ &+\frac{i}{2\pi}\delta'(x-y)(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc}).\end{aligned} \tag{23}$$

A similar procedure gives the corresponding relationship for left-moving currents.

How can the SO(5) symmetry currents be expressed in the Majorana (vector) representation $\xi$? A vector representation of the SO(5) generators may be easily written down:

$$t_{ij}^{(ab)}=i(\delta_i^a\delta_j^b-\delta_j^a\delta_i^b). \tag{24}$$

The currents (22) can then be represented as follows:

$$l^{ab}(x)=\frac{1}{2}\sum_{i,j=1..5}\xi_i t_{ij}^{(ab)}\xi_j, \tag{25}$$

where the five fermions $\xi_i$ are numbered as in Eq. (16): Note that the $\pi$ operators correspond to bilinears involving one fermion $\xi$ from the spin sector and one from the charge sector, which again fits with the physics since we know that they create objects with both spin and charge.

In the low-energy sector of the model (1), the spin-charge sector can be represented by an $\mathrm{SO}(5)_1$ WZW model, perturbed away from the perfectly symmetric point by current-current interactions (the only interactions present in the spin and charge sectors in our model). So Zhang's idea of using an SO(5) $\sigma$-model representation with symmetry breaking interactions[1] is explicitly seen to be valid for this model, and the Hamiltonian for the spin and charge sectors can be written in terms of the SO(5) currents in the Sugawara form, analogous to the form proposed in Ref. 1:

$$\begin{aligned}H_{cs}&=H_{0s}+H_{0c}+V_c\\&=\frac{\pi v_F}{4}\int dx\sum_{a<b}\{(l^{ab})^2+(\overline{l}^{ab})^2\}\\&\quad+\int dx\{g_2 l^{15}\overline{l}^{15}+g_4((l^{15})^2+(\overline{l}^{15})^2)\}.\end{aligned} \tag{26}$$

One notable difference between this system and that considered by Zhang is that here we are working away from half-filling; the $\pi$ operators are defined slightly differently, to carry momentum $(\pm 2k_F,\pi)$. The chemical potential term in the Hamiltonian, which in two dimensions breaks SO(5), here (due to perfect nesting) simply renormalizes the wave vector $k_F$; momentum is still conserved and the algebra (23) still closes because operators carrying $2k_F$ only give nonzero expectation values when combined with operators carrying $-2k_F$. If we were to work at half-filling, there would be an additional Umklapp scattering which would lead to a Mott charge gap.[18]

Since there are $\pi$ operators in this system, one may also ask whether there is a well-defined $\pi$ resonance as claimed in two dimensions.[2,1] If this is so, the commutator of the Hamiltonian with the $\Pi$ operator will be proportional to the $\Pi$ operator.[1] It can easily be seen that this will not be the case. Later on, in Sec. V, we obtain bosonised forms for these operators. Then one can see that in Fourier space, the correlator of $\pi$ operators does not have a simple pole; their effect is not to generate a single well-defined triplet excitation but a shower of unconfined spinons and holons.

## V. BOSONIZATION

The low-energy sector of the model (1), i.e., the perturbed SO(5) WZW model of Eq. (26) is exactly solvable, in the sense that we may find the exact energy levels of low-lying states and the long-distance correlations of various operators. We will first indicate the physical content of the SO(5) WZW model without perturbations, and then see explicitly how the interactions $g_2$ and $g_4$ separate spin and charge sectors, affect correlation exponents and deform low-energy SO(5) multiplets.

In the language of conformal field theory,[26] particularly useful when dealing with critical theories, each WZW model contains a finite number of primary fields, having well-defined conformal dimensions $\Delta$ and $\overline{\Delta}$. An operator $A$ with such conformal dimensions has the following dynamical correlations:

$$\langle A(x,t)A(0,0)\rangle\sim\frac{1}{(x-vt)^{2\Delta}(x+vt)^{2\overline{\Delta}}}. \tag{27}$$

The level-1 SO(5) WZW model has two primary fields: a five-component vector field $\xi$ of conformal dimension $\Delta=\frac{1}{2}$ and a four-component spinor field $h$ of conformal dimension $\frac{5}{16}$. Under SO(5) rotations, these fields transform, respectively, in the vector and spinor representations of SO(5). Of course, the field $\xi$ is made of the five Majorana fermions (16), whereas the field $h$ is what is left of the original electron fields $R_\mu$, originally in a spinor representation of SO(8), after the flavor sector has been gapped out. Freezing out the flavor part has had the effect of decreasing the conformal dimension of the spinor field from $\frac{1}{2}$ [in SO(8)] to $\frac{5}{16}$ [in SO(5)], thus making it more relevant. In addition to these primary fields, the ten SO(5) currents (22) also play a crucial role in the theory and their correlations may also be exactly calculated.

### A. Spin-charge separation

When the interactions of Eq. (26) are turned on, the SO(5) symmetry is explicitly broken and the spin and charge sectors of the theory separate. The spin sector, unaffected by the interactions, becomes a level-1 SO(3) WZW model, which is the same as a level-2 SU(2) WZW model. The $\mathrm{SU}(2)_2$ WZW theory contains two primary fields: a spin triplet (or vector) $\xi_i^s$ $(i=1,2,3)$ with conformal dimension $\frac{1}{2}$, and a spin-$\frac{1}{2}$ (or spinor) field $g_\alpha$ $(\alpha=\uparrow,\downarrow)$, of conformal dimension $\frac{3}{16}$. Products of the left- and right-moving parts of $g$ are commonly arranged in a $2\times 2$ matrix:

$$G=\begin{pmatrix}g_\uparrow\overline{g}_\uparrow & g_\uparrow\overline{g}_\downarrow\\ g_\downarrow\overline{g}_\uparrow & g_\downarrow\overline{g}_\downarrow\end{pmatrix}. \tag{28}$$

The charge sector becomes a U(1) theory, which may be described by a single boson field $\Phi_c$. The effect of the interactions on the charge boson is simply to change the spectrum of anomalous dimensions $(K_c\neq 1)$ and the theory remains critical. Since Abelian bosonization is fairly standard,[30–32,24] we shall only state a few results. The charge boson $\Phi_c$ may be written as the sum of right and left parts:

$\Phi_c=\phi_c+\overline{\phi}_c$. Defining the dual field $\theta_c$ as $\theta_c=\phi_c-\overline{\phi}_c$, the charge Hamiltonian may be written as

$$H_c=\frac{v_c}{2}\int dx\left[K_c(\partial_x\theta_c)^2+\frac{1}{K_c}(\partial_x\Phi_c)^2\right], \tag{29}$$

where

$$K_c=\sqrt{\frac{\pi v_F+g_4-g_2}{\pi v_F+g_4+g_2}},$$

$$v_c=\sqrt{\left(v_F+\frac{g_4}{\pi}\right)^2-\left(\frac{g_2}{\pi}\right)^2}. \tag{30}$$

For more general lattice Hamiltonians that are Luttinger liquids in the low-energy sector, the parameters $v_c$ and $K_c$ depend in a more complicated way upon the original couplings, so in the following analysis, one can just treat them as independent parameters the precise values of which depend upon the original model.

