Spinless particle in a rapidly fluctuating random magnetic field

V. G. Benza and B. Cardinetti
*Dipartimento di Fisica, Università di Milano, Via Celoria 16, 20133 Milano, Italy
and INFM, unità di Milano, Via Celoria 16, 20133 Milano, Italy*
(Received 17 March 1998)

We study a two-dimensional spinless particle in a disordered Gaussian magnetic field with short-time fluctuations, by means of the evolution equation for the density matrix $\langle \mathbf{x}^{(1)}|\hat{\rho}(t)|\mathbf{x}^{(2)}\rangle$; in this description the two coordinates are associated with the retarded and advanced paths, respectively. In the classical limit the baricentric coordinate $\mathbf{r}=(1/2)(\mathbf{x}^{(1)}+\mathbf{x}^{(2)})$ is the particle position and the dual of the relative coordinate $\mathbf{x}=\mathbf{x}^{(1)}-\mathbf{x}^{(2)}$ its momentum. The vector potential correlator is assumed to grow with distance with a power h: when $h=0$ it corresponds to a δ-correlated magnetic field, when $h=2$ to a magnetic field with infinite range fluctuations. We find that the value $h=2$ separates two different propagation regimes, of diffusion and logarithmic growth, respectively. When $h<2$, $\mathbf{r}$ undergoes diffusion with a coefficient D_r proportional to x^{-h}. As $h>2$, the magnetic-field fluctuations grow with distance and D_r scales as x^{-2}. The width in r of the density matrix then grows for large times proportionally to $\ln(t/x^2)$. [S0163-1829(98)02734-9]

I. INTRODUCTION

In recent years various studies have been devoted to propagation in the presence of a stochastic drift term. The most known example comes from fluid dynamics: the transport of a passive scalar in a velocity field. It has been shown that this problem can be solved in various physically relevant situations. In particular, Gawedzky and Kupiainen[1] gave a solution in the three-dimensional case by means of an explicit resummation of the perturbative series. Correlators of the scalar field of any order can be obtained. The basic ingredient is the assumption of zero correlation time for the velocity, which allows for an effective evolution generator. Quotation of previous work on the passive scalar can be found, e.g., in Ref. 2. A second, related example is quantum motion in a disordered magnetic field. This topic has been raised in various physical contexts: Mott-Hubbard systems,[3] vortex lines in superconductors,[4,5] quantum Hall effect at filling factor 1/2.[6,7] In the last case the magnetic field is static with zero mean and numerically there is evidence of a delocalization transition,[8–10] at odds with a supersymmetric treatment leading to a sigma model with unitary symmetry.[11] Fluctuations in the topological density have also been held responsible for the delocalization transition.[12] Recent work related the occurrence of delocalized states with random antisymmetric disorder, or with symmetric disorder as well, provided a peculiar sublattice decomposition is allowed.[13] Quantum mechanics with an imaginary magnetic field and a disordered potential, introduced by Hatano and Nelson,[14] recently attracted much attention (see, e.g., Efetov[15]), showing how nonhermiticity can sustain extended states. In a previous work we studied the imaginary magnetic field that arises after averaging over magnetic disorder. We further interpreted the effective action in terms of Coulomb gas, and discussed how its strong coupling regime can enhance phase coherence among trajectories.[16] Static disorder obviously produces an effective action nonlocal in time. Everything simplifies with time-dependent disorder: when the correlation time goes to zero the action is local and in principle identifies an effective generator for time evolution. One can think of motion in a sea of high energy particles carrying flux lines, or to a vortex in a rapidly varying magnetic field. Cooperon dynamics with time-dependent magnetic fluctuations has been studied by Aronov and Wolfle[17] in the context of high-T_c conductors. In the present work we analyze rapidly fluctuating magnetic disorder on an otherwise free particle: our aim is to determine whether the ballistic behavior is frustrated in such a case, and what kind of presumably slower propagation sets in. By taking the correlation length of the fluctuations as the large scale of the system we establish the effective (annealed) dynamics for various power-law disorder correlators. In Sec. II we determine the time evolution generator for the density matrix from the Feynman path integral. This is a two-particle description, since both retarded and advanced paths must be taken into account. Averaging over disorder leads to self-interaction as well as to mutual interaction between the paths: the former amounts to a renormalization of the single-particle dynamics (i.e., it influences the quantum amplitude), while the latter is the interference contribution to probability. In Sec. III we discuss the particle-antiparticle relative dynamics by averaging over the baricentric variables; this is related with motion in the momentum space, as will be illustrated. In Sec. IV we analyze the full problem and give our main results on the particle diffusion and subdiffusion. In Sec. V we compare with behaviors similar to ours in different physical contexts.

