Field-theoretic approach to the Lifshitz point

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We study the renormalization of the field theory that describes the Lifshitz point (LP). Our motivation was an old controversy on the order-ϵ^2 values of critical exponents for this multicritical point. First we analyze the Green functions at the LP where some simplifications occur. The primitively divergent diagrams are identified and renormalization prescriptions that eliminate ultraviolet divergences to all orders of perturbation are found. The Green functions in the neighborhood of the LP are expanded in terms of the Green functions calculated at the LP. This enables us to derive the renormalization-group equation satisfied by the renormalized Green functions and by analyzing its solutions we find expressions for the critical exponents that hold to all orders of perturbation. Finally, we obtain generalized scaling relations for the exponents associated with the LP. [S0163-1829(98)03234-2]

I. INTRODUCTION

The Lifshitz point (LP) (Ref. 1) is a multicritical point that occurs in magnetic systems,[2] liquid crystals,[3] charge-transfer salts,[4] structural phase transitions,[5] domain-wall instabilities,[6] and ferroelectric crystals.[7] Hornreich[8] and Selke[9] reviewed most of the work related to this special point. In order to see how a LP arises, consider the Landau free energy of a system described by a scalar order parameter M:[8]

$$F=a_2M^2+a_4M^4+a_6M^6+\cdots+c_1(\nabla M)^2+c_2(\nabla^2 M)^2+\cdots, \tag{1}$$

where the coefficients a_i and c_i depend on the temperature T and on an external parameter p. The system has a LP if, as we move along the critical line $T_c(p)$ (obtained from the condition $a_2=0$), the coefficient $c_1(T,p)$ changes sign. The point (T_L,p_L) on the critical line at which $c_1=0$ is the LP. In this case, the c_2 term becomes relevant and has to be kept.

A simple model with these properties is the axial next-nearest-neighbor Ising (ANNNI) model.[10] It consists of a spin-$\frac{1}{2}$ Ising model on a cubic lattice with nearest-neighbor ferromagnetic couplings and next-nearest-neighbor competing antiferromagnetic couplings along a single lattice axis. Its phase diagram, in the p-T plane, where p is the ratio between the competing couplings, is divided into three regions. In addition to the usual paramagnetic and ferromagnetic phases, due to the competition there is a region with modulated phases, which are spatially modulated structures characterized by a wave vector **k**. High-temperature series techniques were utilized to study the neighborhood of the LP in the three-dimensional ANNNI model by Redner and Stanley, Oitmaa, and Mo and Ferer.[11] The critical exponents β_ℓ, γ_ℓ, and ν_ℓ were estimated from Monte Carlo data by Selke and Fisher.[12]

The first renormalization-group calculation of critical exponents associated with the LP was performed by Hornreich, Luban, and Shtrikman[1] using the Ginsburg-Landau-Wilson Hamiltonian

$$H=\frac{1}{2}\int_q v(q)\vec{\Phi}_{-q}\cdot\vec{\Phi}_q+\frac{\lambda}{4!}\int_{q_1}\int_{q_2}\int_{q_3}(\vec{\Phi}_{q_1}\cdot\vec{\Phi}_{q_2})\times(\vec{\Phi}_{q_3}\cdot\vec{\Phi}_{-q_1-q_2-q_3}),$$

$$v(q)=r_0+q_\beta^2+c_0q_\alpha^2+(q_\alpha^2)^2, \tag{2}$$

$$q_\alpha^2\equiv\sum_{\mu=1}^{m}q_\mu^2,\quad q_\beta^2\equiv\sum_{\mu=m+1}^{d}q_\mu^2,$$

where $\vec{\Phi}_q$ is an n-component order parameter. Note that the space is divided into two isotropic subspaces: an α subspace of dimension m, and a β subspace of dimension $d-m$. A LP is associated with a wave-vector instability in m directions of the α subspace. A large class of models is described by the Hamiltonian (2), each one parametrized by different values of n and m, $1\leqslant m\leqslant 8$.[1] The ANNNI model corresponds to the $m=n=1$ case. At the Lifshitz point, both r_0 and c_0 go to zero and the q_α^4 term has to be kept. The upper critical dimension $d_u(m)$, above which classical critical behavior is expected, is obtained by means of the Ginsburg criterion and is given by[1]

$$d_u(m)=4+\frac{m}{2},\quad m\leqslant 8. \tag{3}$$

Using renormalization-group techniques and an ϵ expansion about $d_u(m)$, Hornreich, Luban, and Shtrikman[1] calculated, for all m, the exponents $\nu_{\ell 2}$ and $\nu_{\ell 4}$ to order ϵ, and, for $m=8$, $\nu_{\ell 2}$, $\nu_{\ell 4}$, and $\eta_{\ell 4}$ to order ϵ^2, where the subscript $\ell 4$ ($\ell 2$) refers to the α subspace (β subspace). Mukamel[13] determined $\eta_{\ell 2}$ and $\eta_{\ell 4}$ to order ϵ^2 for all m, and β_k to order ϵ^2 for $m<6$ (one does not expect helical long-range order for $m\geqslant 6$). Hornreich and Bruce[14] calculated, for $m=1$, the exponents $\eta_{\ell 2}$ and $\eta_{\ell 4}$ to order ϵ^2 and the exponent β_k to order ϵ^2 and their result agrees with Mukamel's. However, Sak and Grest[15] performed an independent calculation, for $m=2$ and $m=6$, of $\eta_{\ell 2}$, $\eta_{\ell 4}$, and β_k to order ϵ^2, obtaining results that are different from Mukamel's. As emphasized in a review of the papers[8,9] cited above, the reason for this discrepancy is not clear.

All renormalization-group calculations mentioned above use the Wilson-Fisher momentum space technique.[16] Since the Hamiltonian (2) is not rotationally invariant and the propagator contains a quartic term, the two-loop integrals over momentum shells are extremely involved, and in all calculations performed so far different approximations were used. Due to the difficulty in analyzing which approximation gives the correct two-loop corrections, we have resorted to a different approach. We decided to use field theory to calculate exactly the order ϵ^2 contributions to the critical exponents for the Lifshitz point. In order to do this, we first had to analyze the renormalization of the theory described by the Hamiltonian (2), and then to adapt to our problem a formalism introduced by Weinberg[17] and applied to critical phenomena by Zinn-Justin.[18] A clear presentation of this technique can be found in Amit's book.[19] In its original formulation, the critical behavior of the ϕ^4 theory is obtained by expanding all Green functions in terms of the massless Green functions calculated at the critical point. In our case we expand about the LP. Our formalism applies to all values of m in the Hamiltonian (2) to all orders in perturbation theory, and allows us to identify the critical exponents in terms of the renormalization constants. It is important to mention that field theory has already been applied to study other properties of the Lifshitz point. Nasser and Folk[20] studied crossover phenomena, Abdel-Hady and Folk[21] analyzed tricritical Lifshitz points, and Nasser, Abdel-Hady, and Folk[22] calculated universal amplitude ratios. In the present work, a thorough study of the renormalization of the field theory that describes the LP is made and used to obtain expressions for the critical exponents.