If $g_2=0$, there are no anomalous exponents ($K_c=1$) and the scaling fields $e^{\pm i\sqrt{4\pi}\phi_c}$ represent right-moving fermions of conformal dimensions $(\frac{1}{2},0)$. When $g_2$ is turned on, the original charge current ($I^{15}=J_R$) is no longer conserved but becomes a linear combination of currents that are still conserved (see, e.g., Ref. 21):

$$J_R=j_R\cosh\vartheta-j_L\sinh\vartheta \quad (K_c=e^{-2\vartheta}), \tag{31}$$

where $j_R$ and $j_L$ have, respectively, conformal dimensions (1,0) and (0,1). The chiral components $\phi_c$ and $\overline{\phi}_c$ mix through the same Bogoliubov transformation and the original fermion operators $e^{\pm i\sqrt{4\pi}\phi_c}$ acquire a left conformal dimension:

$$\Delta=\tfrac{1}{8}(K_c+1/K_c)+\tfrac{1}{4}, \qquad \overline{\Delta}=\tfrac{1}{8}(K_c+1/K_c)-\tfrac{1}{4}. \tag{32}$$

Although the above results are very well known, it is worth pausing over them for a moment. They show that the interaction strengths $g_2$ and $g_4$ are not relevant energy scales in the low-energy theory. They only appear as ratios $g_{2,4}/v_F$ in the renormalization of the velocity $v_c$ and the anomalous exponent $K_c$. This is a very nonperturbative result. If we recall the exact solution of the one-dimensional Hubbard model away from half-filling, we know that it is a Luttinger liquid for all $U>0$ from 0 to $\infty$.[33] In this range, $K_c$ varies from its noninteracting value of 1, to 1/2 at $U=\infty$. Even when the on-site repulsion is infinite, its effect in the low-energy sector is just a fairly small renormalization of the anomalous exponent! The theory is still critical with gapless spin and charge excitations.

Some critics of the Zhang SO(5) proposal have claimed that because of the strong on-site repulsion in the Hubbard model, the $\pi$ operators cannot create low-energy excitations.[3] The argument is essentially that one is forced to put two electrons on the same site, which has an energy cost of order $U$. The reason that the criticism[3] may be too simplistic is first of all that it is a single-particle argument, whereas the low-energy excitations of this system are many-body collective phenomena, and secondly that it is a short-length scale argument that may have some validity in the UV; but we are interested in the low-energy IR behavior that is quite different in a non-Fermi-liquid such as the TL liquid. Even if much of the spectral weight is shifted to high energies there is still some at low energies and this is what dominates the low-energy theory. Given that the two-dimensional cuprates are examples of non-Fermi-liquids, it cannot be ruled out *a priori* using these arguments that even in the presence of strong on-site repulsion, the $\pi$-operators may generate low-energy excitations (at least when one is slightly away from half-filling).

Let us then see what happens to the SO(5) currents and primary fields after spin-charge separation. Three of the five components of $\xi$ become a spin triplet ($\xi_s$) and the remaining two are simply $\cos(\sqrt{4\pi}\phi_c)$ and $\sin(\sqrt{4\pi}\phi_c)$. Out of the 10 SO(5) currents, six—the $\pi$ operators—are no longer conserved currents and may then be expressed as products of SU$(2)_2^{\rm sp.}$ fields with charge fields. Schematically,

$$\pi,\pi^\dagger\sim e^{\pm i\sqrt{4\pi}\phi_c(z)}\otimes\xi_s(z). \tag{33}$$

When $K_c\neq1$, the conformal dimensions of the $\pi$ operators are no longer (1,0), but rather, from Eq. (32) and since the field $\xi_s(z)$ has conformal dimensions $(\frac{1}{2},0)$,

$$\Delta=\tfrac{1}{8}(K_c+1/K_c)+\tfrac{3}{4} \qquad \overline{\Delta}=\tfrac{1}{8}(K_c+1/K_c)-\tfrac{1}{4}. \tag{34}$$

Thus, the $\pi$ operators are no longer conserved currents, as expected.

As mentioned above, the spinor representation **4** of SO(5) factorizes into a pair of SU(2) doublets of charges $\pm1$ when SO(5) is broken. The spinor field $h$ may thus be factorized as

$$h(x)\sim(g_\uparrow,g_\downarrow)\otimes\begin{pmatrix}\cos(\sqrt{\pi}\phi_c)\\ \sin(\sqrt{\pi}\phi_c)\end{pmatrix}, \tag{35}$$

where $g$ is the SU(2) spinor mentioned above and the boson factors have conformal dimension $\frac{1}{4}$. The decomposition described here can be rigorously proven by checking the corresponding commutators with the currents. We obtained it differently, by the method of affine characters (see Ref. 26), which we will not explain in detail here, since the coincidence of conformal dimensions and components is sufficiently convincing for our purpose.

