

## Reflection coefficient and localization length of waves in one-dimensional random media

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We develop a powerful method of exactly calculating various transport characteristics of waves in one-dimensional random media with (or without) coherent absorption or amplification. Using the method, we compute the probability densities of the reflectance and of the phase of the reflection coefficient, together with the localization length, of electromagnetic waves in sufficiently long random dielectric media. We find substantial differences between our exact results and the previous results obtained using the random phase approximation (RPA). The probability density of the phase of the reflection coefficient is highly nonuniform when either disorder or absorption (or amplification) is strong. The probability density of the reflectance when the absorption or amplification parameter is large is also quite different from the RPA result. We prove that the probability densities in the amplifying case are related to those in the absorbing case with the same magnitude of the imaginary part of the dielectric permeability by exact dual relationships. From the analysis of the average reflectance that shows a nonmonotonic dependence on the absorption or amplification parameter, we obtain a useful criterion for the applicability of the RPA. In the parameter regime where the RPA is invalid, we find that the exact localization length is substantially larger than the RPA localization length. [S0163-1829(98)07034-9]

### I. INTRODUCTION

Propagation and localization of waves of various kinds in one-dimensional disordered media have been studied intensively for several decades. Examples of particular interest are *classical* electromagnetic wave propagation in random dielectric media<sup>1,2</sup> and *quantum* electron transport in disordered solids.<sup>3-5</sup> Much recent attention has been paid to the problem of wave propagation in coherently absorbing or amplifying random media.<sup>6-15</sup> It has been demonstrated that the wave is more strongly localized in both types of media than in unitary (or elastic) random media with no absorption and amplification.

A large number of previous studies have been based on the so-called random phase approximation (RPA), where it is assumed that the phase of the reflected wave relative to that of the incident wave is uniformly and randomly distributed over all angles. One can easily prove that this approximation is equivalent to assuming large wave energy or weak disorder and, at the same time, weak absorption or amplification. When these conditions are not met, an exact calculation beyond the random phase approximation is required.

Recently, several authors have performed numerical calculations of the probability densities of the transmittance, the reflectance, and the phase of the complex reflection coefficient in absorbing, amplifying, and unitary random media using methods that go beyond the random phase approximation.<sup>7,14-17</sup> Their results clearly indicate that the random phase approximation can often lead to qualitatively wrong conclusions on the behavior of waves in random media. Above all, it has been demonstrated that the distribution of the phase of the reflection coefficient is generally nonuniform in all of absorbing, amplifying, and unitary media. The distribution of the reflectance is also quite different from the RPA result. For example, the average reflectance in the RPA is a monotonically decreasing function of the absorption pa-

rameter, whereas the exact average reflectance is a nonmonotonic one.<sup>7</sup>

In the present work, we develop a powerful method of *exactly* calculating various disorder-averaged quantities including the reflectance, the transmittance, and their probability densities in absorbing, amplifying, and unitary media. We also calculate the probability density of the phase of the reflection coefficient and the localization length exactly. In addition, we derive a couple of exact dual relationships that relate the probability densities in the absorbing case to those in the amplifying case with the same magnitude of the imaginary part of the dielectric permeability. From the quantitative analysis of the average reflectance in the absorbing case, we obtain a useful criterion for the applicability of the RPA.

The outline of the paper is as follows. In the next section, we introduce the model. In Sec. III, we describe our method of calculating the probability densities and the localization length in detail. We also prove the exact dual relationships that relate the probability densities in the amplifying case to those in the absorbing case. Results of our calculation are presented in Sec. IV. Finally, we summarize the paper in Sec. V.

### II. MODEL

We are interested in the propagation of a monochromatic electromagnetic wave of frequency  $\omega$  in disordered dielectric media. For the sake of simplicity, we consider the one-dimensional case, where the dielectric permeability  $\epsilon$  varies only in one direction in space and the wave propagates in the same direction. We take this direction as the  $z$  axis and assume the medium lies in  $0 \leq z \leq L$ . Then the complex amplitude of the electric field  $E$  satisfies the Helmholtz equation with wave number  $k = \omega/c$ , where  $c$  is the speed of light in a vacuum,

$$\left[ \frac{d^2}{dz^2} + k^2 \epsilon(z) \right] E(z) = 0. \quad (1)$$

The dielectric permeability  $\epsilon(z)$  is equal to 1 for  $z < 0$  and  $z > L$  and  $\epsilon(z) = 1 + a + \delta\epsilon(z) + i\gamma$  for  $0 \leq z \leq L$ , where  $a$  and  $\gamma$  are assumed to be real constants and  $\delta\epsilon(z)$  is a real random function of  $z$ . We suppose  $\delta\epsilon(z)$  to be a Gaussian random function with zero mean and a white-noise spectrum:

$$\langle \delta\epsilon(z) \delta\epsilon(z') \rangle = g \delta(z - z'), \quad \langle \delta\epsilon(z) \rangle = 0, \quad (2)$$

where  $\langle \dots \rangle$  denotes an average over disorder and the constant  $g$  is a measure of the strength of randomness. The imaginary part of the dielectric permeability,  $\gamma$ , makes the medium absorb ( $\gamma > 0$ ) or amplify ( $\gamma < 0$ ) the wave without destroying its phase coherence. For simplicity,  $\gamma$  is assumed to be uniform. The constant  $1 + a$  is the disorder average of the real part of the dielectric permeability and, in general, is not equal to 1. Our method is applicable to the cases where  $a$  is an arbitrary nonrandom function, e.g., a periodic function, of  $z$ , but in this paper, we restrict our attention to the case where  $a$  is a constant.