This paper is organized as follows. In Sec. II we review briefly the field theory formalism emphasizing the modifications that have to be done to apply it to the Hamiltonian (2). In Sec. III we derive the renormalization-group equations, identify the critical exponents, and demonstrate that they satisfy generalized scaling relations. In Sec. IV we present our conclusions. Finally, in the Appendix we show in some detail the cancellation mechanism of the divergences due to the insertion of the two-point function into other diagrams. This cancellation is more involved than in the usual ϕ^4 theory.

II. PERTURBATIVE FIELD THEORY AND CRITICAL PHENOMENA

In this section we present a brief review of renormalized field theory and its relation to critical phenomena.[19] The starting point consists in using the Ginsburg-Landau-Wilson effective Hamiltonian (2) with an extra parameter σ_0. Thus, instead of $v(q)$ given in Eq. (2), we shall use

$$v(q)=r_0+q_\beta^2+c_0q_\alpha^2+\sigma_0(q_\alpha^2)^2. \tag{4}$$

The dimensionless parameter σ_0, as we are going to show below, plays an important role in the renormalization of the two-point Green function. As a consequence of our choosing it dimensionless, the α components of the momentum q have dimension of square root of mass, $[q_\alpha]=[\kappa^{1/2}]$, where κ has dimension of mass. The parameters r_0 and c_0 are related to the temperature and to p by $r_0=T-T_{0L}$ and $c_0\sim p-p_{0L}$, where T_{0L} and p_{0L} are the mean-field coordinates of the Lifshitz point in the p-T plane. In momentum space there is an ultraviolet cutoff, $\Lambda\simeq 1/a$, where a is the lattice spacing in the original system.

All equilibrium properties can be obtained from the one-particle irreducible (1PI) Green functions $\Gamma^{(N,L)}(k_1,\ldots,k_N,p_1,\ldots,p_L;\sigma_0,c_0,r,\lambda,\overline{\phi},\Lambda)$, which contain N external legs, L insertions of $\phi^2(p_i)$ operators, and that are renormalized in such a way that the corresponding renormalized functions $\Gamma_R^{(N,L)}$ are finite in the infinite cutoff limit when the space dimension $d\leqslant d_u(m)$. The magnetization $\overline{\phi}$ is zero in the paramagnetic phase and not null in the ferromagnetic phase in zero magnetic field. In this paper we shall be concerned with the calculation of critical exponents for the LP in the paramagnetic and ferromagnetic regions. In this case the magnetization is constant, and can be used with a single component order parameter. The dependence on n, the number of components of $\vec{\Phi}_q$, is contained only in the combinatorial factors of the Feynman diagrams and can be inserted in the last stage of calculations.

The inverse of the zero-field susceptibility χ is proportional to $\Gamma^{(2,0)}$ calculated at zero external momenta:[19]

$$\chi^{-1}=\beta^{-1}\Gamma^{(2,0)}(0,0,\sigma_0,c_0,r_0,\lambda,\overline{\phi}=0,\Lambda). \tag{5}$$

At criticality, χ diverges, and the equation that determines the critical line $T_c(p)$ is given by

$$\Gamma^{(2,0)}(0,0,\sigma_0,c_0,r_0,\lambda,0,\Lambda)=0. \tag{6}$$

At the Lifshitz point the coefficient of k_α^2 is zero and, in addition to Eq. (6), the coordinates (c_L,r_L) of the Lifshitz point also satisfy

$$\left.\frac{\partial}{\partial k_\alpha^2}\Gamma^{(2,0)}(k,-k,\sigma_0,c_0,r_0,\lambda,0,\Lambda)\right|_{k_\alpha^2=0}=0. \tag{7}$$

Recall that $r_0=T-T_{0L}$ and $c_0\sim p-p_{0L}$, and, to lowest order in perturbation theory (mean-field approximation), $r_L=T_L-T_{0L}=0$ and $c_L\sim p_L-p_{0L}=0$. As we take fluctuations into account, T_L and p_L move away from their mean-field values. The corrections are determined by expanding r_L and c_L in the coupling constant λ, inserting these expansions in Eqs. (6) and (7) and solving them perturbatively. When we expand the propagators in the Feynman diagrams about $r_L=c_L=0$ we obtain integrals without any dimensional parameters. These integrals in the dimensional regularization scheme vanish and all corrections to the mean-field coordinates of the Lifshitz point are exactly zero. Thus, Green functions at the LP are calculated with the propagator $(\sigma_0q_\alpha^4+q_\beta^2)^{-1}$. From now on we shall use dimensional regularization, calculating integrals in dimension $d=d_u(m)-\epsilon$, and taking the limit $\Lambda\to\infty$.

The identification of the primitively divergent 1PI functions is not straightforward. Due to the fact that the α components of momenta in the propagator are raised to the fourth power and the β components to the second power in the propagators, naive power counting does not give the correct degree of divergence δ of the diagrams. To obtain the correct

δ, we first integrate over the β components. Consider a general diagram that contributes to $\Gamma^{(N,L)}$ with N external legs, L insertions of operators ϕ^2, I internal lines (propagators), ℓ loops (ℓ integrations over internal momenta), and v vertices. These variables are not independent. A ϕ^4 vertex has four lines, a ϕ^2 insertion has two lines, and each internal line is shared between two vertices, or two insertions, or between a vertex and an insertion. Thus, we have the relation

$$4v+2L=2I+N. \tag{8}$$

According to the Feynman rules there is a momentum associated with each internal line and integration over this internal momentum. However, the conservation δ functions at vertices and insertions eliminate $v+L$ integrations. There remains only one δ function that expresses the overall momentum conservation of the diagram, and we obtain

$$\ell=I-v-L+1. \tag{9}$$

Each propagator in this diagram has the form

$$\left[\sigma_0\left(\sum_i q_{i\alpha}+K_\alpha\right)^4+\left(\sum_i q_{i\beta}+K_\beta\right)^2\right]^{-1}, \tag{10}$$

where K stands for the sum of the external momenta that flow through the propagator and the sum is over internal momenta. We can use Feynman parameters to put all I propagators together, obtaining a single term in the denominator raised to the power I. After integrating over $q_{i\beta}$, $i=1,2,\ldots,\ell$, the resulting term in the denominator, which now only contains the $q_{i\alpha}$ components, is raised to the power $I-\ell d_\beta/2$. Using naive power counting for the remaining α components, we obtain $\delta=\ell d_\alpha-4(I-\ell d_\beta/2)$. Using Eqs. (8) and (9), we rewrite this expression as

$$\delta=\left[d-\left(4+\frac{d_\alpha}{2}\right)\right]I+\left(L-\frac{N}{2}\right)\left(\frac{d_\alpha}{2}+d_\beta\right). \tag{11}$$