### B. The SO(5) order parameter

One of the interesting operators of the SO(5) WZW model (and of its perturbed version) is a continuum version of Zhang's five-component order parameter $n_a$ ($a=1,\ldots,5$).[1] This operator can be defined in terms of the original electron fields. The components $n_{2,3,4}$ correspond to the staggered magnetization and the components $n_{1,5}$ to the $d$-wave superconducting order parameter. The staggered magnetization is defined as

$$\mathbf{n}_\mathbf{Q}=\sum_{\mathbf{k},\alpha,\beta}c^\dagger_\alpha(\mathbf{k}+\mathbf{Q})\boldsymbol{\sigma}_{\alpha\beta}c_\beta(\mathbf{k}). \tag{36}$$

Picking $\mathbf{Q}=(2k_F,\pi)$ and using Eq. (4), we find

$$\mathbf{n}_\mathbf{Q}=\sum_{k,\alpha,\beta}(R^\dagger_{\alpha,1}\boldsymbol{\sigma}_{\alpha\beta}L_{\beta,1}-R^\dagger_{\alpha,2}\boldsymbol{\sigma}_{\alpha\beta}L_{\beta,2}). \tag{37}$$

The $\mathbf{Q}=(-2k_F,\pi)$ component of the magnetization is just the Hermitian conjugate of the above. The $d$-wave order parameter $\mathcal{D}$ is defined as

$$\int dx\mathcal{D}^\dagger=\sum_{\mathbf{m},\mathbf{n}} g_{\mathbf{m},\mathbf{n}}c_\uparrow^\dagger(\mathbf{m})c_\downarrow^\dagger(\mathbf{n}), \tag{38}$$

where $g_{\mathbf{m},\mathbf{n}}$ has been introduced in Eq. (19). Taking the continuum limit, we find

$$\mathcal{D}^\dagger=\sum_{i=1,2}(R_{\uparrow,i}^\dagger L_{\downarrow,i}^\dagger+L_{\uparrow,i}^\dagger R_{\downarrow,i}^\dagger)-R_{\uparrow,1}^\dagger L_{\downarrow,2}^\dagger-L_{\uparrow,1}^\dagger R_{\downarrow,2}^\dagger-R_{\uparrow,2}^\dagger L_{\downarrow,1}^\dagger -L_{\uparrow,2}^\dagger R_{\downarrow,1}^\dagger. \tag{39}$$

The combinations $\mathcal{D}+\mathcal{D}^\dagger$ and $i(\mathcal{D}-\mathcal{D}^\dagger)$ then correspond to $n_1$ and $n_5$, respectively.

An expression of the order parameter $n_a$ in terms of the scaling fields $h$ or $\xi$ would be more useful, since the flavor part is not explicitly absent from the above. Such an expression is difficult to obtain in a systematic way from the above expressions; but one can infer what it has to be (this result can be confirmed by Abelian bosonization; see the Appendix). Clearly, $n_a$ should be a bilinear in $h$ or $\xi$, with equal left and right conformal dimensions. Let us consider the following SO(5) tensor products:

$$\mathbf{4}\otimes\mathbf{4}=\mathbf{1}\oplus\mathbf{5}\oplus\mathbf{10},$$

$$\mathbf{5}\otimes\mathbf{5}=\mathbf{1}\oplus\mathbf{10}\oplus\mathbf{14}. \tag{40}$$

This means that a bilinear in $\xi$ (five components) cannot transform as a vector of SO(5), whereas a bilinear in $h$ (four components) can. We thus seek an order parameter of the form

$$n_a=\Gamma_{ij}^a h_i\bar{h}_j, \tag{41}$$

where the five $4\times4$ matrices $\Gamma^a$ must transform as a vector of SO(5) when $h$ and $\bar{h}$ are acted upon by a $4\times4$ unitary representation of SO(5). If we denote by $\ell^{ab}$ a $4\times4$ representation of the SO(5) generators, this requirement amounts to

$$[\ell^{ab},\Gamma^c]=i(\delta^{ac}\Gamma^b-\delta^{bc}\Gamma^a). \tag{42}$$

Experience with the Lorentz group and Dirac matrices may guide us here. If a set of five matrices $\Gamma^a$ obey the Clifford algebra $\{\Gamma^a,\Gamma^b\}=2\delta^{ab}$, then it is a simple matter to show that the above commutation relations are satisfied if we define

$$\ell^{ab}=-\frac{i}{4}[\Gamma^a,\Gamma^b]. \tag{43}$$

Moreover, the matrices thus defined do obey the SO(5) algebra,

$$[\ell^{ab},\ell^{cd}]=i(\delta^{ac}\ell^{bd}+\delta^{bd}\ell^{ac}-\delta^{ad}\ell^{bc}-\delta^{bc}\ell^{ad}). \tag{44}$$

Let us adopt the following representation for the Clifford algebra:

$$\Gamma^1=1\otimes\sigma_3, \tag{45a}$$

$$\Gamma^2=\sigma_1\otimes\sigma_2, \tag{45b}$$

$$\Gamma^3=\sigma_2\otimes\sigma_2, \tag{45c}$$

$$\Gamma^4=\sigma_3\otimes\sigma_2, \tag{45d}$$

$$\Gamma^5=-1\otimes\sigma_1. \tag{45e}$$

Then the charge and $S_z$ matrices take the form

$$\ell^{51}=\frac{1}{2}\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix},\qquad \ell^{23}=\frac{1}{2}\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}. \tag{46}$$

With the above generators $Q$ and $S_z$, the factorization of the chiral field $h$ in terms of the $\mathrm{SU}(2)_2^{\mathrm{sp.}}$ field $g$ and of the charge boson $\phi_c$ must be

$$h=(g_\uparrow\cos(\sqrt{\pi}\phi_c),g_\uparrow\sin(\sqrt{\pi}\phi_c),g_\downarrow\cos(\sqrt{\pi}\phi_c),g_\downarrow \times\sin(\sqrt{\pi}\phi_c)), \tag{47}$$

$$\bar{h}=(\bar{g}_\uparrow\cos(\sqrt{\pi}\bar{\phi}_c),\bar{g}_\uparrow\sin(\sqrt{\pi}\bar{\phi}_c),\bar{g}_\downarrow\cos(\sqrt{\pi}\bar{\phi}_c), -\bar{g}_\downarrow\sin(\sqrt{\pi}\bar{\phi}_c)). \tag{48}$$