II. DENSITY-MATRIX EVOLUTION

We assume that the correlation time of magnetic fluctuations can be neglected and take the following Gaussian correlator for the vector potential, in the transverse gauge:

$$D_{\alpha,\beta}(\mathbf{k})\cdot\delta(t)\equiv\langle A_\alpha A_\beta\rangle(\mathbf{k},t)=\Delta\,\delta^T_{\alpha,\beta}(\mathbf{k})\frac{1}{(k^2+k_0^2)^{1+h/2}}\,\delta(t), \tag{1}$$

where k_0^{-1} is the correlation length, h is positive, and $\delta^T_{\alpha,\beta} \equiv \delta_{\alpha,\beta} - (k_\alpha k_\beta)/k^2$. We will take into account the range $0 < h < 4$; the limit $h \to 0, k_0 \to 0$ gives a magnetic field B delta correlated in space, $h=3$ can be associated with the anomalous skin effect;[17] $h=2, k_0 \to 0$ gives a free massless B. This case incidentally corresponds to the enstrophy cascade in two-dimensional turbulence, if the vector potential is identified with velocity.[18] Notice that, as $h>2$, the correlator of B grows as a power of distance. We write the Feynman path-integral representation for the product of amplitudes:

$$F(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}';t) \equiv \langle \mathbf{x} | U^{(1)}(t) | \mathbf{y} \rangle \langle \mathbf{x}' | U^{(2)}(t) | \mathbf{y}' \rangle^*, \tag{2}$$

where $U^{(i)}(t), i=1,2$ are the evolution operators of two identical copies of single-particle systems in the presence of the magnetic field. The time-evolved density matrix is given by

$$\langle \mathbf{x} | \hat{\rho}(t) | \mathbf{x}' \rangle = \int d\mathbf{y}\, d\mathbf{y}' F(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}';t) \langle \mathbf{y} | \hat{\rho}(0) | \mathbf{y}' \rangle. \tag{3}$$

The representation of Eq. (2) involves a two-particle action: advanced and retarded path, respectively. Averaging over the vector potential couples the paths and generates the following effective Lagrangian, $\mathcal{L}_{eff}(\mathbf{X},\dot{\mathbf{X}}), \mathbf{X} \equiv (\mathbf{x}^{(1)},\mathbf{x}^{(2)})$:

$$\mathcal{L}_{eff} = \frac{m}{2} \dot{\mathbf{X}} \sigma_3 \dot{\mathbf{X}} + \frac{ig^2}{2\hbar} \dot{\mathbf{X}} \sigma_3 \hat{D} \sigma_3 \dot{\mathbf{X}} \equiv \frac{1}{2} \dot{\mathbf{X}} \hat{M} \dot{\mathbf{X}}. \tag{4}$$

Here $g = e/c$, the Pauli matrix σ_3 acts on the particle indices, and the (4×4) matrix $\hat{D} \equiv \hat{D}^{(i,j)}(i,j=1,2)$ corresponds to the static part of the correlator defined in Eq. (1): $\hat{D}^{(1,1)} = \hat{D}^{(2,2)} = D_{\alpha,\beta}(\mathbf{x}^{(1)} = \mathbf{x}^{(2)}); \hat{D}^{(1,2)} = \hat{D}^{(2,1)} = D_{\alpha,\beta}(\mathbf{x}^{(1)} - \mathbf{x}^{(2)})$; $\alpha,\beta \equiv x,y$. The evolution of the density matrix is given by Eq. (3), where F is converted into its average over disorder; in the effective action the imaginary part of $\hat{M}$ is obviously related with dissipation. Since, contrary to the static case (see, e.g., Ref. 16), we have locality in time, it is possible to extract a time evolution generator $\hat{H}$ from Eq. (3). This can be performed with no ambiguity in a flat metrics, while in general operator ordering prescriptions are needed. As discussed at length in Ref. 17, if one evaluates the metrics at the midpoint between the initial and final configurations, and consistently normalizes the intermediate Gaussian integral, one obtains a symmetrized generator. We have $\hat{H} = \frac{1}{2}(\nabla \hat{F}^{-1} \nabla)_s$, where $\hat{F} = (1/i\hbar)\hat{M}$ and $(\nabla \hat{A} \nabla)_s \equiv \frac{1}{4}(\nabla\nabla\hat{A} + 2\nabla\hat{A}\nabla + \hat{A}\nabla\nabla)$. Here the sum over indices is understood and the gradient is with respect to $\mathbf{X}$. The kernel $\hat{F}^{-1}$ is given by

$$\hat{F}^{-1} = \left[\frac{i\hbar}{m} \sigma_3 + \left(\frac{g}{m} \right)^2 \hat{D} \right] \hat{G}^{-1},$$
$$G_{\alpha,\beta} = \delta_{\alpha,\beta} + \left(\frac{g^2}{m \cdot \hbar} \right)^2 [D^2_{\alpha,\beta}(\mathbf{0}) - D^2_{\alpha,\beta}(\mathbf{x}^{(1)} - \mathbf{x}^{(2)})]. \tag{5}$$