Recalling that $d_\alpha=m$, we see that the term that depends on I in the equation above cancels when d is equal to the upper critical dimension d_u, see Eq. (3). At $d=d_u$, the only 1PI functions with primitive divergences ($\delta\geqslant 0$) are $\Gamma^{(2,0)}$, $\Gamma^{(4,0)}$, $\Gamma^{(2,1)}$, and $\Gamma^{(0,2)}$. These are the same as in the ϕ^4 theory and here we can also neglect $\Gamma^{(0,1)}$, which gives an infinite constant.

As in the usual ϕ^4 theory, which describes the criticality of the Ising model, all $\Gamma^{(N,L)}$ at the LP are renormalized multiplicatively except $\Gamma^{(0,2)}$, which also requires additive renormalization. We have checked this point by performing a two-loop calculation of the primitively divergent Green functions for $m=2$ and 6. However, there are differences. For example, the divergent part of $\Gamma^{(2,0)}$ has the structure

$$\Gamma^{(2,0)}=\frac{A\sigma_0}{\epsilon}k_\alpha^4+\frac{B}{\epsilon}k_\beta^2+\mathcal{O}(\epsilon^0), \tag{12}$$

with $A\neq B$. Thus, besides field renormalization we need the renormalization of the σ_0 parameter to eliminate the poles of $\Gamma^{(2,0)}$. In general, the relations between $\Gamma^{(N,L)}$ and $\Gamma_R^{(N,L)}$ at the LP are given by

$$\Gamma_R^{(N,L)}(k_i,p_i,\sigma,g,\kappa)$$

$$=Z_\phi^{N/2}Z_{\phi^2}^L[\Gamma^{(N,L)}(k_i,p_i,\sigma_0,\lambda)-\delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}(p,-p,\sigma_0,\lambda)\Big|_{\substack{\sigma p_\alpha^4=\kappa^2\\ p_\beta^2=0}}], \tag{13}$$

where g is the renormalized coupling constant, $\sigma=Z_\sigma\sigma_0$ is the renormalized σ_0 parameter, Z_σ, Z_ϕ, and Z_{ϕ^2} are renormalization constants, and κ is an arbitrary momentum scale. Bare parameters and renormalization constants are calculated through the renormalization conditions

$$\frac{\partial}{\partial k_\alpha^4}\Gamma_R^{(2,0)}(k,-k,\sigma,g,\kappa)\Big|_{\substack{\sigma k_\alpha^4=\kappa^2\\ k_\beta^2=0}}=\sigma, \tag{14}$$

$$\frac{\partial}{\partial k_\beta^2}\Gamma_R^{(2,0)}(k,-k,\sigma,g,\kappa)\Big|_{\substack{\sigma k_\alpha^4=0\\ k_\beta^2=\kappa^2}}=1, \tag{15}$$

$$\Gamma_R^{(4,0)}(k_1,\ldots,k_4,\sigma,g,\kappa)|_{sp_\alpha}=g, \tag{16}$$

$$\Gamma_R^{(2,1)}(k_1,k_2,p,\sigma,g,\kappa)|_{\overline{sp_\alpha}}=1, \tag{17}$$

$$\Gamma_R^{(0,2)}(p,-p,\sigma,g,\kappa)\Big|_{\substack{\sigma p_\alpha^4=\kappa^2\\ p_\beta^2=0}}=0, \tag{18}$$

where the renormalization points are defined as follows: sp_α means $\sigma^{1/2}k_{i\alpha}k_{j\alpha}=\kappa(4\delta_{ij}-1)/4$; $\overline{sp_\alpha}$ means $\sigma^{1/2}k_{i\alpha}^2=3\kappa/4$, $\sigma^{1/2}k_{1\alpha}k_{2\alpha}=-\kappa/4$, $\sigma^{1/2}(k_1+k_2)_\alpha^2=\sigma^{1/2}p_\alpha^2=\kappa$, and, except in Eq. (15), the external momenta at which the values of the Green functions are evaluated have no components in the β subspace. This choice of renormalization points will make bare parameters and renormalization constants σ independent, as we are going to show below.

Let us discuss in more detail the dependence of $\Gamma_R^{(N,L)}$ on σ. In order to do that we first determine the dependence of $\Gamma^{(N,L)}$ on σ_0. In perturbation theory, $\Gamma^{(N,L)}$ is a sum of infinite 1PI diagrams. Consider one of these diagrams, with v vertices, I propagators, L insertions of operators ϕ^2, and ℓ loops. If we make the change of variable $q_{i\alpha}\to\sigma_0^{-1/4}q_{i\alpha}$, for the α components of all ℓ internal momenta q_i, then $d^dq_i\equiv d^{d_\alpha}q_{i\alpha}d^{d_\beta}q_{i\beta}\to\sigma_0^{-d_\alpha/4}d^dq_i$ and the whole diagram is multiplied by a factor $\sigma_0^{-\ell d_\alpha/4}$. In the propagators, see Eq. (10), after changing variables, only the α components of the external momenta are multiplied by $\sigma_0^{1/4}$. Combining Eqs. (8) and (9) we obtain $\ell=v-N/2+1$, and the global factor can also be written as $(\sigma_0^{-d_\alpha/4})^{v-N/2+1}$. Part of it $(\sigma_0^{-vd_\alpha/4})$ multiplies the coupling constants λ since each vertex has a factor λ. Thus, the α components of all external momenta in the 1PI Green functions are multiplied by $\sigma_0^{1/4}$ and the coupling constant by $\sigma_0^{-d_\alpha/4}$. There remains a global factor $(\sigma_0^{d_\alpha/4})^{N/2-1}$. This analysis holds for all diagrams of $\Gamma^{(N,L)}$ and we can finally write

$$\Gamma^{(N,L)}(k_i,p_i,\sigma_0,\lambda)=(\sigma_0^{d_\alpha/4})^{N/2-1}\Gamma^{(N,L)}(\sigma_0^{1/4}k_{i\alpha},k_{i\beta},\sigma_0^{1/4}p_{i\alpha},p_{i\beta},1,\lambda\sigma_0^{-d_\alpha/4})$$
$$=(\sigma^{d_\alpha/4})^{N/2-1}\Gamma^{(N,L)}(\sigma^{1/4}k_{i\alpha},k_{i\beta},\sigma^{1/4}p_{i\alpha},p_{i\beta},Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4}), \quad (19)$$

where we used the expression $\sigma_0=Z_\sigma^{-1}\sigma$ to write the last equality in Eq. (19). If we can show that the renormalization constants Z_ϕ, Z_{ϕ^2}, and Z_σ do not depend on σ, then Eqs. (19) and (13) can be used to give the dependence of $\Gamma_R^{(N,L)}$ on σ.