The explicit expression for the order parameter $n_a=\mathrm{Tr}(\Gamma^a h\bar{h})$ in terms of the spin matrix field (28) and of the charge boson is then

$$n_1=\mathrm{Tr}\,(G)\cos(\sqrt{\pi}\theta), \tag{49a}$$

$$n_2=i\mathrm{Tr}\,(G\sigma_1)\sin(\sqrt{\pi}\Phi), \tag{49b}$$

$$n_3=i\mathrm{Tr}\,(G\sigma_2)\sin(\sqrt{\pi}\Phi), \tag{49c}$$

$$n_4=i\mathrm{Tr}\,(G\sigma_3)\sin(\sqrt{\pi}\Phi), \tag{49d}$$

$$n_5=-\mathrm{Tr}\,(G)\sin(\sqrt{\pi}\theta). \tag{49e}$$

We first notice that $n_{1,5}$ form a spin singlet and are the real and imaginary parts of the complex $d$-wave order parameter $\mathrm{Tr}\,(G)\exp(-i\sqrt{\pi}\theta)$, whereas $n_{2,3,4}$ form a spin triplet. We also see how the scaling dimensions of $n_{1,5}$ diverge from those of $n_{2,3,4}$ when $K_c$ is different from unity: the fields $\cos(\sqrt{\pi}\theta)$ and $\sin(\sqrt{\pi}\theta)$ have conformal dimensions $\Delta=\bar{\Delta}=1/(8K_c)$ while $\cos(\sqrt{\pi}\Phi)$ and $\sin(\sqrt{\pi}\Phi)$ have conformal dimensions $\Delta=\bar{\Delta}=K_c/8$. Thus

$$\Delta(n_1)=\Delta(n_5)=3/16+1/8K_c,$$

$$\Delta(n_2)=\Delta(n_3)=\Delta(n_4)=3/16+K_c/8. \tag{50}$$

We may also consider the SO(5) singlet $h_i\bar{h}_i$, which becomes simply $\mathrm{Tr}(G)\cos(\sqrt{\pi}\Phi)$ in this representation. This field is conjectured to be the charge-density-wave order parameter.[34] Within this model it has the same scaling dimension as the staggered magnetization (or spin density wave), but in a more general model with a spin gap the two fields could have different correlation lengths since spin singlet and triplet states would not necessarily have the same excitation energy.

To summarize, the components of the SO(5) order parameter have power-law correlations governed by the above conformal dimensions. When full SO(5) symmetry is present, $\Delta(\vec{n})=5/16$; and $\vec{n}$ is the vector primary field of the $\mathrm{SO}(5)_1$ WZW model. When $K_c\neq 1$, the SO(5) symmetry is broken and the staggered magnetization is less $(K_c>1)$ or more $(K_c<1)$ relevant than the $d$-wave order parameter. The behavior of the other possible fermion bilinears can be checked by a combination of Abelian bosonization and an Ising model representation of bosonic exponents—we find that their correlations decay exponentially as a result of the gap in the flavor sector (for details see the Appendix). Of course, since we are in one dimension, there are no real phase transitions, just enhanced fluctuations. Thus the fluctuations in the superspin channel are enhanced whereas other tendencies are suppressed. If a weak interladder coupling were added to form a two- or three-dimensional system, then a mean-field treatment would lead to a phase transition in the channel that has the highest susceptibility (i.e., the most fluctuations), i.e., an ordered phase for the most relevant operator. Thus, this approach predicts $d$-wave superconductivity for weakly coupled ladders with attractive effective interactions $(K_c>1)$. Finally, since the charge and spin sectors of this model are described by conformal field theories, one can also recover the finite-temperature behavior of the correlation functions in the standard way.[26]

### C. Lowest-energy states

In the absence of interactions and interchain coupling, the low-energy sector of the model is especially simple: the theory is a $\mathrm{SO}(8)_1$ WZW model. The states fall into two representations of the Kac-Moody algebra: that of the identity, which contains states with even charge, and that of the spinor (electron) field $R_\mu$, which contain states of odd charge. Remember, this is just a complicated way of representing noninteracting fermion excited states. As SO(8) is broken into $\mathrm{SO}(5)_1\otimes\mathrm{SU}(2)_2^{\mathrm{fl.}}$, these representations break into a finite number of representations of $\mathrm{SO}(5)_1\otimes\mathrm{SU}(2)_2^{\mathrm{fl.}}$, as indicated in Eq. (12). When the flavor sector is gapped, the low-lying states must all be flavor singlets and so many of those representations become irrelevant, in particular all the representations of odd charge coming from the spinor $R_\alpha$.

The only surviving Kac-Moody representation in the $\mathrm{SO}(5)_1$ theory is that of the identity. Such a representation contains an infinite number of energy levels, and at each level the states fall into SO(5) multiplets. In the pure WZW model (before spin-charge separation) the excited states may be obtained from the vacuum by applying ladder operators associated with the SO(5) currents. Let us explain: in a system of finite length $L$, the currents may be Fourier expanded as follows:

$$l^{ab}(x)=\sum_n e^{2\pi inx/L}l_n^{ab}, \tag{51}$$

where the sum runs over all integers (positive and negative). From the commutation relations of the currents, one may infer commutation relations for the modes $l_n^{ab}$ and show that, for $n<0$, $l_n^{ab}$ is a raising operator for the energy in the WZW model. Of course, the $l_0^{ab}$ are nothing but the SO(5) generators and allow us to navigate within a multiplet.

The multiplet content at each energy level may be easily obtained from the representation theory of Kac-Moody algebras, in particular by the method of affine characters.[26] Schematically, in the case at hand, the multiplet content may be expressed in terms of a spectrum-generating function $X(q)$:

$$\begin{aligned}X(q)&=\mathbf{1}+q\mathbf{10}+q^2(\mathbf{14}+\mathbf{10}+\mathbf{5}+\mathbf{1})\\&\quad+q^3(\mathbf{35}+\mathbf{14}+3\cdot\mathbf{10}+\mathbf{5}+\mathbf{1})+\cdots,\end{aligned} \tag{52}$$

where the coefficient of $q^\Delta$ indicates the multiplet content of states with conformal dimension $\Delta$. For instance, a term like $2q^3\cdot\mathbf{10}$ in $X(q)$ means that the multiplet $\mathbf{10}$ of SO(5) occurs twice with conformal dimension $\Delta=3$ in the right-moving sector. The full low-energy Hilbert space is a left-right product, encapsulated in the generating function $X(q)X(\overline{q})$. For instance, the term $Nq^\Delta\mathbf{10}\otimes\overline{q^{\bar{\Delta}}\mathbf{5}}$ stands for a left-right tensor product of multiplets, occurring $N$ times at the energy level $(2\pi v_F/L)(\Delta+\bar{\Delta})$, with momentum $(2\pi/L)(\Delta-\bar{\Delta})$ ($v_F$ is the common spin and charge velocity before spin-charge separation).