The above model is also relevant to the electron transport problem in disordered quasi-one-dimensional solids. In that case, the electron wave function  $\psi(z)$  plays the same role as  $E(z)$  and  $\epsilon(z)$  is replaced by  $1 - U(z)/E_0$ , where  $U(z)$  is the random potential experienced by electrons and  $E_0$  is the kinetic energy of incident electrons.

### III. METHOD

We consider a plane wave of unit magnitude  $E(z) = e^{ik(L-z)}$  incident on the medium from the right. The quantities of main interest are the complex reflection and transmission coefficients  $r = r(L)$  and  $t = t(L)$  defined by the wave function outside the medium:

$$E(z) = \begin{cases} e^{ik(L-z)} + r(L)e^{ik(z-L)}, & z > L, \\ t(L)e^{-ikz}, & z < 0. \end{cases} \quad (3)$$

Using the so-called *invariant imbedding* method,<sup>18,19</sup> we derive exact differential equations for  $r$  and  $t$  with respect to  $L$ :

$$\begin{aligned} \frac{dr(L)}{dL} &= 2ikr(L) + \frac{ik}{2} [a + i\gamma + \delta\epsilon(L)][1 + r(L)]^2, \\ r(L=0) &= 0, \end{aligned} \quad (4)$$

$$\frac{dt(L)}{dL} = ikt(L) + \frac{ik}{2} [a + i\gamma + \delta\epsilon(L)][1 + r(L)]t(L),$$

$$t(L=0) = 1. \quad (5)$$

These stochastic differential equations can be used in calculating the disorder averages of various physical quantities consisting of  $r$  ( $\equiv \sqrt{R}e^{i\theta}$ ) and  $t$  ( $\equiv \sqrt{T}e^{i\phi}$ ), where the reflectance  $R = |r|^2$  and the transmittance  $T = |t|^2$  as well as the phases  $\theta$  and  $\phi$  are functions of  $L$ . In the present work, we are mainly interested in computing the *exact* probability densities  $P_R(R)$  and  $P_\theta(\theta)$  in semi-infinite ( $L \rightarrow \infty$ ) absorbing and amplifying random dielectric media. We will also compute the *exact* localization length  $\xi$  of the wave in absorbing random media defined by

$$\lim_{L \rightarrow \infty} \langle \ln T \rangle = -L/\xi. \quad (6)$$

The unitary case with no absorption or amplification ( $\gamma = 0$ ) is also of great interest. In that case, the probability density  $P_R(R)$  in the  $L \rightarrow \infty$  limit is clearly equal to  $\delta(R - 1)$  regardless of the strength of disorder  $g$ . But the probability density  $P_\theta(\theta)$  and the localization length  $\xi$  depend on  $g$  in a nontrivial manner and will be studied in the present work.

#### A. Probability distribution of the reflectance

We obtain the probability density  $P_R(R)$  from the moments  $\langle R^n \rangle$  for all integers  $n$ . An infinite number of coupled nonstochastic ordinary differential equations satisfied by these moments are obtained using Eq. (4) and Novikov's formula.<sup>20</sup> It turns out that in order to compute  $\langle R^n \rangle = \langle r^n r^{*n} \rangle$ , one needs to compute the moments  $Z_{n\tilde{n}} \equiv \langle r^n r^{*\tilde{n}} \rangle$  for all integers  $n$  and  $\tilde{n}$ . In other words, the moments  $Z_{n\tilde{n}}$  with  $n = \tilde{n}$  are coupled to  $Z_{n\tilde{n}}$  with  $n \neq \tilde{n}$ . Using the notation  $l = L/\xi_0$ ,  $C = k\xi_0$ ,  $\alpha = k\xi_0 a$ , and  $\beta = k\xi_0 \gamma$ , where  $\xi_0 = 4/gk^2$  is the localization length with no absorption or amplification in the random phase approximation, we obtain the *nonrandom* equation satisfied by  $Z_{n\tilde{n}}$ :

$$\begin{aligned} \frac{dZ_{n\tilde{n}}}{dl} &= [i(2C + \alpha)(n - \tilde{n}) - \beta(n + \tilde{n}) + 4n\tilde{n} - 3n^2 - 3\tilde{n}^2]Z_{n\tilde{n}} + n[\frac{1}{2}(i\alpha - \beta) - 2n + 2\tilde{n} - 1]Z_{n+1,\tilde{n}} \\ &+ n[\frac{1}{2}(i\alpha - \beta) - 2n + 2\tilde{n} + 1]Z_{n-1,\tilde{n}} - \tilde{n}[\frac{1}{2}(i\alpha + \beta) - 2n + 2\tilde{n} + 1]Z_{n,\tilde{n}+1} - \tilde{n}[\frac{1}{2}(i\alpha + \beta) - 2n + 2\tilde{n} - 1]Z_{n,\tilde{n}-1} \\ &+ n\tilde{n}Z_{n+1,\tilde{n}+1} + n\tilde{n}Z_{n-1,\tilde{n}-1} + n\tilde{n}Z_{n+1,\tilde{n}-1} + n\tilde{n}Z_{n-1,\tilde{n}+1} - \frac{1}{2}n(n+1)Z_{n+2,\tilde{n}} - \frac{1}{2}n(n-1)Z_{n-2,\tilde{n}} - \frac{1}{2}\tilde{n}(\tilde{n}+1)Z_{n,\tilde{n}+2} \\ &- \frac{1}{2}\tilde{n}(\tilde{n}-1)Z_{n,\tilde{n}-2}, \end{aligned} \quad (7)$$