This simple form results from $D_{\alpha,\beta}$ being diagonal when its argument is equal to $\mathbf{0}$. The operator $\hat{G}$ is diagonal in the particle indices and, as one easily verifies, is even under particle exchange; it obviously commutes with $\hat{D} \equiv \hat{D}(\mathbf{x}^{(1)} - \mathbf{x}^{(2)})$. By taking into account the transversality condition one ends up with the following evolution equation for the density matrix $\rho(\mathbf{x}^{(1)},\mathbf{x}^{(2)};t)$ in coordinate representation:

$$\frac{\partial \rho}{\partial t} = \hat{H} \rho,$$

$$\begin{aligned}
\hat{H} = {} & \frac{i\hbar}{2m} (G^{-1})_{\alpha,\beta} (\partial^{(1)}_\alpha \partial^{(1)}_\beta - \partial^{(2)}_\alpha \partial^{(2)}_\beta) \\
& + \frac{1}{2} \left(\frac{g}{m} \right)^2 D^0_{\alpha,\gamma} (G^{-1})_{\gamma,\beta} (\partial^{(1)}_\alpha \partial^{(1)}_\beta + \partial^{(2)}_\beta \partial^{(2)}_\alpha) \\
& + \frac{1}{2} \left(\frac{g}{m} \right)^2 D^x_{\alpha,\gamma} (G^{-1})_{\gamma,\beta} (\partial^{(1)}_\alpha \partial^{(2)}_\beta + \partial^{(2)}_\alpha \partial^{(1)}_\beta) \\
& + \frac{i\hbar}{2m} [\partial_\alpha (G^{-1})_{\alpha,\beta}] (\partial^{(1)}_\beta + \partial^{(2)}_\beta) \\
& + \frac{1}{2} \left(\frac{g}{m} \right)^2 (D^0 - D^x)_{\alpha,\gamma} [\partial_\alpha (G^{-1})_{\gamma,\beta}] (\partial^{(1)}_\beta - \partial^{(2)}_\beta) \\
& + \frac{1}{4} \left(\frac{g}{m} \right)^2 \partial_\gamma [(D^0 - D^x)_{\alpha,\beta} \partial_\alpha (G^{-1})_{\beta,\gamma}].
\end{aligned} \tag{6}$$

Here the partial derivatives, unless labeled with the particle index, are taken with respect to the relative coordinate $\mathbf{x} \equiv \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$ and D^0, D^x refer to the static part of the correlator at separation $\mathbf{0}$ and $\mathbf{x}$, respectively. Notice that the particle mass is renormalized, and an effective magnetic field [proportional to $(g/m)^2$] acts with opposite signs on the two trajectories. The remaining terms are dissipative; the drift term acts as an imaginary magnetic field (proportional to $\hbar/m$). We conclude this section by noticing that if one disregards the $\mathbf{A}^2$ term, one can directly average the two-particle evolution operator emerging from Eq. (2). Along the lines of Ref. 1, one starts with the Neumann series for this operator, takes the $\mathbf{A} \cdot \mathbf{p}$ coupling as perturbation, then uses Wick's theorem and averages each term. The resulting series can be reexponentiated to an effective two-particle generator that coincides with Eq. (6), but with $\hat{G} = \hat{1}$. This case is fortunate since the transversality condition guarantees that the generator $\hat{H}$ does not depend on ordering prescriptions.

III. PARTICLE-ANTIPARTICLE RELATIVE DYNAMICS

It is readily verified from Eq. (6), when written in terms of the relative coordinate and of the baricentric coordinate $\mathbf{r} \equiv \frac{1}{2}(\mathbf{x}^{(1)} + \mathbf{x}^{(2)})$ [see Eq. (16) in the next section], that it has a purely derivative dependence on the latter variable. This fact, which follows from the translational invariance of disorder, makes $\hat{H}$ into a divergence term with respect to $\mathbf{r}$ plus an operator in $\mathbf{x}$. Upon integrating over $\mathbf{r}$ with suitable boundary conditions, the divergence term disappears and one is left with the evolution in the reduced $\mathbf{x}$ space in closed form. This evolution, which isolates the effective interaction between the retarded and advanced paths, is directly associated with the dynamics in the space of momenta [see Eq. (21)]. Furthermore, the average over the baricentric coordinate gives the equation of the Cooperon amplitude.[17] In summary, the operator $\hat{H}$ reduces to

$$\hat{H}=\left(\frac{g}{m}\right)^2(D^0-D^x)_{\alpha,\gamma}(G^{-1})_{\gamma,\beta}\partial_\alpha\partial_\beta+\left(\frac{g}{m}\right)^2(D^0-D^x)_{\alpha,\gamma}[\partial_\alpha(G^{-1})_{\gamma,\beta}]\partial_\beta+\frac{1}{4}\left(\frac{g}{m}\right)^2\partial_\gamma[(D^0-D^x)_{\alpha,\beta}\partial_\alpha(G^{-1})_{\beta,\gamma}]. \tag{7}$$