The eigenvalues of $S_z$ and $Q$ and the energy of each state in the right-moving sector may be encoded in a more general spectrum-generating function $X(q,x,y)$:

$$X(q,x,y)=\sum_{\text{states}} q^\Delta x^{2S_z}y^Q. \tag{53}$$

The advantage of spectrum-generating functions is that tensor products translate into ordinary products of functions, and direct sums into ordinary sums. Anticipating spin-charge separation, it is possible to write the function $X(q.x.y)$ as a combination of spin-charge products:

$$X(q,x,y)=X_{\mathrm{sp}}^{(0)}(q,x)X_{\mathrm{c}}^{(0)}(q,y)+X_{\mathrm{sp}}^{(2)}(q,x)X_{\mathrm{c}}^{(2)}(q,y), \tag{54}$$

where $X_{\mathrm{sp}}^{(j)}(q,x)$ is the spectrum-generating function for the spin-$j$ Kac-Moody representation of $\mathrm{SU}(2)_2$ and $X_{\mathrm{c}}^{(0,2)}$ is the analog for the charge sector. The lowest terms of these functions are

$$\begin{aligned}X_{\mathrm{sp}}^{(0)}(q,x)&=1+q(1+x^2+x^{-2})\\&\quad+q^2(3+2x^2+2x^{-2}+x^4+x^{-4})+\cdots,\\X_{\mathrm{sp}}^{(2)}(q,x)&=q^{1/2}(1+x^2+x^{-2})+q^{3/2}(2+x^2+x^{-2})\\&\quad+q^{5/2}(4+3x^2+3x^{-2}+x^4+x^{-4})+\cdots,\\X_{\mathrm{c}}^{(0)}(q,y)&=1+q+q^2(2+y^4+y^{-4})\\&\quad+q^3(3+y^4+y^{-4})+\cdots,\\X_{\mathrm{c}}^{(2)}(q,y)&=q^{1/2}(y^2+y^{-2})+q^{3/2}(y^2+y^{-2})\\&\quad+q^{5/2}(2y^2+2y^{-2})+\cdots.\end{aligned} \tag{55}$$

Again, the exponent of $x$ is twice the value of $S_z$ and that of $y$ is the charge $Q$. A term like $4x^2y^{-2}q^3$ in Eq. (54) would stand for four states with $\Delta=3$, $S_z=1$, and $Q=-2$. The charge states represented in $X_{\mathrm{c}}^{(0)}$ have charge $Q=0$ (modulo 4) and those in $X_{\mathrm{c}}^{(2)}$ have charge $Q=2$ (modulo 4). From the above expressions and relation (54), the full spectrum of energies and quantum numbers may be recovered. Of course, we must consider the left-right product $X(q,x,y)X(\overline{q},x,y)$. That the expression (54) is a sum of products, instead of being a simple product of spin and charge factors, means that one cannot consider the charge and spin spectra independently: there are ‘‘glueing conditions’’ between charge and spin states, conditions encoded in Eq. (54).

When spin and charge separate, the energy levels shift in two ways. First, because of different spin and charge velocities, $X_{\rm sp}^{(j)}(q,x)$ and $X_{\rm c}^{(n)}(q,y)$ become, respectively, $X_{\rm sp}^{(j)}(q_s,x)$ and $X_{\rm c}^{(n)}(q_c,y)$, where $q_s=q^{v_s/v_F}$ and $q_c=q^{v_c/v_F}$. Second, anomalous charge exponents change the conformal dimensions in the charged sector, the structure of which deserves a more detailed explanation: excited states in the charge sector may be obtained either (i) by applying the creation operators associated with the charge boson $\phi_c$ (this does not change the charge $Q$) or (ii) by applying exponentials $\exp(iQ\sqrt{\pi}\phi_c)$ on the vacuum, where $Q$ is the charge thus created. The generating functions in the charge sector may be expressed as

$$X_{\rm c}^{(\ell)}(q,y)=X_{\rm bos.}(q)\sum_{Q=4r+\ell} q^{Q^2/8}y^Q, \tag{56}$$

where $r$ runs over all integers, $\ell=0$ or 2, and $X_{\rm bos.}(q)$ is the spectrum generating function associated with the boson creation operators only:

$$X_{\rm bos.}(q)=\prod_{r=1}^{\infty}\frac{1}{1-q^r}=1+q+2q^2+3q^3+5q^4+7q^5+\cdots. \tag{57}$$

When $K_c$ changes from its initial value of unity, left and right boson creation operators mix through some Bogoliubov transformation and the conformal dimensions associated with the exponentials of $\phi_c$ and $\overline{\phi}_c$ become

$$\Delta(Q,\overline{Q})=\frac{1}{32}\left[\frac{1}{\sqrt{K_c}}(Q+\overline{Q})+\sqrt{K_c}(Q-\overline{Q})\right]^2,$$

$$\overline{\Delta}(Q,\overline{Q})=\frac{1}{32}\left[\frac{1}{\sqrt{K_c}}(Q+\overline{Q})-\sqrt{K_c}(Q-\overline{Q})\right]^2. \tag{58}$$

Left-right products of spectrum-generating functions in the charge sector then become

$$X_{\rm c}^{(\ell)}(q_c,y)X_{\rm c}^{(\ell')}(\overline{q}_c,y)=X_{\rm bos.}(q_c)X_{\rm bos.}(\overline{q}_c)\times\sum_{\substack{Q=4r+\ell\\ \overline{Q}=4r'+\ell'}} aq^{\Delta(Q,\overline{Q})}\overline{q}^{\overline{\Delta}(Q,\overline{Q})}y^{Q-\overline{Q}}. \tag{59}$$

The above expression, combined with Eqs. (54,55,58) allows us to extract the energy, momentum, charge, and spin of the whole low-energy sector for arbitrary values of $v_s$, $v_c$, and $K_c$.