with the conditions  $Z_{00}=1$  and  $Z_{n\tilde{n}}(l=0)=0$  for  $n, \tilde{n}>0$ . The random phase approximation applies to the case where  $C \gg 1$  and  $C \gg \beta$ , that is,  $\gamma \ll 1$ . Then it is clear that  $Z_{n\tilde{n}}$  with  $n \neq \tilde{n}$  can be neglected. This leads to the much simpler equation for  $Z_n \equiv Z_{nn} = \langle R^n \rangle$  obtained previously in Ref. 6:

$$\frac{dZ_n}{dl} = -2n(n+\beta)Z_n + n^2Z_{n+1} + n^2Z_{n-1}, \quad (8)$$

with the conditions  $Z_0=1$  and  $Z_{n>0}(l=0)=0$ . In this paper, we go beyond the random phase approximation and solve the exact equation (7) directly.

We consider the absorbing ( $\beta>0$ ) case first. When  $\beta$  is positive, the moments  $Z_{n\tilde{n}}$  with  $n, \tilde{n} \geq 0$  are coupled to one another and are well behaved for all  $l$ . Furthermore, the magnitude of the moment  $Z_{n\tilde{n}}$  decays (more rapidly for larger  $\beta$  values) as either  $n$  or  $\tilde{n}$  increases. Based on this crucial observation, we solve the infinite number of coupled equations, Eq. (7), by a simple truncation method, which was developed and applied successfully to the problem of computing the electronic properties of quasi-one-dimensional Peierls systems by the author and collaborators in previous publications.<sup>21,22</sup> We assume  $Z_{n\tilde{n}}=0$  for either  $n$  or  $\tilde{n}$  greater than some large positive integer  $N$  and solve a *finite* number [ $=N(N+2)$ ] of coupled equations numerically for given values of  $C, \alpha$ , and  $\beta$ . We increase the cutoff  $N$ , repeat a similar calculation, and then compare the newly obtained  $Z_{n\tilde{n}}$  with the value of the previous step. If there is no change in the values of  $Z_{n\tilde{n}}$  within an allowed numerical error, we conclude that we have obtained the exact solution of  $Z_{n\tilde{n}}$ . In the present work, we will limit our attention further and consider only the  $l \rightarrow \infty$  limit, where we expect  $dZ_{n\tilde{n}}/dl=0$ . Then the left-hand side of Eq. (7) vanishes and we have a set of coupled *algebraic* equations. We solve these equations by the truncation method described above to find  $\langle R^n \rangle$  for every integer  $n>0$ .

It is possible to get the probability density  $P_R(R)$  from the moments by several different methods. We find it is efficient to use the expansion of  $P_R(R)$  in terms of the shifted Legendre polynomials:

$$P_R(R) = \sum_{m=0}^{\infty} (2m+1) \langle P_m^*(R) \rangle P_m^*(R), \quad (9)$$

where  $P_m^*(R)$  is the shifted Legendre polynomial of order  $m$  defined over the interval  $0 \leq R \leq 1$ . The average value  $\langle P_m^*(R) \rangle$  is computed using the moments  $\langle R^n \rangle$  for  $0 \leq n \leq m$  and turns out to be a rapidly decreasing function of  $m$ .

The amplifying ( $\beta<0$ ) case is a little trickier. Our method fails since  $|Z_{n\tilde{n}}|$  does not decay as  $n$  or  $\tilde{n}$  increases to large positive values. For sufficiently large values of  $l$ , however, Eq. (7) is well-defined if  $n$  and  $\tilde{n}$  are negative. From Eq. (7), we can easily prove that  $Z_{n\tilde{n}}(\beta=\beta_0) = Z_{-\tilde{n},-n}(\beta=-\beta_0)$ . Thus we obtain

$$\langle R^n \rangle_{\beta=\beta_0} = \langle R^{-n} \rangle_{\beta=-\beta_0}, \quad (10)$$

which is equivalent to

$$P_R(R, \beta = -\beta_0) = \frac{1}{R^2} P_R\left(\frac{1}{R}, \beta = \beta_0\right). \quad (11)$$

We observe that Eq. (11) suggests  $P_R(R)$  to be nonzero only for  $1 \leq R \leq \infty$  in the amplifying case. This result does not make much sense for  $l \ll 1$  because in that case, we expect  $P_R(R)$  for  $\beta<0$  as well as  $P_R(R)$  for  $\beta>0$  to have a sharp peak around  $R=0$ . Nevertheless, we conjecture that Eq. (11) is true for all  $l \geq 1$ , that is for all  $L \geq \xi_0$ . Once we get  $P_R(R)$  for  $\beta<0$ , we can compute the moments  $\langle R^n \rangle$  using the definition

$$\langle R^n \rangle_{\beta<0} = \int_1^{\infty} dR R^n P_R(R, \beta<0). \quad (12)$$