We now show that Eq. (7) in the case of long-range fluctuations ($k_0\ll 1$) further simplifies, and the drift and curvature contributions can be neglected: indeed the kernel $\hat{F}^{-1}$ in this limit satisfies the transversality condition, and no ordering prescription is needed. Let us examine the correlator. In space coordinates its static part has the form

$$D_{\alpha,\beta}(\mathbf{x})=\left(\frac{\Delta}{4\pi}\right)[\mathcal{I}_0(x)\,\delta_{\alpha,\beta}+\mathcal{I}_2(x)\Sigma_{\alpha,\beta}],$$

$$\Sigma_{1,1}=-\Sigma_{2,2}=\cos(2\psi),\quad \Sigma_{1,2}=\Sigma_{2,1}=\sin(2\psi), \tag{8}$$

$$\mathcal{I}_i(x)=\int_0^\infty\frac{k\,dk}{(k^2+k_0^2)^{1+h/2}}J_i(kx),$$

where x and ψ are the polar coordinates of $\mathbf{x}$ and the J_i's are Bessel functions. The small k_0 behavior of Eq. (8) is readily obtained:

$$D_{\alpha,\beta}=\left(\frac{\Delta}{k_0^h}\right)\left(\frac{1}{4\pi h}\,\delta_{\alpha,\beta}+\mathcal{A}_{\alpha,\beta}\right),$$

$$\hat{\mathcal{A}}=\left(\frac{\xi}{2}\right)^h\hat{R},0<h<2,\xi\equiv k_0x,$$

$$\hat{\mathcal{A}}=(\xi)^2\hat{T},\quad 2<h<4, \tag{9}$$

$$\hat{R}=\frac{1}{8\pi}\frac{\Gamma(1-h/2)}{\Gamma(1+h/2)}[-(1+2/h)\hat{1}+\hat{\Sigma}],$$

$$\hat{T}=\frac{1}{8\pi}\frac{1}{h(h-2)}(-\hat{1}+\tfrac{1}{2}\hat{\Sigma}).$$

The explicit form of $\hat{G}$ is then

$$\hat{G}=\hat{1}-\left(\frac{g^2}{m\cdot\hbar}\frac{\Delta}{k_0^h}\right)^2\hat{\mathcal{A}}\left(\frac{1}{2\pi\hbar}\hat{1}+\hat{\mathcal{A}}\right). \tag{10}$$

The subtracted correlator has a power-law behavior, and is $O(k_0^0)$ in the limit $k_0\to 0$, when $h<2$:

$$\hat{D}^0-\hat{D}^x=-\frac{\Delta}{k_0^h}\hat{\mathcal{A}}. \tag{11}$$

The ratio Δ/k_0^h has the dimensions of a momentum, let us call it Π. The order of magnitude of the $\mathbf{A}^2$ contribution can be estimated by means of the adimensional coupling constant $r_e\Pi/\hbar$, where r_e is the classical radius of the electron ($r_e\equiv g^2/m$). We now define the large fluctuations regime, in which $\hat{G}$ can be approximated with its second term:

$$\frac{g^2\Delta}{m\hbar k_0^h}\equiv\frac{r_e\Pi}{\hbar}\gg 1. \tag{12}$$

Notice that this relates the amplitude of the fluctuations (associated with Δ) with the correlation length k_0^{-1}. Finally the kernel $\hat{F}^{-1}$, which is a function of Π, h, and of the quantum flux unit $\hbar/g\equiv\Phi_0/(2\pi)$, reduces to

$$\hat{F}^{-1}=\left(\frac{\hbar}{g}\right)^2\frac{k_0^h}{\Delta\gamma}\left(\hat{1}-\frac{1}{\gamma}\hat{\mathcal{A}}\right),\qquad \gamma\equiv\frac{1}{2\cdot\pi\cdot h}. \tag{13}$$

It is easily verified that in both regimes ($h<2$, $h>2$) the matrix $\hat{\mathcal{A}}$ satisfies the transversality condition; hence the generator is

$$\hat{H}=\frac{1}{2}\left(\frac{\hbar}{g}\right)^2\frac{1}{\Pi\gamma}\left(\delta_{\alpha,\beta}-\frac{1}{\gamma}\mathcal{A}_{\alpha,\beta}\right)\partial_\alpha\partial_\beta. \tag{14}$$