Let us consider, for instance, the first excited states. According to Eq. (52), they fall into the multiplet **10** of SO(5). The spin and charge content of such a multiplet is easily read from the corresponding $S_z-Q$ diagram of Fig. 1. The multiplet **10** consists of three spin triplets (of charge $-2$, 0, and 2, respectively), plus a neutral spin singlet. After spin-charge separation and if $K_c\neq 1$, the contribution of this multiplet to the spectrum-generating function is, according to Eqs. (54,55),

$$q_c+q_s(1+x^2+x^{-2})+q_s^{1/2}q_c^{\Delta(2,0)}\overline{q}_c^{\overline{\Delta}(2,0)}(1+x^2+x^{-2})(y^2+y^{-2}). \tag{60}$$

Thus, the energies of these states split in the following fashion:

$$E(Q=0,j=0)=\frac{2\pi v_c}{L},$$

$$E(Q=0,j=1)=\frac{2\pi v_s}{L},$$

$$E(Q=\pm 2,j=1)=\frac{\pi v_s}{L}+\frac{\pi v_c}{2L}\left(K_c+\frac{1}{K_c}\right). \tag{61}$$

The last of these states are in fact created by applying $\pi$ operators [see Eq. (34)]. The energy levels are proportional to the scaling dimensions of the operators in the conformal field theory.[26]

To conclude, the eigenstates, in particular the lowest-energy states, fall into deformed SO(5) multiplets. The amount of deformation is exactly determined by the renormalized charge velocity $v_c$ and anomalous charge exponent $K_c$.

## VI. LUTTINGER LIQUID CASE

As we mentioned earlier, the case of two coupled Luttinger liquids is different from that of two Luttinger models. Let us consider, as an example of a Luttinger liquid, the one-chain Hubbard model at weak coupling $U\ll t$:

$$H_{\rm Hub}=-t\sum_{r,\alpha}(c_{r,\alpha}^\dagger c_{r+1,\alpha}+c_{r+1,\alpha}^\dagger c_{r,\alpha})+U\sum_r n_{r,\uparrow}n_{r,\downarrow}. \tag{62}$$

If we linearize about the right and left Fermi points as in Eq. (4), and use the charge currents (3) and the corresponding spin currents

$$\mathbf{j}_R=\frac{1}{2}\sum_{\alpha,\beta}R_\alpha^\dagger\boldsymbol{\sigma}_{\alpha\beta}R_\beta,\quad \mathbf{j}_L=\frac{1}{2}\sum_{\alpha\beta}L_\alpha^\dagger\boldsymbol{\sigma}_{\alpha\beta}L_\beta, \tag{63}$$

we find the Hamiltonian density ($v_F\sim ta_0$):

$$\mathcal{H}_{\rm Hub}\approx -iv_F\sum_\alpha(R_\alpha^\dagger\partial_x R_\alpha-L_\alpha^\dagger\partial_x L_\alpha)+\frac{U}{4}(j_R^2+j_L^2)+\frac{U}{2}j_Rj_L-\frac{U}{3}(\mathbf{j}_R^2+\mathbf{j}_L^2)-2U\mathbf{j}_R\cdot\mathbf{j}_L. \tag{64}$$

This Hamiltonian is not equivalent to a Tomonaga Luttinger model because the last two terms are not pure density-density interactions. The $(\mathbf{j}_R^2+\mathbf{j}_L^2)$ term will only renormalize the spin velocity $v_s$ and is not very important. However, the marginally irrelevant $\mathbf{j}_R\cdot\mathbf{j}_L$ term cannot simply be absorbed in this way. It is this term that gives rise to logarithmic corrections to the correlation functions in Luttinger liquids—although otherwise it does not drastically change their properties, hence the utility of the Luttinger liquid concept.

As this weak-coupling bosonization suggests, the coupling constant of this term is of the same order as the other couplings in the theory, so it cannot necessarily be jettisoned when we consider more complicated models such as the two-chain ladder. Let us therefore consider as a model for the generic Luttinger liquid two-chain ladder, the Hamiltonian (1) perturbed by marginally irrelevant spin current interactions in each chain:

$$\mathcal{H}_{\text{liq}}=\mathcal{H}+\mathcal{H}_{\text{marg}},$$

$$\mathcal{H}_{\text{marg}}=-\lambda(\mathbf{j}_{R1}\cdot\mathbf{j}_{L1}+\mathbf{j}_{R2}\cdot\mathbf{j}_{L2}), \tag{65}$$

where $\lambda>0$, and $\mathbf{j}_{Ri},\mathbf{j}_{Li}$ are the right and left moving spin currents in chains $i=1,2$. It is instructive to write the perturbation in terms of the Majorana (vector) fermions $\xi_i$. We find

$$\mathcal{H}_{\text{marg}}=-\frac{\lambda}{2}\left(\mathbf{J}_R\cdot\mathbf{J}_L-\sum_{i=1,2,3}(\xi_i^s\overline{\xi_i^s})\xi_3^f\overline{\xi_3^f}\right). \tag{66}$$

The first term is a marginally irrelevant interaction in the total spin sector [$\mathbf{J}_R=\mathbf{j}_{R1}+\mathbf{j}_{R2}$ as defined in Eq. (6)]. But it is the second term that is most significant. It couples the fermions of the spin sector ($\xi_i^s$=right moving, $\overline{\xi_i^s}$=left moving, $i=1,2,3$) to one of the fermions in the flavor sector. So the spin and flavor sectors are no longer genuinely decoupled.

Suppose that the flavor sector becomes gapped. Then

$$\langle\xi_3^f\overline{\xi_3^f}\rangle\neq 0. \tag{67}$$

To first approximation, we can then replace $\xi_3^f\overline{\xi_3^f}$ in Eq. (66) by its expectation value, and we see that the effect of a gap in the flavor sector is to generate a mass term for the fermions of the spin sector—a spin gap. This is a crude argument but it is borne out by the RG analysis of Refs. 17,13 as well as numerical work[20] (in the notation of Ref. 17 a finite backscattering corresponds to $g_i^{(1)}\neq 0$) that shows the existence of strong-coupling regimes with a spin gap in a model of two Hubbard chains coupled by a small hopping. In general it is hard to estimate the size of this gap. If it is large, the-low energy physics will be as described in Refs. 17,20. It could be, however, that in some models (with small $\lambda$, for example) the spin gap is very small, in which case for intermediate energy scales the behavior will still be described approximately by the model (1).