Equation (19) enables us to rewrite the renormalization conditions [Eqs. (14) through (18)] as

$$\frac{\partial}{\partial k_\alpha^4}\left[Z_\phi\Gamma^{(2,0)}(\sigma k_\alpha^4,k_\beta^2,Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4})\right]_{\substack{\sigma k_\alpha^4=\kappa^2\\ k_\beta^2=0}}=\sigma, \quad (20)$$

$$\frac{\partial}{\partial k_\beta^2}\left[Z_\phi\Gamma^{(2,0)}(\sigma k_\alpha^4,k_\beta^2,Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4})\right]_{\substack{\sigma k_\alpha^4=0\\ k_\beta^2=\kappa^2}}=1, \quad (21)$$

$$\sigma^{d_\alpha/4}Z_\phi^2\Gamma^{(4,0)}(\sigma^{1/4}k_{i\alpha},k_{i\beta},Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4})|_{sp_\alpha}=g, \quad (22)$$

$$Z_\phi Z_{\phi^2}\Gamma^{(2,1)}(\sigma^{1/4}k_{i\alpha},k_{i\beta},\sigma^{1/4}p_\alpha,p_\beta,Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4})|_{\overline{sp_\alpha}}=1, \quad (23)$$

$$\Gamma^{(0,2)}(\sigma p_\alpha^4,p_\beta^2,Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4})-\Gamma^{(0,2)}(\sigma p_\alpha^4,p_\beta^2,Z_\sigma^{-1},\lambda\sigma^{-d_\alpha/4})\Big|_{\substack{\sigma p_\alpha^4=\kappa^2\\ p_\beta^2=0}}=0. \quad (24)$$

At this stage it is convenient to introduce the dimensionless coupling constants u_0 and u such that

$$u\kappa^{4-D}=g\sigma^{-d_\alpha/4},$$
$$u_0\kappa^{4-D}=\lambda, \quad (25)$$
$$D\equiv d_\alpha/2+d_\beta.$$

Equations (20)–(24) can be satisfied by expressing $u_0\sigma^{-d_\alpha/4}$ and renormalization constants Z_σ, Z_ϕ, and Z_{ϕ^2} as power series in u. In fact, due to the rotational symmetry in each subspace, $\Gamma^{(N,L)}$ depends only on the external momenta through scalar products of their α and β components separately. Recall that the α components of the momenta are always multiplied by $\sigma^{1/4}$. With our choice for the renormalization points [see the definitions after Eq. (18)] this dependence on σ disappears. This is less obvious for Eq. (20). In this case rotational invariance implies that $\Gamma_R^{(2,0)}=\Gamma_R^{(2,0)}(\sigma k_\alpha^4,k_\beta^2)$. After calculating its derivative with respect to k_α^4 and evaluating it at the renormalization point $\sigma k_\alpha^4=\kappa^2$, $k_\beta^2=0$, a global factor σ remains. However, this factor is canceled out by the σ on the right-hand side of Eq. (20). Finally, according to Eq. (19), when σ is factored out the coupling constant $\lambda=\kappa^{4-D}u_0$ is multiplied by $\sigma^{-d_\alpha/4}$. Expanding the product $u_0\sigma^{-d_\alpha/4}$ in powers of u, instead of expanding only u_0 as in the usual ϕ^4 theory, we eliminate the last dependence on σ in Eqs. (20)–(24). In this way, we can satisfy these equations by expressing $u_0\sigma^{-d_\alpha/4}$ and renormalization constants as a power series in u only, as stated above. An alternative choice for the renormalization points consists in choosing, except in Eq. (20), the external momenta without components in the α subspace. In this case, it is clear again that Eqs. (20)–(24) do not depend on σ. However, we verified that the resulting two-loop integrals are more involved than in the previous case. We have calculated,[23] for $m=6$, the critical exponents using both choices for the external momenta. The results are the same and confirm the independence on σ.

Since the renormalization constants do not depend on σ, Eqs. (13) and (19) imply that

$$\Gamma_R^{(N,L)}(k_i,p_i,\sigma,u,\kappa)$$
$$=(\sigma^{d_\alpha/4})^{N/2-1}\Gamma_R^{(N,L)}(\sigma^{1/4}k_{i\alpha},k_{i\beta},\sigma^{1/4}p_{i\alpha},p_{i\beta},1,u,\kappa)$$
$$=Z_\phi^{N/2}Z_{\phi^2}^L\Bigg[\Gamma^{(N,L)}(k_i,p_i,\sigma_0,u_0\kappa^{4-D})$$
$$-\delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}(p,-p,\sigma_0,u_0\kappa^{4-D})\Big|_{\substack{\sigma p_\alpha^4=\kappa^2\\ p_\beta^2=0}}\Bigg], \quad (26)$$

where u is defined in Eq. (25) and all dependence of $\Gamma_R^{(N,L)}$ on σ is in the α components of the external momenta and in an overall multiplicative factor.