The dominant contribution is pure diffusion with a coefficient $O(k_0^h)$, while the first correction has scaling form, and in both regimes it corresponds to a faster spreading. When $0<h<2$, the correction is of order k_0^{2h} and, taken alone, would give superdiffusion ($t\approx x^{2-h}$); when $2<h<4$, it has order k_0^{2+h} with $t\approx(\ln x)^2$. Further work on Eq. (14) is needed in order to understand the interplay between diffusion and this faster mechanism. We end this section with some comments on the regime of weak fluctuations ($r_e\Pi/\hbar\ll 1$). When $\hat{G}$ reduces to the identity, $\hat{F}^{-1}$, being proportional to $\hat{\mathcal{A}}$, has a scaling form

$$\hat{F}^{-1}=-\left(\frac{g}{m}\right)^2\Pi\hat{\mathcal{A}}. \tag{15}$$

Recall that while in the $h<2$ range this is $O(k_0^0)$, in the complementary range it diverges as k_0^{2-h} in the limit $k_0\to 0$. Corrections in the small fluctuation parameter come from the expansion of $\hat{G}^{-1}$; since only the dominant term preserves the transversality condition, the operator $\hat{H}$ will have a drift and a curvature term. The dominant term, which occurred as the first correction in Eq. (14), gives superdiffusion ($x\approx t^{1/(2-h)}$) when $0<h<2$ and time exponential behavior when $2<h<4$. As already pointed out, to approximate $\hat{G}$ with the identity means to disregard the $\mathbf{A}^2$ term. This corresponds to the passive scalar problem studied by Gawedzki and Kupiainen,[1] or, more properly, to the quantum version of it. In conclusion, the $\mathbf{A}^2$ term reduces an otherwise strong tendency of the two trajectories towards separation, from fluctuation-supported superdiffusion to diffusion.

IV. GENERAL CASE

Let us write Eq. (6) in terms of the relative coordinate and of the baricentric coordinate:

$$\frac{\partial\rho}{\partial t}=\hat{H}\rho,$$

$$\hat{H}=\frac{i\hbar}{2m}(G^{-1})_{\alpha,\beta}\left(\frac{\partial}{\partial r_\alpha}\frac{\partial}{\partial x_\beta}+\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial r_\beta}\right)+\frac{i\hbar}{2m}\left(\frac{\partial}{\partial x_\alpha}(G^{-1})_{\alpha,\beta}\right)\frac{\partial}{\partial r_\beta}+\frac{1}{4}\left(\frac{g}{m}\right)^2[(D^0+D^x)G^{-1}]_{\alpha,\beta}\frac{\partial}{\partial r_\alpha}\frac{\partial}{\partial r_\beta}$$
$$+\left(\frac{g}{m}\right)^2[(D^{(0)}-D^{(x)})G^{-1}]_{\alpha,\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}+\left(\frac{g}{m}\right)^2\left(\frac{\partial}{\partial x_\alpha}(D^0-D^x)G^{-1}\right)_{\alpha,\beta}\frac{\partial}{\partial x_\beta}+\frac{1}{4}\left(\frac{g}{m}\right)^2\frac{\partial}{\partial x_\alpha}\frac{\partial}{x_\beta}[(D^0-D^x)G^{-1}]_{\alpha,\beta}. \tag{16}$$

Here the limit $k_0\to 0$ gives a finite nonzero result provided that $h<2$. As shown in the previous section, the diffusion term for $\mathbf{x}$ is $O(k_0^h)$; similarly the other terms go to zero in the limit. From Eq. (16) we get then

$$\hat{H}\to-\frac{1}{4\Delta}\left(\frac{\hbar}{g}\right)^2\left(\frac{x}{2}\right)^{-h}(\hat{R}^{-1})_{\alpha,\beta}\frac{\partial}{\partial r_\alpha}\frac{\partial}{\partial r_\beta}. \tag{17}$$

If one disregards the anisotropy, this implies an evolution through the combination $r^2\cdot x^h/t$. Let us assume then

$$\hat{H}\approx\mathcal{D}(x/2)^{-h}\nabla_{\mathbf{r}}^2, \tag{18}$$

where $\mathcal{D}$ is constant. By chosing as initial condition a Gaussian in both particle coordinates with equal width

$$\rho(\mathbf{r},\mathbf{x})=\left(\frac{w}{2\pi}\right)^2\exp[-w\{r^2+(x/2)^2\}] \tag{19}$$

one gets

$$\rho(\mathbf{r},\mathbf{x},t)=\frac{w}{2(2\pi)^2}\frac{1}{1/2w+2\mathcal{D}(2/x)^h t}\exp\left[-\left(\frac{w}{4}\right)x^2-\left(\frac{1}{2}\right)\frac{r^2}{(1/2w)+2\mathcal{D}(2/x)^h t}\right]. \tag{20}$$

For a more direct physical interpretation, it is convenient to Fourier transform with respect to $\mathbf{x}$, obtaining the Wigner function, whose classical limit is the phase space distribution

$$W(\mathbf{r},\mathbf{k},t)\equiv\frac{1}{(2\pi)}\int d^2\mathbf{x}\exp(-i\mathbf{k}\cdot\mathbf{x})\rho(\mathbf{r},\mathbf{x},t). \tag{21}$$