The unitary case requires a separate consideration. We expect all  $Z_{n\tilde{n}}$ 's with  $n=\tilde{n}$  are equal to 1 and  $P_R(R, \beta=0) = \delta(R-1)$  in the  $l \rightarrow \infty$  limit. Therefore we have to set  $Z_{nn}$  for  $n=N+1$  equal to 1 instead of 0 in solving Eq. (7). This changes the right-hand side of the last one of our  $N(N+2)$  algebraic equations, by solving which we obtain all moments of the form  $\langle e^{i(n-\tilde{n})\theta} \rangle$ . The result will be used in Secs. III B and III C in calculating the probability density of the phase of the reflection coefficient and the localization length exactly.

## B. Probability distribution of the phase of the reflection coefficient

The probability density  $P_\theta(\theta)$  is most easily obtained using the Fourier series expansion

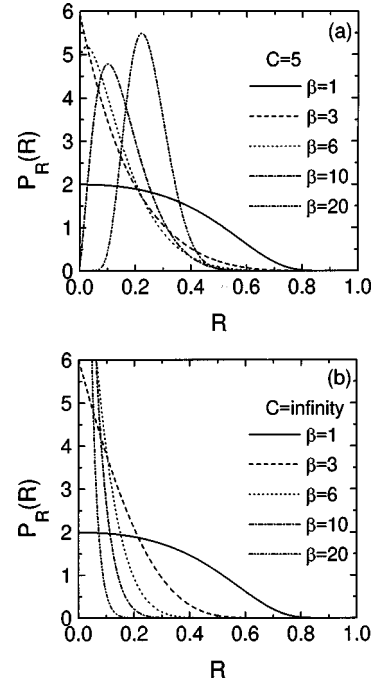


FIG. 1. Probability density of the reflectance in the absorbing case for  $\alpha=0$ ,  $\beta=1, 3, 6, 10, 20$  and (a)  $C=5$  (exact result) or (b)  $C=\infty$  (RPA).

$$P_\theta(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \langle \cos(m\theta) \rangle \cos(m\theta) + \frac{1}{\pi} \sum_{m=1}^{\infty} \langle \sin(m\theta) \rangle \sin(m\theta). \quad (13)$$

Therefore we need the averages  $\langle \cos(m\theta) \rangle$  and  $\langle \sin(m\theta) \rangle$  for

$$\begin{aligned} \frac{dY_{n\tilde{n}}}{dl} = & [i(2C + \alpha)(n + \tilde{n}) - \beta(n - \tilde{n}) - 4n\tilde{n} - 3n^2 - 3\tilde{n}^2]Y_{n\tilde{n}} + n[\frac{1}{2}(i\alpha - \beta) - 2n - 2\tilde{n} - 1]Y_{n+1,\tilde{n}} \\ & + n[\frac{1}{2}(i\alpha - \beta) - 2n - 2\tilde{n} + 1]Y_{n-1,\tilde{n}} + \tilde{n}[\frac{1}{2}(i\alpha + \beta) - 2n - 2\tilde{n} - 1]Y_{n,\tilde{n}+1} \\ & + \tilde{n}[\frac{1}{2}(i\alpha + \beta) - 2n - 2\tilde{n} + 1]Y_{n,\tilde{n}-1} - n\tilde{n}Y_{n+1,\tilde{n}+1} - n\tilde{n}Y_{n-1,\tilde{n}-1} - n\tilde{n}Y_{n+1,\tilde{n}-1} - n\tilde{n}Y_{n-1,\tilde{n}+1} \\ & - \frac{1}{2}n(n+1)Y_{n+2,\tilde{n}} - \frac{1}{2}n(n-1)Y_{n-2,\tilde{n}} - \frac{1}{2}\tilde{n}(\tilde{n}+1)Y_{n,\tilde{n}+2} - \frac{1}{2}\tilde{n}(\tilde{n}-1)Y_{n,\tilde{n}-2}, \end{aligned} \quad (14)$$

with the condition  $Y_{00}=1$ . We find that when  $C \gg 1$ ,  $\beta$  or, more precisely, when  $|2C + \alpha| \gg 1$ ,  $\beta$ , all  $Y_{n\tilde{n}}$ 's except  $Y_{00}$  are negligible, which obviously means that the phase  $\theta$  is distributed randomly and uniformly over the interval  $0 \leq \theta \leq 2\pi$ . We solve Eq. (14) in the  $l \rightarrow \infty$  limit for  $\beta > 0$  using the truncation method described in the previous section. In order to obtain  $P_\theta(\theta)$ , we also need to find  $\langle e^{im\theta} \rangle$  for odd  $m$ . For this purpose, we make a conjecture that  $\langle e^{im(\theta - \pi)} \rangle$  is a smooth function of  $m$ , the validity of which can be tested directly in the unitary case where we can compute  $\langle e^{im\theta} \rangle$  for every integer  $m$ . Then the averages  $\langle e^{im\theta} \rangle$  for odd  $m$  can be obtained by a numerical interpolation between the averages for even  $m$ . Once we get  $P_\theta(\theta)$  for  $\beta > 0$ , it is trivial to find  $P_\theta(\theta)$  for  $\beta < 0$ . We easily see from Eq. (14) that  $Y_{n\tilde{n}}(\beta = \beta_0) = Y_{\tilde{n}n}(\beta = -\beta_0)$ . This implies

$$P_\theta(\theta, \beta = \beta_0) = P_\theta(\theta, \beta = -\beta_0). \quad (15)$$