In an analogous way we derive the expression for the renormalized connected Green function $G_{cR}^{(N,L)}$ (see the Appendix for more details),

$$G_{cR}^{(N,L)}(k_i,p_i,\sigma,u,\kappa)$$
$$=Z_{\phi}^{-N/2}Z_{\phi^2}^{L}\bigg[G^{(N,L)}(k_i,p_i,\sigma_0,u_0\kappa^{4-D}) - \delta_{N,0}\delta_{L,2}G^{(0,2)}(p,-p,\sigma_0,u_0\kappa^{4-D})\bigg|_{\substack{\sigma p_\alpha^4=\kappa^2\\ p_\beta^2=0}}\bigg]. \quad (27)$$

The renormalization-group equation can now be obtained in the standard[19] way by first moving $Z_{\phi}^{N/2}$ and $Z_{\phi^2}^{L}$ to the left-hand side of Eq. (26) and then applying the operator $(\kappa\partial/\partial\kappa)_{\lambda,\sigma_0}$ to the resulting expression. In this way we obtain

$$\left\{\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}+\gamma_\sigma(u)\sigma\frac{\partial}{\partial\sigma}-\frac{N}{2}\gamma_\phi(u)+L\gamma_{\phi^2}(u)\right\}\times\Gamma_R^{(N,L)}(k_i,p_i,\sigma,u,\kappa)=\delta_{N,0}\delta_{L,2}\kappa^{D-4}B(u), \quad (28)$$

where

$$\kappa^{D-4}B(u)=-Z_{\phi^2}^2\kappa\frac{\partial}{\partial\kappa}\Gamma^{(0,2)}(p,-p,\sigma_0,u_0\kappa^{4-D})\bigg|_{\substack{\sigma p_\alpha^4=\kappa^2,\\ p_\beta^2=0}} \quad (29)$$

$$\beta(u)=\left(\kappa\frac{\partial u}{\partial\kappa}\right)_{\lambda,\sigma_0}, \quad (30)$$

$$\gamma_\sigma(u)=\left(\kappa\frac{\partial\ln Z_\sigma}{\partial\kappa}\right)_{\lambda,\sigma_0}, \quad (31)$$

$$\gamma_\phi(u)=\left(\kappa\frac{\partial\ln Z_\phi}{\partial\kappa}\right)_{\lambda,\sigma_0}, \quad (32)$$

$$\gamma_{\phi^2}(u)=-\left(\kappa\frac{\partial\ln Z_{\phi^2}}{\partial\kappa}\right)_{\lambda,\sigma_0}. \quad (33)$$

Green functions with $T\neq T_L$ can be expanded about T_L. This technique is analogous to the expansion of the renormalized ϕ^4 above and below T_c in terms of the massless critical theory introduced by Weinberg[17] and applied to critical phenomena by Zinn-Justin.[18] In our case, we expand Green functions about $T=T_L$ and $\overline{\phi}=0$. It can be shown[19] that

$$\Gamma^{(N,L)}(k_i,p_i,\sigma_0,c_0=0,\delta r,\lambda,\overline{\phi})=\sum_{I,J}\frac{(\delta r)^I(\overline{\phi})^J}{I!J!}\Gamma^{(N+J,L+I)}\times(k_i,l_i=0,p_i,q_i=0,\sigma_0,c_0=0,\delta r=0,\lambda,\overline{\phi}=0). \quad (34)$$

Note that the c_0 parameter was kept fixed and equal to zero, which is equivalent to keeping $p=p_L$. In this way, our analysis is restricted to the line $p=p_L$ in the p-T plane. This is irrelevant for the determination of the exponents $\eta_{\ell 2}$ and $\eta_{\ell 4}$ that are calculated precisely at the Lifshitz point. On the other hand, $\nu_{\ell 2}$ and $\nu_{\ell 4}$ require the determination of Green functions in the neighborhood of the Lifshitz point. However, we expect the exponents to be the same if we cross the boundary between the paramagnetic and the ferromagnetic phase, through the Lifshitz point, along any direction in the p-T plane. This is the case for the one-loop corrections. Our results agree with the one-loop results of Hornreich, Luban, and Shtrikman.[1] We expect that this invariance with direction also holds for our two-loop calculation of $\nu_{\ell 2}$ and $\nu_{\ell 4}$.

If we subtract the term $\delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}|_{\sigma p_\alpha^4=\kappa^2,p_\beta^2=0}$ from both sides of Eq. (34), multiply the resulting expression by $Z_\phi^{N/2}Z_{\phi^2}^{L}$, we define

$$t=Z_{\phi^2}^{-1}\delta r,\quad M=Z_\phi^{-1/2}\overline{\phi}, \quad (35)$$

where t and M are finite, introduce the dimensionless couplings u_0 and u [see Eqs. (25)], and use Eq. (13) we obtain

$$Z_\phi^{N/2}Z_{\phi^2}^{L}\bigg[\Gamma^{(N,L)}(k_i,p_i,\sigma_0,\delta r,u_0\kappa^{4-D},\overline{\phi}) - \delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}(p,-p,\sigma_0,0,u_0\kappa^{4-D},0)\bigg|_{\substack{\sigma p_\alpha^4=\kappa^2\\ p_\beta^2=0}}\bigg],$$
$$=\sum_{I,J}\frac{t^IM^J}{I!J!}\Gamma_R^{(N+J,L+I)}(k_i,l_i=0,p_i,q_i=0,\sigma,u,\kappa)$$
$$\equiv\Gamma_R^{(N,L)}(k_i,p_i,\sigma,t,u,M), \quad (36)$$

where the double sum in Eq. (36) defines $\Gamma_R^{(N,L)}$ in the neighborhood of the LP. Thus, we can renormalize Green functions away from T_L using the renormalization constants calculated at the LP solving Eqs. (20)–(24).

Recalling that each $\Gamma_R^{(N,L)}$ in the right-hand side of Eq. (36) satisfies the renormalization-group equation (28), it is simple to check that the Green functions away from T_L satisfy the renormalization-group equation

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta(u)\frac{\partial}{\partial u}+\gamma_\sigma(u)\sigma\frac{\partial}{\partial\sigma}-\frac{1}{2}\gamma_\phi(u)\left(N+M\frac{\partial}{\partial M}\right)+\gamma_{\phi^2}(u)\left(L+t\frac{\partial}{\partial t}\right)\right]\Gamma_R^{(N,L)}(k_i,p_i,\sigma,t,u,M,\kappa)$$
$$=\delta_{N,0}\delta_{L,2}\kappa^{D-4}B(u), \quad (37)$$

where the terms that appear in Eq. (37) were defined in Eqs. (29)–(33).

Finally, Eqs. (36) and (26) give us the dependence of the renormalized 1PI Green functions on σ,

$$\Gamma_R^{(N,L)}(k_i,p_i,\sigma,t,u,M,\kappa)=(\sigma^{d_\alpha/4})^{N/2-1}\Gamma_R^{(N,L)}\times(\sigma^{1/4}k_{i\alpha},k_{i\beta},\sigma^{1/4}p_{i\alpha},p_{i\beta},1,t,u,M\sigma^{d_\alpha/8},\kappa). \quad (38)$$

Equation (38) is valid in the broken symmetry phase. Above T_L, in the paramagnetic phase, one obtains an analogous expression with $M=0$.