This can be easily evaluated at zero momentum: in the large time limit we disregard $1/(2w)$ with respect to $2\mathcal{D}(2/x)^h t$ and obtain

$$W(\mathbf{r},\mathbf{k}=\mathbf{0},t)=\frac{1}{(2\pi)^2}\frac{w}{2^{h+2}\mathcal{D}t}\int_0^\infty dx\,x^{h+1}\exp(-(w/4)x^2-Qx^h)\approx\frac{w}{(2\pi)^2}\frac{1}{2^{h+2}\mathcal{D}t}\int_0^{1/(w)^{1/2}}dx\,x^{h+1}\exp(-Qx^h),$$

$$Q\equiv r^2(2^{h+2}\mathcal{D}t)^{-1},\quad \mathcal{D}t>\frac{1}{(2w^{1/2})^{2+h}},\quad \frac{r^2}{\mathcal{D}t}<2^h w^{h/2}.$$

The integral can be explicitly written in terms of the truncated gamma function, and is a function of $r^2/\mathcal{D}t$. In other words at zero momentum one gets diffusion behavior. One expects anomalous behavior, with an h dependence, at large momenta, but unfortunately the $\mathbf{k}$ dependence of W is not easily obtained. We merely point out the following: the density matrix has a stretched exponential in x for every initial condition, Gaussian shaped in $\mathbf{r}$. Recall that, for real p, $\exp(-a|p|^h)$ is the Fourier transform of the Levy distribution $L_h(z)\approx 1/z^{(h+1)}$ $(z\gg 1)$.[19] It is reasonable to assume that the large-$\mathbf{k}$ behavior of W is only marginally influenced by the initial $\mathbf{x}$ dependence $(x\ll 1)$, if sufficiently smooth. The $\mathbf{k}$ dependence of the Wigner function is then approximated by the Levy distribution in $\mathbf{k}$, or more precisely by its derivative with respect to the parameter Q, which takes into account the factor x^h in the integrand. So far we have been dealing with the $k_0\to 0$ limit; as k_0 is small but different from zero we must take into account other terms, starting with the first correction, which gives pure diffusion in $\mathbf{x}$ (see the previous section). A situation of this sort can be studied, at least in the regime $2<h<4$. We have that D_r is $O(k_0^{h-2})$; this, together with the next to leading term gives $[k_0^h/(\xi)^2\ll 1]$

$$\hat{H}\approx\left(\frac{\hbar}{g}\right)^2\frac{1}{\Pi}\left((-1/4)\frac{1}{(k_0x)^2}(\hat{T}^{-1})_{\alpha,\beta}\frac{\partial}{\partial r_\alpha}\frac{\partial}{\partial r_\beta}+(2\pi h)\nabla_{\mathbf{x}}^2\right). \tag{22}$$

Again we disregard anisotropy; a Fourier transform with respect to $\mathbf{r}$ leads to a particle in a centrifugal barrier:

$$\frac{\partial\rho}{\partial\tau}=\left[-\left(\frac{q}{\xi}\right)^2+\nabla_\xi^2\right]\rho,$$
$$\tau=\left(\frac{\hbar}{g}\right)^2\frac{k_0^h 2\pi h}{\Delta}t, \tag{23}$$
$$q^2=(4/3)(h-2)\frac{K^2}{k_0^2},$$

where $\mathbf{K}$ is the momentum with respect to $\mathbf{r}$. If we search for a similarity solution, $\rho=f(u,q)$, $u\equiv\xi^2/\tau$, the result is

$$f''+\left(\frac{1}{u}+1/4\right)f'-\left(\frac{q}{2u}\right)f=0. \tag{24}$$

Two independent solutions (provided q is not an integer) are $f(u,q)=u^sF(s,2s+1;-u/4)$, where $s=q/2,-q/2$ and F is the confluent hypergeometric function. In order to avoid sin-

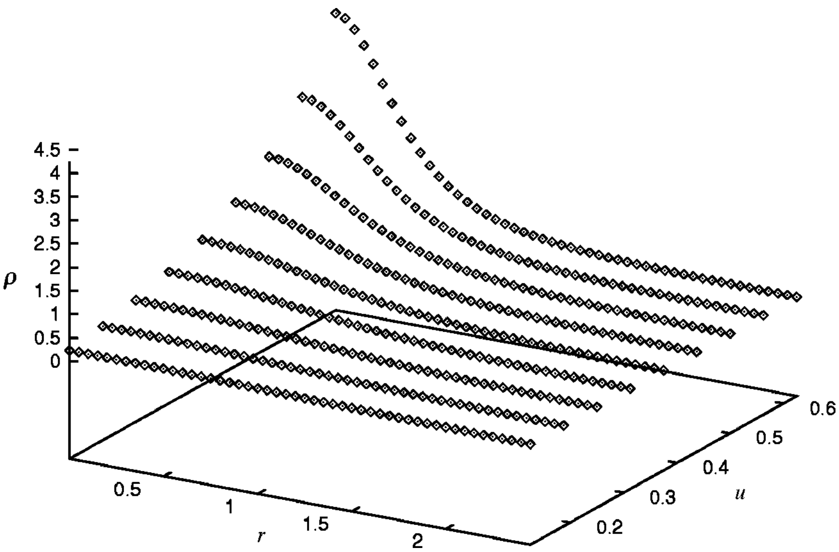

FIG. 1. Plot of the solution $\rho(r,u)$ [see Eq. (24)] at different values of u as u approaches zero (large time behavior).