### C. Localization length

In order to compute the localization length as defined by Eq. (6), we need to compute the average  $W \equiv \langle \ln T \rangle$  in the  $l \rightarrow \infty$  limit. The nonrandom differential equation satisfied by  $W$  in the absorbing ( $\beta > 0$ ) and unitary ( $\beta = 0$ ) cases is obtained using Eqs. (4) and (5) and Novikov's formula in a straightforward manner:

$$\frac{dW}{dl} = -(1 + \beta) - \text{Re}[(2 + \beta - i\alpha)Z_{10} + Z_{20}]. \quad (16)$$

In the case of the random phase approximation,  $Z_{10}$  and  $Z_{20}$  vanish and the RPA localization length becomes<sup>6</sup>

$$\xi_{\text{RPA}} = \frac{\xi_0}{1 + \beta}. \quad (17)$$

$Z_{10}$  and  $Z_{20}$ , however, do not vanish in general and the exact localization length is obtained from

every positive integer  $m$ . Unfortunately, we are not aware of how to compute these averages directly when  $m$  is an odd integer unless  $\beta = 0$ . For even integers  $m = 2n$ , we can calculate  $\langle e^{2in\theta} \rangle = \langle \cos(2n\theta) \rangle + i \langle \sin(2n\theta) \rangle$  using the relation  $\langle e^{2in\theta} \rangle = Z_{n,-n} = \langle (r/r^*)^n \rangle$ . Since  $Z_{n,-n}$  is coupled to every  $Z_{n\tilde{n}}$  with  $n \geq 0$  and  $\tilde{n} \leq 0$ , we need to calculate  $Y_{n\tilde{n}} \equiv Z_{n,-\tilde{n}}$  with  $n, \tilde{n} \geq 0$ . The equation satisfied by  $Y_{n\tilde{n}}$  is obtained from Eq. (7) by a simple substitution:

$$\frac{\xi_0}{\xi} = 1 + \beta + \text{Re}[(2 + \beta - i\alpha)Z_{10}(l \rightarrow \infty) + Z_{20}(l \rightarrow \infty)], \quad (18)$$

where the asymptotic values of  $Z_{10}$  and  $Z_{20}$  found in Sec. III A are used.

## IV. RESULTS

All of our results were obtained for the cutoff  $N = 60$ . In other words, we have solved  $60 \times 62 = 3720$  coupled algebraic equations numerically. We have confirmed that the results presented in this section are exact for all practical purposes.

### A. Probability distribution of the reflectance

We consider the absorbing case first. Figure 1(a) shows the probability density of the reflectance  $P_R(R)$  in the large distance limit for  $C = 5$ ,  $\alpha = 0$ , and  $\beta = 1, 3, 6, 10, 20$ . Figure 1(b) is the probability density of the reflectance for the same  $\beta$  values in the random phase approximation (that is,  $C = \infty$ ), the analytical form of which was obtained previously in Ref. 9:

$$P_R(R, C = \infty) = \frac{2\beta \exp[-2\beta R/(1-R)]}{(1-R)^2} \quad \text{for } 0 \leq R \leq 1. \quad (19)$$

For small  $\beta$  values ( $\beta = 1, 3$ ), we find that the exact  $P_R(R)$  agrees pretty well with the RPA probability density. As  $\beta$  increases, however, there appear remarkable differences between the exact and RPA probability densities. Most notably, the exact  $P_R(R)$  develops a sharp peak at nonzero  $R = R_{\text{max}}$ , while the RPA probability density has a peak at  $R = 0$  for all  $\beta \geq 1$ . This  $R_{\text{max}}$  increases and the half-width of the peak decreases as  $\beta$  increases further. We have observed a similar behavior for other values of  $C$ . In the parameter range we have explored in detail ( $1 \leq C$ ,  $\beta \leq 10$ ,  $\alpha = 0$ ),

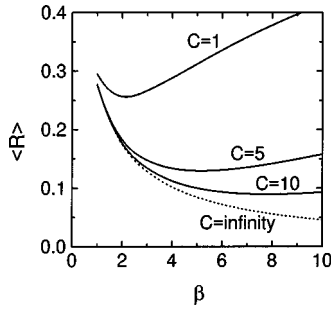


FIG. 2. Average reflectance vs  $\beta$  in the absorbing case for  $\alpha=0$  and  $C=1, 5, 10, \infty$ .

we do not find a double-peaked structure in the exact probability density reported in Ref. 7.