III. IDENTIFICATION OF THE CRITICAL EXPONENTS

In field theory the critical behavior is obtained combining the solutions of the renormalization-group equation at the fixed point with dimensional analysis. In our case, parameters, fields, and 1PI Green functions without the momentum conserving δ function have the following dimensions: $[r_0]=[\kappa^2]$, $[c_0]=[\kappa^0]$, $[\sigma_0]=[\kappa^0]$, $[k_\beta]=[\kappa]$, $[x_\beta]=[\kappa^{-1}]$, $[k_\alpha]=[\kappa^{1/2}]$, $[x_\alpha]=[\kappa^{-1/2}]$, $[\lambda]=[\kappa^{4+m/2-d}]$, $[\phi(x)]=[\kappa^{-1+d_\alpha/4+d_\beta/2}]$, and $[\Gamma^{(N,L)}(k_i,p_i,\dots)]=[\kappa^{(d_\alpha/2+d_\beta)(1-N/2)+N-2L}]$. Note that our choosing σ_0 dimensionless leads to a coupling constant λ that is dimensionless at the upper critical dimension, as usual.

The exponents $\eta_{\ell 2}$, $\eta_{\ell 4}$, $\nu_{\ell 2}$, $\nu_{\ell 4}$, and $\gamma_{\ell 4}$ are determined from the renormalization-group equation for $\Gamma_R^{(2,0)}$ at the fixed point $u=u^*$. It suffices to consider the case $T\geqslant T_L$ for which $M=0$. Replacing u^* for u in Eq. (37) with $N=2$, $L=0$, and recalling[19] that $\beta(u^*)=0$, we obtain

$$\left[\kappa\frac{\partial}{\partial\kappa}+\gamma_\sigma^*\sigma\frac{\partial}{\partial\sigma}+\gamma_2^*t\frac{\partial}{\partial t}-\gamma_1^*\right]\Gamma_R^{(2,0)}(k_\alpha^4,k_\beta^2,\sigma,t,u^*,\kappa)=0, \tag{39}$$

where $\gamma_\sigma^*\equiv\gamma_\sigma(u^*)$, $\gamma_1^*\equiv\gamma_\phi(u^*)$, $\gamma_2^*\equiv\gamma_{\phi^2}(u^*)$. The definitions of $\gamma_\sigma(u)$, $\gamma_\phi(u)$, and $\gamma_{\phi^2}(u)$ are given in Eqs. (31), (32), and (33), respectively. Rotational invariance in each subspace guarantees that $\Gamma_R^{(2,0)}(k,t,u^*,\kappa)=\Gamma_R^{(2,0)}(k_\alpha^4,k_\beta^2,t,u^*,\kappa)$.

The general solution of Eq. (39) is given by

$$\Gamma_R^{(2,0)}(k_\alpha^4,k_\beta^2,\sigma,t,u^*,\kappa)=\kappa^{\gamma_1^*}\Phi^{(2,0)}(k_\alpha^4,k_\beta^2,\sigma\kappa^{-\gamma_\sigma^*},t\kappa^{-\gamma_2^*},u^*), \tag{40}$$

where $\Phi^{(2,0)}$ is an arbitrary function. Combining Eq. (38), which gives the dependence of $\Gamma^{(2,0)}$ on σ, and Eq. (40), we obtain

$$\Gamma_R^{(2,0)}(k_\alpha^4,k_\beta^2,\sigma,t,u^*,\kappa)=\kappa^{\gamma_1^*}\Phi^{(2,0)}(\sigma\kappa^{-\gamma_\sigma^*}k_\alpha^4,k_\beta^2,1,t\kappa^{-\gamma_2^*},u^*). \tag{41}$$

On the other hand, if ρ is an arbitrary mass parameter, then the dimensional analysis yields

$$\Gamma_R^{(2,0)}(k_\alpha^4,k_\beta^2,\sigma,t,u^*,\kappa)=\rho^2\Gamma_R^{(2,0)}\left(\frac{\sigma k_\alpha^4}{\rho^2},\frac{k_\beta^2}{\rho^2},\frac{t}{\rho^2},u^*,\frac{\kappa}{\rho}\right). \tag{42}$$

Combining Eqs. (41) and (42), we finally obtain

$$\Gamma_R^{(2,0)}(\sigma k_\alpha^4,k_\beta^2,t,u^*,\kappa)=\rho^{2-\gamma_1^*}\kappa^{\gamma_1^*}\Phi^{(2,0)}\left(\frac{\sigma k_\alpha^4}{\rho^2}\left(\frac{\kappa}{\rho}\right)^{-\gamma_\sigma^*},\frac{k_\beta^2}{\rho^2},\frac{t}{\rho^2}\left(\frac{\kappa}{\rho}\right)^{-\gamma_2^*},u^*\right). \tag{43}$$

The exponent $\eta_{\ell 4}$ is obtained putting $t=0$ and $k_\beta=0$ in Eq. (43), and choosing

$$\rho=\sigma^{1/(2-\gamma_\sigma^*)}\kappa^{-\gamma_\sigma^*/(2-\gamma_\sigma^*)}|k_\alpha|^{4/(2-\gamma_\sigma^*)}. \tag{44}$$

In this way,

$$\Gamma_R^{(2)}(\sigma k_\alpha^4,k_\beta^2,u^*,\kappa)=\sigma^{(2-\gamma_1^*)/(2-\gamma_\sigma^*)}\kappa^{(2\gamma_1^*-2\gamma_\sigma^*)/(2-\gamma_\sigma^*)}\times|k_\alpha|^{(8-4\gamma_1^*)/(2-\gamma_\sigma^*)}\Phi_R^{(2,0)}(1,0,0,u^*), \tag{45}$$

and from Eq. (45) we identify

$$\eta_{\ell 4}=4\left(\frac{\gamma_1^*-\gamma_\sigma^*}{2-\gamma_\sigma^*}\right). \tag{46}$$

In an analogous way, putting $t=0$, $k_\alpha=0$, and choosing $\rho=|k_\beta|$ in Eq. (43), we obtain the exponent $\eta_{\ell 2}$,

$$\eta_{\ell 2}=\gamma_1^*. \tag{47}$$

The exponents γ, $\nu_{\ell 2}$, and $\nu_{\ell 4}$ are also obtained from Eq. (43), keeping $t\neq 0$ and choosing

$$\rho=t^{1/(2-\gamma_2^*)}\kappa^{-\gamma_2^*/(2-\gamma_2^*)}. \tag{48}$$

Thus,

$$\Gamma_R^{(2,0)}(\sigma k_\alpha^4,k_\beta^2,1,t,u^*,\kappa)=\kappa^{\gamma_1^*}(t\kappa^{-\gamma_2^*})^{(2-\gamma_1^*)/(2-\gamma_2^*)}\Phi^{(2,0)}(\sigma\kappa^{-\gamma_\sigma^*}k_\alpha^4(t\kappa^{-\gamma_2^*})^{-(2-\gamma_\sigma^*)/(2-\gamma_2^*)},k_\beta^2(t\kappa^{-\gamma_2^*})^{-2/(2-\gamma_2^*)},1,u^*). \tag{49}$$