gular behavior at small u we exclude the negative s case. We numerically computed the Fourier transform in order to reconstruct the **r** dependence; in Fig. 1 we show $\rho(r,u)$ for a set of values $u<1$, i.e., in the large time regime. We find that the width in r increases, as u approaches zero, linearly in the logarithm of u^{-1}, up to four orders of magnitude (see Fig. 2). Disregarding the next to leading term would result, similarly to the previous case, in the combination r^2x^2/t. In conclusion then, when the fluctuations of the magnetic field grow with distance, the propagation is strongly inhibited, and the spread of the particle position grows with the logarithm of time.

V. CONCLUSIONS

We studied the two-dimensional dynamics of a particle in the presence of a stochastic magnetic field in the fast fluctuation regime. We have found a transition by varying the exponent of the disorder correlator. The reduced dynamics after averaging over the baricentric coordinate, which is related with the motion in momentum space, gives diffusion as the dominant behavior in the $k_0 \ll 1$ regime. Superdiffusion arises in the first correction, with the scaling $t \approx x^{2-h}$ when $h<2$ and $t \approx (\ln x)^2$ when $2<h<4$. The dominance of diffusion originates in the $\mathbf{A}^2$ term; if this term is neglected we obtain fluctuation-sustained superdiffusion with the previously mentioned power laws. The tendency of the retarded and advanced trajectories to spread apart very rapidly (favored by the $\mathbf{A}\cdot\mathbf{p}$ term) is partly stabilized by the quadratic term. In the general case, the limit $k_0 \to 0$ can be performed exactly in the range $h<2$. The solution [see Eq. (20)] shows diffusion in **r**, with the dependence $r \approx (t/x^h)^{1/2}$. The Wigner function W at zero momentum, i.e., the average over particle separation, describes diffusion ($t \approx r^2$); at **k** large it appears in the form $W \approx W(r^2/t,\mathbf{k})$, its **k** behavior being associated with the Levy distribution in the momentum space. Random

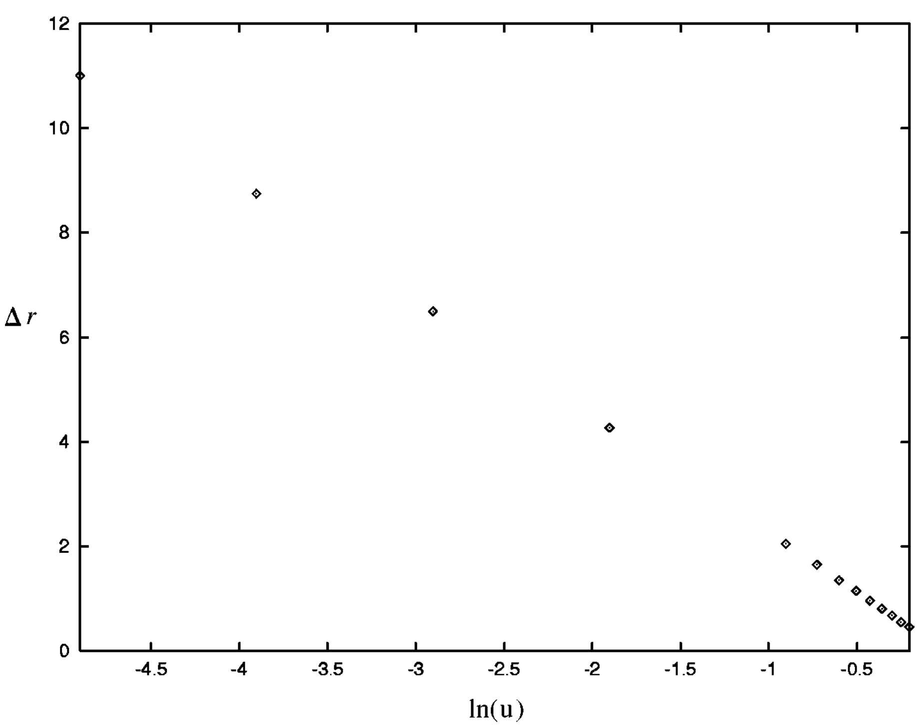

FIG. 2. Width in Δr of the solution vs ln(u).