## VII. CONCLUSIONS

In this paper we have studied a system of two TL models coupled by a small interchain hopping. We have shown that this critical (at $T=0$) theory can be represented much more symmetrically than in the standard Abelian bosonization representation as an $SO(5)_1$ WZW model, or equivalently as a system of five Majorana fermions, perturbed by symmetry-breaking interactions. We have obtained the correlations of fermion bilinears in this theory and demonstrated that the components of the ''superspin''[1] have power-law correlations, and are enhanced, while other tendencies are suppressed. Conformal field theory allows us to obtain the exact energy levels in a finite size system and observe how the degeneracy of the SO(5) multiplets is broken by spin-charge separation ($v_c\neq v_s$) and the presence of an anomalous exponent ($K_c\neq 1$) in the charge sector. Except in the trivial noninteracting case, there is no exact SO(5) symmetry. In Sec. VI we showed briefly how the inclusion of backscattering results in the appearance of a spin gap.

In light of these results, Zhang has recently shown[34] that a whole class of ladder systems with more general interactions have Hamiltonians with microscopic SO(5) symmetry. Knowing the Hamiltonian does not tell us about the strong-coupling behavior at low energies. But a continuous symmetry like SO(5) cannot be spontaneously broken in one dimension and must therefore be present in the low-energy theory as well. The latter must be described by a SO(5) WZW model, perturbed by various primary fields, perhaps with a critical point or line in the space of coupling constants (a conformal field theory with Lie-group symmetry is necessarily a WZW model). This is the simplest class of low-energy theories with SO(5) symmetry in 1+1 dimensions.

The SO(5) symmetric description of ladder models, which are clearly related to popular models of the cuprates, and the similarity of the form of the theory in ladders to that proposed by S.C. Zhang[1] for the two-dimensional Cuprates, is certainly encouraging and suggestive. Nonetheless, in view of the many special features of one-dimensional theories we are cautious about drawing more general conclusions.

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## APPENDIX: ABELIAN BOSONIZATION OF FERMION BILINEARS

We want to know the long-distance (low-energy) behavior of the correlations of various bilinears. In Sec. V we demonstrated that the correlations of the ''superspin'' order parameter could be deduced within the framework of non-Abelian bosonization from a careful analysis of the operators of the conformal field theory. We can further justify this analysis and find the behavior of the other fermion bilinears explicitly by using Abelian bosonization.

We introduce an Abelian boson $\Phi_\alpha^i$ for each species of fermion $i=1,2$, $\alpha=\uparrow,\downarrow$. Then we introduce linear charge and spin, bonding and antibonding combinations of these fields:

$$\Phi_\alpha^{\pm}=\frac{1}{\sqrt{2}}(\Phi_\alpha^1\pm\Phi_\alpha^2),$$

$$\Phi_{c,s}^i=\frac{1}{\sqrt{2}}(\Phi_\uparrow^i\pm\Phi_\downarrow^i). \tag{A1}$$

If one carefully applies Abelian bosonization to the original Hamiltonian (1), taking full account of the anticommutation factors,[32] one can identify each of these Bose fields with two

TABLE I. Fermion bilinears.

| | Operator | $X$ | $\Delta=\bar{\Delta}$ |
|---|---|---|---|
| (a) | $R^\dagger_{1\uparrow}L^\dagger_{1\downarrow}$ | $\Phi_s^+-\theta_c^+-\Phi_s^--\theta_c^-$ | |
| (b) | $R^\dagger_{1\uparrow}L^\dagger_{2\downarrow}$ | $\Phi_s^+-\theta_c^+-\theta_s^-+\Phi_c^-$ | $\frac{3}{16}+\frac{1}{8K_c}$ |
| (c) | $R^\dagger_{1\uparrow}L_{1\downarrow}$ | $\Phi_c^+-\theta_s^++\Phi_c^--\theta_s^-$ | $\frac{3}{16}+\frac{K_c}{8}$ |
| (d) | $R^\dagger_{1\uparrow}L_{1\uparrow}$ | $\Phi_c^++\Phi_s^++\Phi_c^-+\Phi_s^-$ | $\frac{3}{16}+\frac{K_c}{8}$ |
| (e) | $R^\dagger_{1\uparrow}L_{2\downarrow}$ | $\Phi_c^+-\theta_s^+-\theta_c^-+\Phi_s^-$ | |
| (f) | $R^\dagger_{1\uparrow}L_{2\uparrow}$ | $\Phi_c^++\Phi_s^+-\theta_s^--\theta_c^-$ | |

of the Majorana fermions introduced in Sec. IV. The $\Phi_c^+$ field is simply associated with the two charge fermions, the $\Phi_c^-$ field represents two of the flavor fermions. $\Phi_s^+$ represents two of the spin fermions, and $\Phi_s^-$ comprises one flavor fermion and one spin fermion.

After bosonization, the various fermion bilinears have the form

$$\hat{O}\sim e^{-i\sqrt{\pi}X}, \tag{A2}$$

where the different $X$'s are given in Table I. We will briefly describe how we arrive at the long-distance behavior of their correlations. We find straightforwardly

$$\Delta(e^{\pm i\sqrt{\pi}\Phi_s^+})=\Delta(e^{\pm i\sqrt{\pi}\theta_s^+})=\frac{1}{8},$$

$$\Delta(e^{\pm i\sqrt{\pi}\Phi_c^+})=\frac{K_c}{8},$$

$$\Delta(e^{\pm i\sqrt{\pi}\theta_c^+})=\frac{1}{8K_c}. \tag{A3}$$

(The scaling dimension is $D=\Delta+\bar{\Delta}$ and here $\Delta=\bar{\Delta}$ so $D=2\Delta$.) But the charge and spin $(-)$ fields are a little more subtle. In our model, the flavor sector acquires a gap. Since we start with $K_c<1$ this strong coupling regime corresponds to the limit $K_c\to 0$ in the c$-$ sector, whence to leading approximation we can replace the complex exponents by their expectation values:

$$\langle e^{i\sqrt{\pi}\Phi_c^-}\rangle\neq 0,$$

$$\langle e^{i\sqrt{\pi}\theta_c^-}\rangle=0. \tag{A4}$$

Higher-order corrections will die away exponentially. Thus the bilinears (a), (e), and (f) in Table I die away exponentially and the corresponding tendencies are suppressed.