Inspecting Eq. (49) we note that $\Phi^{(2,0)}$ depends only on the combinations $|k_\alpha|\xi_{\ell 4}$ and $|k_\beta|\xi_{\ell 2}$ with

$$\xi_{\ell 4}\sim t^{(1/4)[(2-\gamma_\sigma^*)/(2-\gamma_2^*)]},$$

$$\xi_{\ell 2}\sim t^{1/(2-\gamma_2^*)}. \tag{50}$$

The correlation lengths $\xi_{\ell 4}\sim t^{-\nu_{\ell 4}}$ and $\xi_{\ell 2}\sim t^{-\nu_{\ell 2}}$ define the exponents $\nu_{\ell 4}$ and $\nu_{\ell 2}$. Thus,

$$\nu_{\ell 4}=\frac{1}{4}\left(\frac{2-\gamma_\sigma^*}{2-\gamma_2^*}\right),$$

$$\nu_{\ell 2}=\frac{1}{2-\gamma_2^*}. \tag{51}$$

Finally, putting $k_\alpha=k_\beta=0$ in Eq. (49), and recalling that $\Gamma_R^{(2,0)}(0,0,u^*,t,\kappa)\sim\chi^{-1}\sim t^\gamma$, we identify the exponent γ_ℓ:

$$\gamma_{\ell}=\frac{2-\gamma_1^*}{2-\gamma_2^*}. \tag{52}$$

The exponent δ_{ℓ} is obtained from the renormalization equation for $H(t,u,M,\kappa)=\Gamma_R^{(1,0)}(t,u,M,\kappa)$. The general solution of Eq. (37) for $N=1$ and $L=0$ at the fixed point $u=u^*$ is given by

$$H(\sigma,t,u^*,M,\kappa)=\kappa^{\gamma_1^*/2}h(\sigma\kappa^{-\gamma_\sigma^*},t\kappa^{-\gamma_2^*},u^*,M\kappa^{\gamma_1^*/2}). \tag{53}$$

Taking into account the dependence of $H=\Gamma^{(1,0)}$ on σ [see Eq. (38)], we obtain

$$H(\sigma,t,u^*,M,\kappa)=\sigma^{-d_\alpha/8}\kappa^{(1/2)\gamma_1^*+(d_\alpha/8)\gamma_\sigma^*}\times h(1,t\kappa^{-\gamma_2^*},u^*,\sigma^{d_\alpha/8}M\kappa^{(1/2)\gamma_1^*-(d_\alpha/8)\gamma_\sigma^*}). \tag{54}$$

Using dimensional analysis and recalling that $[H]=[\kappa^{1+d_\alpha/4+d_\beta/2}]\equiv[\kappa^{1+D/2}]$, $[M]=[\kappa^{-1+d_\alpha/4+d_\beta/2}]\equiv[\kappa^{-1+D/2}]$, and $[\rho]=[\kappa]$, we obtain

$$H(\sigma,t,u^*,M,\kappa)=\sigma^{-d_\alpha/8}\rho^{1+D/2}\left(\frac{\kappa}{\rho}\right)^{(1/2)\gamma_1^*+(d_\alpha/8)\gamma_\sigma^*}\times h\left(1,\frac{t}{\rho^2}\left(\frac{\kappa}{\rho}\right)^{-\gamma_2^*},u^*,\sigma^{-d_\alpha/8}\frac{M}{\rho^{-1+D/2}}\times\left(\frac{\kappa}{\rho}\right)^{(1/2)\gamma_1^*-(d_\alpha/8)\gamma_\sigma^*}\right). \tag{55}$$

In order to calculate the exponents β_{ℓ} and δ_{ℓ} we choose ρ such that

$$\sigma^{-d_\alpha/8}\frac{M}{\rho^{-1+D/2}}\left(\frac{\kappa}{\rho}\right)^{(1/2)\gamma_1^*-(d_\alpha/8)\gamma_\sigma^*}=1. \tag{56}$$

In this way, Eq. (55) becomes

$$H(\sigma,t,u^*,M,\kappa)=\kappa^{(1/2)\gamma_1^*+(d_\alpha/8)\gamma_\sigma^*}(M\kappa^{(1/2)\gamma_1^*-(d_\alpha/8)\gamma_\sigma^*})^{[D+2-\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*]/[D-2+\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*]}\times h(1,t\kappa^{-\gamma_2^*}(M\kappa^{1/2\gamma_1^*-d_\alpha/8\gamma_\sigma^*})^{(-4+2\gamma_2^*)/[D-2+\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*]},u^*,1). \tag{57}$$

Putting $t=0\Leftrightarrow T=T_L$ in Eq. (57), and recalling that on this line $H\sim M^{\delta_{\ell}}$, we identify

$$\delta_{\ell}=\frac{D+2-\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*}{D-2+\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*}. \tag{58}$$

The exponent β is calculated by making $H=0$ and $t<0$ in Eq. (57). The resulting equation can only be satisfied if

$$x_0\equiv t\kappa^{-\gamma_2^*}(M\kappa^{(1/2)\gamma_1^*-(d_\alpha/8)\gamma_\sigma^*})^{(-4+2\gamma_2^*)/[D-2+\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*]}, \tag{59}$$

is such that $h(1,x_0,u^*,1)=0$. Near the LP we expect $M\sim(-t)^{\beta_{\ell}}$. Thus, from Eq. (59) we extract

$$\beta_{\ell}=\frac{1}{2}\left(\frac{D-2+\gamma_1^*-(d_\alpha/4)\gamma_\sigma^*}{2-\gamma_2^*}\right). \tag{60}$$