In Fig. 2, we show the average reflectance  $\langle R \rangle$  as a function of  $\beta$  for  $C=1, 5, 10, \infty$  and  $\alpha=0$ . In the RPA case,  $\langle R \rangle$  is a monotonically decreasing function of  $\beta$ . Since  $1-\langle R \rangle$  is the amount of average absorption, this means that absorption increases monotonically as the dimensionless absorption parameter  $\beta$  gets bigger. Though this may sound reasonable, it is actually a false conclusion. As pointed out in Ref. 7, the medium with a sufficiently large  $\beta$  behaves as a reflector rather than as an absorber. In agreement with Ref. 7, we have found that  $\langle R \rangle$  reaches a minimum at  $\beta=\beta_{\min}$  and is an increasing function of  $\beta$  for  $\beta>\beta_{\min}$ . We have checked numerically that, in the  $\beta\rightarrow\infty$  limit,  $\langle R \rangle\rightarrow 1$  and  $P_R(R)\rightarrow\delta(R-1)$ . As is obvious from Fig. 2,  $\beta_{\min}$  increases as  $C$  increases and can be used as a useful criterion for the region of validity of the random phase approximation.<sup>7,8</sup> That is, the RPA is approximately valid when  $|\beta|<\beta_{\min}$  [and  $C\gg 1$  (see Sec. IV B)]. In Fig. 3, we plot  $\beta_{\min}$  versus  $C$  for  $\alpha=0$ . It is a monotonically increasing function and is fitted fairly well by a power law function  $\beta_{\min}=aC^b$  with  $a\approx 2.04\pm 0.03$  and  $b\approx 0.58\pm 0.01$ .

Next we show the probability density of the reflectance in the amplifying case, which we obtain quite easily using the dual relationship (11). In Fig. 4(a), we show  $P_R(R)$  for  $C=5$ ,  $\alpha=0$ , and  $\beta=-1, -3, -6, -10, -20$ . Figure 4(b) is the RPA probability density for the same  $\alpha$  and  $\beta$  values. In the RPA case,  $P_R(R)$  is proportional to  $1/R^2$  in the  $R\rightarrow\infty$  limit for all  $\beta<0$ , since  $P_R(R=0)$  is a finite constant for all  $\beta>0$ . This implies that the average reflectance in the amplifying case is always divergent. The RPA result is wrong, however, because when  $\beta\rightarrow-\infty$ , the medium has to behave as a pure reflector with  $\langle R \rangle=1$ . In Fig. 1(a), we observe that

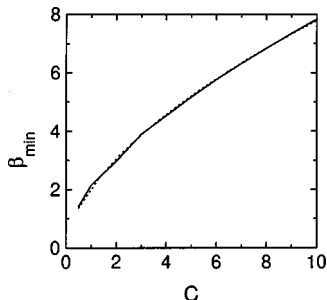


FIG. 3.  $\beta_{\min}$  vs  $C$  for  $\alpha=0$ .  $\beta_{\min}$  is the value of  $\beta$  at which the average reflectance takes the minimum value. The dotted line is a numerical fit:  $\beta_{\min}=aC^b$  with  $a\approx 2.04\pm 0.03$  and  $b\approx 0.58\pm 0.01$ .

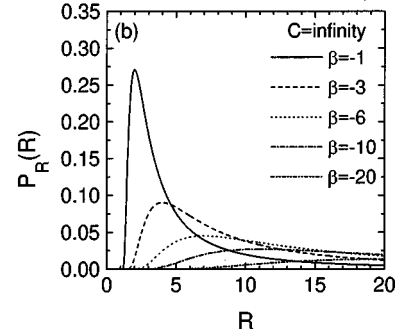
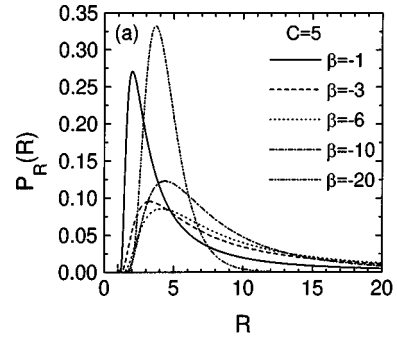


FIG. 4. Probability density of the reflectance in the amplifying case for  $\alpha=0, \beta=-1, -3, -6, -10, -20$  and (a)  $C=5$  (exact result) or (b)  $C=\infty$  (RPA).

$P_R(R)$  is finite at  $R=0$  for  $\beta=1, 3, 6$ , but goes to zero as  $R\rightarrow 0$  for sufficiently large  $\beta$  values. This observation and the dual relationship (11) ensure that  $\langle R \rangle$  is finite for sufficiently large negative  $\beta$ 's.

Finally, in Figs. 5(a) and 5(b), we show the probability density of the reflectance for  $\beta=5, -5, \alpha=0$ , and  $C$

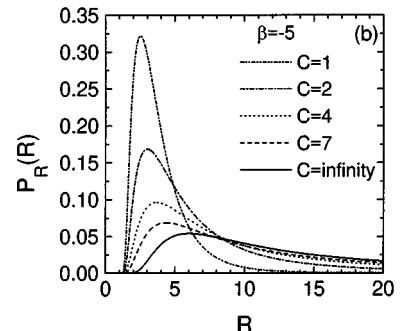
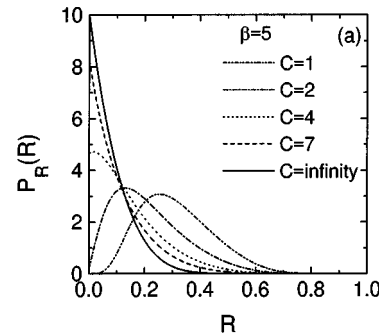


FIG. 5. Probability density of the reflectance (a) in the absorbing case ( $\beta=5$ ) and (b) in the amplifying case ( $\beta=-5$ ) for  $\alpha=0$  and  $C=1, 2, 4, 7, \infty$ .