walk in a quantum mechanical-system was found in the Harper model at its critical point.[20,21] It was also derived from a two-dimensional anisotropic random lattice model, describing edge propagation along the surface of a quantum Hall multilayer[22–25]. In that case backscattering is allowed in the direction of the Hall field and neglected in the orthogonal direction; if the latter is interpreted as time one has a one-dimensional chain with time-dependent hopping disorder where the quantum particle undergoes diffusion. As long as hopping mimicks the presence of a magnetic field, this model can be taken as a one-dimensional version of ours. Propagation is further inhibited when $2<h$; by including the leading and next to leading terms we determined an explicit solution in this regime [see Eq. (24) and arguments following it]. From the large time behavior of the density matrix $\rho(\mathbf{r},\mathbf{x},t)$ one extracts $r\approx\ln(t/x^2)$: this is a much slower spread than expected from the leading term alone, giving $r\approx(t/x^2)^{1/2}$. This ultraslow, logarithmic diffusion occurs in biased random motion on percolation clusters or globally isotropic fractals (see Ref. 26 and references therein). Quantum mechanically, it was found by suitably perturbing at a single point an Anderson Hamiltonian in the localized phase.[27] When disorder is static,[28,29] smoothly varying magnetic fields confine nearly free states within a narrow one-dimensional region along the zero-field lines, such lines thus providing a quantum percolation network. The present case can be more properly depicted in terms of abrupt jumps in the space of momenta, with loss of phase coherence. This apparently results in random walk when the correlator of the magnetic field B decays with distance $(h<2)$ and in logarithmic behavior when it grows $(2<h)$.

ACKNOWLEDGMENTS

We thank R. Artuso for pointing to our attention Refs. 21 and 27.

[1] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).
[2] G. Falkovich, I. Kolokolov, V. Lebedev, and A.A. Migdal, Phys. Rev. E **54**, 4896 (1996).
[3] L.B. Ioffe and A.I. Larkin, Phys. Rev. B **39**, 8988 (1989); L.B. Ioffe, and V. Kalmeyer, *ibid.* **44**, 750 (1991); N. Nagaosa and P.A. Lee, Phys. Rev. Lett. **64**, 2450 (1990).
[4] M.V. Feigelman, Physica A **168**, 319 (1990).
[5] E.M. Chudnovsky, Phys. Rev. B **51**, 15 351 (1995).
[6] V. Kalmeyer and S.C. Zhang, Phys. Rev. B **46**, 9889 (1992).
[7] B.I. Halperin, P.A. Lee, and N. Reed, Phys. Rev. B **47**, 7312 (1993).
[8] Y. Avishai, Y. Hatsugai, and M. Kohmoto, Phys. Rev. B **47**, 9561 (1993).
[9] V. Kalmeyer, D. Wei, D.P. Arovas, and S.C. Zhang, Phys. Rev. B **48**, 11 095 (1993).
[10] D.Z. Liu, X.C. Xie, S. Das Sarma, and S.C. Zhang, Phys. Rev. B **52**, 5858 (1995).
[11] A.G. Aronov, A.D. Mirlin, and P. Wolfle, Phys. Rev. B **49**, 16 609 (1994).
[12] S.C. Zhang and D. Arovas, Phys. Rev. Lett. **72**, 1886 (1994).
[13] J. Miller and J. Wang, Phys. Rev. Lett. **76**, 1461 (1996).
[14] N. Hatano and D.R. Nelson, Phys. Rev. Lett. **77**, 570 (1996).
[15] K.B. Efetov, Phys. Rev. B **56**, 9630 (1997).
[16] B. Cardinetti, V.G. Benza, and L. Molinari, Phys. Rev. B **52**, R5499 (1995).
[17] A.G. Aronov and P. Wolfle, Phys. Rev. B **50**, 16 574 (1994).
[18] M. Lesieur, *Turbulence in Fluids* (Kluwer Academic, Dordrecht, 1990).
[19] E.W. Montroll and B.J. West, in *Fluctuation Phenomena*, edited by E.W. Montroll and J.L. Lebowitz (North-Holland, Amsterdam, 1987).
[20] H. Hiramoto and S. Abe, J. Phys. Soc. Jpn. **57**, 230 (1988).
[21] T. Geisel, R. Ketzmerick, and G. Petschel, Phys. Rev. Lett. **66**, 1651 (1991).
[22] J.T. Chalker and A. Dohmen, Phys. Rev. Lett. **75**, 4496 (1995).
[23] L. Balents and M.P.A. Fisher, Phys. Rev. Lett. **76**, 2782 (1996).
[24] L. Balents, M.P.A. Fisher, and M.R. Zirnbauer, Nucl. Phys. B **483**, 601 (1997).
[25] H. Mathur, Phys. Rev. Lett. **78**, 2429 (1997).
[26] J.P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[27] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, Phys. Rev. Lett. **75**, 117 (1995).
[28] J.E. Muller, Phys. Rev. Lett. **68**, 385 (1992).
[29] D.B. Chklovskii and P.A. Lee, Phys. Rev. B **48**, 18 060 (1993).