The exponents of $\Phi_s^-$ are a little more subtle since we know that only one of the Majorana fermions to which $\Phi_s^-$ corresponds is gapped, while the other remains gapless. Here, however, we can make use of their representation in terms of the corresponding Ising order and disorder operators.[35–37] Introducing Ising order and disorder operators $\sigma_f,\mu_f$ corresponding to the Majorana flavor fermion $\xi_3^f$ and $\sigma_s,\mu_s$ corresponding to the spin fermion $\xi_s^3$, we can identify the following approximate operator correspondences:

$$\cos\sqrt{\pi}\Phi_s^-\sim\sigma_f\sigma_s,$$

$$\sin\sqrt{\pi}\Phi_s^-\sim\mu_f\mu_s,$$

$$\cos\sqrt{\pi}\theta_s^-\sim\sigma_f\mu_s,$$

$$\sin\sqrt{\pi}\theta_s^-\sim\mu_f\sigma_s. \tag{A5}$$

When the flavor fermion becomes gapped, either $\langle\sigma_f\rangle=0$, $\langle\mu_f\rangle\neq 0$ or $\langle\sigma_f\rangle\neq 0$, $\langle\mu_f\rangle=0$, depending upon the definitions. Thus to first approximation these can again be replaced by their expectation values. The Ising model corresponding to the spin fermion $\xi_s^3$ remains critical and has scaling dimensions $\Delta=\bar{\Delta}(\sigma_s,\mu_s)=1/16$. In this way we obtain the long-distance asymptotics shown in Table I. The only bilinears that still have power-law correlations are the interchain pairing (39), represented by (b) in Table I, and the staggered magnetization (37), represented by (c) and (d). Thus, it is precisely the components of the unified order parameter $n_a$ that have power-law correlations, while all the other tendencies around $\pm 2k_F$ are suppressed. Note that the scaling dimensions that we find agree with those found in Sec. V from non-Abelian bosonization [cf. Eq. (50)].

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[1] Shou-Cheng Zhang, Science **275**, 1089 (1997).
[2] E.C. Demler and S.C. Zhang, Phys. Rev. Lett. **75**, 4126 (1995).
[3] Martin Greiter, Phys. Rev. Lett. **79**, 4898 (1997); E. Demler, S.C. Zhang, S. Meixner, and W. Hanke, Phys. Rev. Lett. **79**, 4937 (1997); Martin Greiter, cond-mat/9705282 (unpublished).
[4] G. Baskaran and P.W. Anderson, cond-mat/9706076 (unpublished).
[5] S. Rabello, H. Kohno, E. Demler, and S.C. Zhang, Phys. Rev. Lett. **80**, 3586 (1998); C.L. Henley, *ibid.* **80**, 3590 (1998).
[6] E. Dagotto and T.M. Rice, Science **271**, 618 (1996).
[7] U. Uehara, T. Nagata, J. Akimitsu, H. Takahashi, N. Mori, and K. Kinoshita, J. Phys. Soc. Jpn. **65**, 2764 (1996).
[8] M. Isobe *et al.*, Physica C **282**, 811 (1997).
[9] E. Dagotto, J. Riera, and D. Scalapino, Phys. Rev. B **45**, 5744 (1992).
[10] T.M. Rice, S. Gopalan, and M. Sigrist, Europhys. Lett. **23**, 445 (1993).
[11] A.M. Finkel'stein and A. I. Larkin, Phys. Rev. B **47**, 10 461 (1993).
[12] D.V. Khveshchenko and T.M. Rice, Phys. Rev. B **50**, 252 (1994).
[13] L. Balents and M.P.A. Fisher, Phys. Rev. B **53**, 12 133 (1996); H-H. Lin, L. Balents, M.P.A. Fisher, *ibid.* **56**, 6569 (1997).
[14] H.J. Schulz, Phys. Rev. B **53**, R2959 (1996).
[15] M. Fabrizio, A. Parola, and E. Tossatti, Phys. Rev. B **46**, 3159 (1992).
[16] M. Fabrizio, A. Parola, Phys. Rev. Lett. **70**, 226 (1992).
[17] M. Fabrizio, Phys. Rev. B **48**, 15 838 (1993).
[18] T. Giamarchi, Phys. Rev. B **44**, 2905 (1991).
[19] R.M. Noack, S.R. White, and D.J. Scalapino, Phys. Rev. Lett. **73**, 882 (1997).
[20] D.G. Shelton and A.M. Tsvelik, Phys. Rev. B **53**, 14 036 (1996).

[21] J. Voit, Rep. Prog. Phys. **57**, 977 (1994).
[22] A. M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).
[23] A.B. Zamolodchikov and V.A. Fateev, Sov. J. Nucl. Phys. **43**, 657 (1986).
[24] *Bosonization*, edited by M. Stone (World Scientific, Singapore, 1994).
[25] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B **247**, 83 (1984).
[26] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, Berlin, 1997). Representations of Kac-Moody algebras are discussed in Chap. 14.
[27] A.M. Tsvelik, Phys. Rev. B **42**, 10 499 (1990).
[28] E. Witten, Commun. Math. Phys. **92**, 455 (1984).
[29] D.J. Scalapino, S.-C. Zhang, and W. Hanke, Phys. Rev. B **57**, 13 781 (1998).
[30] S. Coleman, Phys. Rev. D **11**, 2088 (1975).
[31] S. Mandelstam, Phys. Rev. D **11**, 3026 (1975).
[32] T. Banks, D. Horn, and H. Neuberger, Nucl. Phys. B **108**, 119 (1976).
[33] F. Woynarovich, J. Phys. A **22**, 4243 (1989); H. Frahm and V.E. Korepin, Phys. Rev. B **42**, 10 553 (1990).
[34] Shou-Cheng Zhang (private communication).
[35] D.G. Shelton, A.A. Nersesyan, and A.M. Tsvelik, Phys. Rev. B **53**, 8521 (1996).
[36] M. Sato, T. Miwa, and M. Jimbo, *Holonomic Quantum Fields V* (Kyoto University Press, Kyoto, 1980), Vol 16, p. 531.
[37] *Statistical Field Theory*, C. Itzykson and J.-M. Drouffe (Cambridge University Press, Cambridge, 1989), Vol. 1.