$=1,2,4,7,\infty$ . In this way, we can clearly see how the exact  $P_R(R)$  departs from the RPA result as  $C$  decreases.

### B. Probability distribution of the phase of the reflection coefficient

As explained in Sec. III B, we have difficulty in obtaining the exact probability distribution of the phase of the reflection coefficient  $P_\theta(\theta)$  for arbitrary  $\beta$  values in the absorbing and amplifying cases. This difficulty does not arise in the unitary case ( $\beta=0$ ). Therefore we show the probability density in this case first in Figs. 6(a)–6(c) for a wide range of  $C$  values and  $\alpha=0$ . For sufficiently small  $C$ 's,  $P_\theta(\theta)$  has a sharp symmetric peak located at  $\theta=\pi$ . We expect that  $P_\theta(\theta)$  approaches  $\delta(\theta-\pi)$  as  $C\rightarrow 0$ . As  $C$  increases from zero, this large peak moves to  $\theta$  slightly bigger than  $\pi$  and a small secondary peak appears at  $\theta$  smaller than  $\pi$ . As  $C$  is increased further, the large peak keeps moving away from  $\theta=\pi$  and another small secondary peak is developed at  $\theta < \pi$ . The overall shape of the probability density becomes broader. When  $C\approx 0.05$ , the small peaks merge and are turned into a flat region. For  $C\gg 1$ ,  $P_\theta(\theta)$  is almost constant with a small and broad peak at  $\theta=5\pi/3$  and a valley at  $\theta = \pi/3$ .

In the absorbing and amplifying cases, we can calculate  $P_\theta(\theta)$  reliably only when  $\langle e^{im(\theta-\pi)} \rangle$  is a smooth function of the integer  $m$ . It turns out that our interpolation method described in Sec. III B works when  $|\beta|$  is sufficiently large compared to  $C$  or  $C$  is sufficiently small, in other words, when the random phase approximation does not work. In Fig. 7, we illustrate the behavior of  $P_\theta(\theta)$  for a rather small value of  $C$  ( $=0.1$ ) and  $\alpha=0$ ,  $|\beta|=0.1,0.6,1,1.5,2,3$ . When  $|\beta|\gg C$ ,  $P_\theta(\theta)$  has a sharp peak at  $\theta=\pi$ . As  $|\beta|$  decreases, this peak becomes lower and broader. When  $\beta\lesssim 2$ , the peak shifts to  $\theta < \pi$  and a new peak appears at  $\theta > \pi$ . This new peak grows and the old peak decays as  $|\beta|$  increases further. At  $\beta=0.1$ ,  $P_\theta(\theta)$  is almost identical to the probability density in the unitary case [Fig. 6(b)].

### C. Localization length

In Fig. 8(a), we plot the inverse (dimensionless) localization length  $\xi_0/\xi$  in the unitary case versus  $\log_{10}C$ . When  $C\gg 1$ ,  $\xi$  is close to the RPA localization length in the unitary case  $\xi_0$ . In the opposite limit  $C\rightarrow 0$ ,  $\xi_0/\xi$  goes to zero or, equivalently,  $\xi/\xi_0$  diverges. Figure 8(b) shows the same data on a log-log plot. We note that the  $C\ll 1$  region is approximately linear and is fitted by a power law function  $\xi_0/\xi = a' C^{b'}$  with  $a'\approx 0.81$  and  $b'\approx 0.65$ .

Finally, in Fig. 9(a), we show the localization length in the absorbing case as a function of  $\beta$  for  $C=5$  and  $C=\infty$ . We note that the exact localization length is always larger than the RPA localization length [ $=\xi_0/(1+\beta)$ ] for the same  $\beta$ . Figure 9(b) shows the localization length as a function of  $C$  for  $\beta=5$ .

## V. CONCLUSION

In this paper, we have presented a numerical method of calculating various transport characteristics of waves in one-dimensional random media with (or without) absorption or

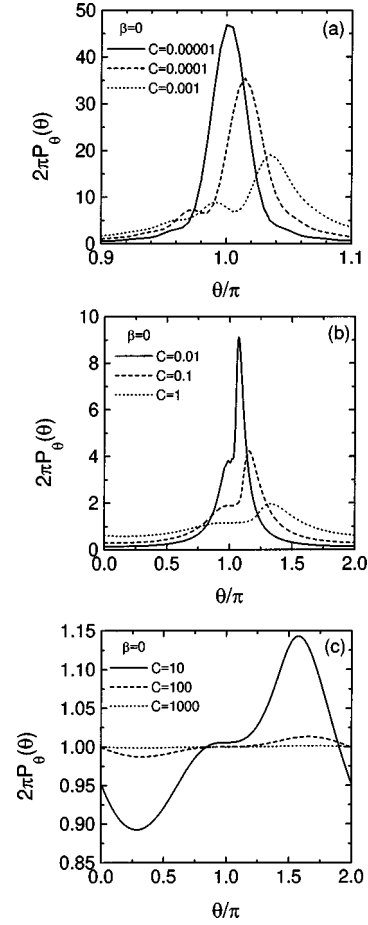


FIG. 6. Probability density of the phase of the reflection coefficient in the unitary case ( $\beta=0$ ) for  $\alpha=0$  and (a)  $C = 0.00001, 0.0001, 0.001$ , (b)  $C = 0.01, 0.1, 1$ , and (c)  $C = 10, 100, 1000$ .

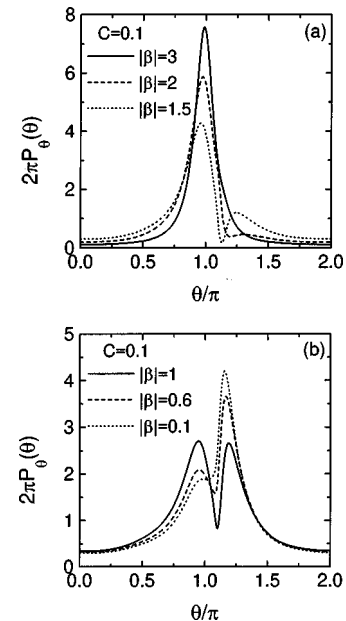


FIG. 7. Probability density of the phase of the reflection coefficient,  $P_\theta(\theta)$ , in the absorbing and amplifying cases for  $C=0.1$ ,  $\alpha=0$  and (a)  $|\beta|=3, 2, 1.5$  and (b)  $|\beta|=1, 0.6, 0.1$ . Note that  $P_\theta(\theta, \beta) = P_\theta(\theta, -\beta)$ .

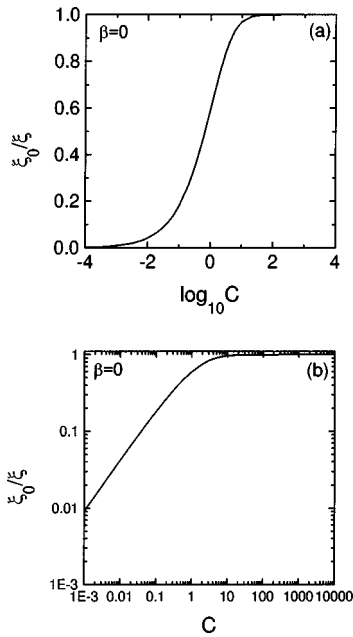


FIG. 8. Inverse dimensionless localization length in the unitary case for  $\alpha=0$  (a) vs  $\log_{10}C$  and (b) vs  $C$  on a log-log plot.  $\xi_0$  is the RPA localization length in the unitary case.

amplification and used it to obtain the probability densities of the reflectance and of the phase of the reflection coefficient in the large distance limit, together with the localization length of waves. Our method is completely beyond the random phase approximation that has been used frequently in previous works and gives essentially exact results in the sense that the numerical error is unnoticeably small in all of the figures presented in this paper. The probability distribution of the phase of the reflection coefficient turns out to be highly nonuniform when either the disorder parameter or the absorption (or amplification) parameter (that is, the magnitude of the imaginary part of the dielectric permeability,  $|\gamma|$ ) is large. When the absorption or amplification parameter is large, the probability distribution of the reflectance shows behavior that is totally different from the RPA behavior. We have also proved a couple of exact dual relationships between the probability densities in the absorbing case and those in the amplifying case with the same  $|\gamma|$ . The probability density of the phase of the reflection coefficient in the amplifying case is the same as that in the absorbing case with the same  $|\gamma|$ . The probability density of the reflectance is obtained from that in the absorbing case with the same  $|\gamma|$  by

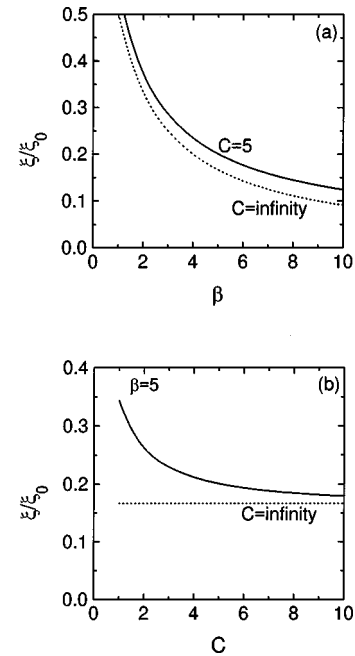


FIG. 9. Dimensionless localization length in the absorbing case with  $\alpha=0$  (a) vs  $\beta$  for  $C=5, \infty$  and (b) vs  $C$  for  $\beta=5$ .  $\xi_0$  is the RPA localization length in the unitary case.

a simple transformation [Eq. (11)]. From the quantitative analysis of the average reflectance that shows a nonmonotonic dependence on the absorption or amplification parameter, we find a criterion for the applicability of the RPA. In the parameter regime where the RPA is invalid, we find that the exact localization length is much larger than the RPA localization length.

In the present work, we have limited our attention to the large distance limit. However, we could have integrated the differential equations (7) and (14) directly to obtain the probability densities of the finite-size system. Our method can also be generalized in a straightforward manner to the calculation of the probability densities of the transmittance and the phase of the transmission coefficient. Work in this direction is in progress and will be presented elsewhere.

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