Finally, the exponent α_{ℓ} is associated with the specific heat at constant field. It can be shown[19] that

$$\Gamma_R^{(0,2)}(0,0,\sigma,t,M=0,u^*,\kappa)\sim t^{-\alpha}. \tag{61}$$

The general solution of Eq. (37) with $N=0$ and $L=2$ at the fixed point $u=u^*$ is given by

$$\Gamma_R^{(0,2)}(0,0,\sigma,t,u^*,\kappa)=\kappa^{-2\gamma_2^*}\Phi^{(0,2)}(\sigma\kappa^{-\gamma_\sigma^*},t\kappa^{-\gamma_2^*},u^*)+\frac{\sigma^{-d_\alpha/4}\kappa^{D-4}B(u^*)}{D-4+2\gamma_2^*-(d_\alpha/4)\gamma_\sigma^*}. \tag{62}$$

Taking into account the dependence of $\Gamma^{(0,2)}$ on σ, given in Eq. (26), we obtain

$$\Gamma_R^{(0,2)}(0,0,\sigma,t,u^*,\kappa)=\sigma^{-d_\alpha/4}\kappa^{-2\gamma_2^*+(d_\alpha/4)\gamma_\sigma^*}\Phi^{(0,2)}(1,t\kappa^{-\gamma_2^*},u^*)+\frac{\sigma^{-d_\alpha/4}\kappa^{D-4}B(u^*)}{D-4+2\gamma_2^*-(d_\alpha/4)\gamma_\sigma^*}. \tag{63}$$

Dimensional analysis allows us to rewrite this equation as

$$\Gamma_R^{(0,2)}(0,0,\sigma,t,u^*,\kappa)=\sigma^{-d_\alpha/4}\rho^{D-4}\left(\frac{\kappa}{\rho}\right)^{-2\gamma_2^*+(d_\alpha/4)\gamma_\sigma^*}\times\Phi^{(0,2)}\left(1,\frac{t}{\rho^2}\left(\frac{\kappa}{\rho}\right)^{-\gamma_2^*},u^*\right)+\frac{\sigma^{-d_\alpha/4}\kappa^{D-4}B(u^*)}{D-4+2\gamma_2^*-(d_\alpha/4)\gamma_\sigma^*}. \tag{64}$$

Choosing ρ such that

$$\frac{t}{\rho^2}\left(\frac{\kappa}{\rho}\right)^{-\gamma_2^*}=1, \tag{65}$$

we obtain

$$\Gamma_R^{(0,2)}(0,0,\sigma,u^*,t,\kappa)=\sigma^{-d_\alpha/4}\kappa^{-2\gamma_2^*+(d_\alpha/4)\gamma_\sigma^*}(t\kappa^{-\gamma_2^*})^{[D-4+2\gamma_2^*-(d_\alpha/4)\gamma_\sigma^*]/(2-\gamma_2^*)}\times\Phi^{(0,2)}(1,1,u^*)+\frac{\sigma^{-d_\alpha/4}\kappa^{D-4}B(u^*)}{D-4+2\gamma_2^*-(d_\alpha/4)\gamma_\sigma^*}. \tag{66}$$

Comparing Eqs. (61) and (66), we find the following expression for α_ℓ:

$$\alpha_\ell=\frac{4-D-2\gamma_2^*+(d_\alpha/4)\gamma_\sigma^*}{2-\gamma_2^*}. \tag{67}$$

It is important to emphasize that the expressions for the critical exponents Eqs. (46), (47), (51), (52), (58), (60), and (67) hold to orders of perturbations. Using these equations, it is a simple task to check that the critical exponents associated with the LP satisfy the generalized scaling relations given below.

Fisher's law:

$$\gamma_\ell=\nu_{\ell 4}(4-\eta_{\ell 4})=\nu_{\ell 2}(2-\eta_{\ell 2}); \tag{68}$$

Widom's law:

$$\gamma_\ell=\beta_\ell(\delta_\ell-1); \tag{69}$$

Rushbrooke's law:

$$\alpha_\ell+2\beta_\ell+\gamma_\ell=2; \tag{70}$$

and Josephson's law (hyperscaling)

$$2-\alpha_\ell=d_\beta\nu_{\ell 2}+d_a\nu_{\ell 4}. \tag{71}$$

These scaling relations were first derived by Hornreich,[1,8] based on a one-loop analysis.

IV. CONCLUSIONS

We have studied the renormalization of the field theory that describes the LP. This has been done by first studying the Green functions at the LP. In this case the propagator simplifies considerably and we are capable of making a thorough analysis of the renormalization structure of the theory. Three points are worth emphasizing: (1) our finding renormalization prescriptions for which the renormalization constants Z_ϕ, Z_{ϕ^2}, and Z_σ depend only on the renormalized constant u and not on the parameter σ; (2) our determining the precise dependence of the renormalized Green functions on σ; (3) the expansion of the Green functions in the neighborhood of the LP in terms of the Green functions calculated at the LP. All three points have allowed us to obtain rather simple renormalization-group equations whose solutions have enabled us to identify the critical exponents. Our expressions are valid for all orders of perturbation and for all values of m. Using this formalism we have rederived the scaling relations first put forward by Hornreich, Luban, and Shtrikman based on a one-loop analysis. Our ideas can probably be adapted for other multicritical points.

Finally, our main motivation was the solution of an old controversy on the values of the critical exponents for the LP with $m=2$ and $m=6$. We have solved this problem and we anticipate the solution. Our technique gives the same exponents $\eta_{\ell 2}$ and $\eta_{\ell 4}$, as obtained by Sak and Grest.[15] In the field-theoretic approach that we have used in order to determine the critical exponents, we first have to calculate dimensional regularization poles of primitively divergent Green functions. We have accomplished this without any approximations. Our calculations are as accurate as the analogous one for the ϕ^4 theory. Since the algebra is rather long, we shall present the details, as well as the values for $\nu_{\ell 2}$ and $\nu_{\ell 4}$, to order ϵ^2 in a forthcoming paper.[23]

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APPENDIX

In this Appendix we illustrate in a simple case the cancellation of singularities that come from the insertion of $\Gamma^{(2,0)}$ into other diagrams. This cancellation is a consequence of the interplay of the renormalization constants Z_ϕ, Z_σ and of the renormalization of the coupling constant λ. The renormalization of λ plays a double role: it cancels the primitive logarithmic divergence of $\Gamma^{(4,0)}$ and, together with Z_ϕ, it eliminates part of the divergences due to the insertions of $\Gamma^{(2,0)}$. It is convenient, following Amit,[19] to analyze both effects separately by extracting a factor Z_ϕ^2 from the renormalized coupling constant g and define

$$g=Z_\phi^2\tilde{g}, \tag{A1}$$

where $\tilde{g}$ is determined in such a way as to eliminate the primitive logarithmic divergence of of $\Gamma^{(4,0)}$, and Z_ϕ^2 takes care of the logarithmic divergence of $\Gamma^{(2,0)}$.

The poles of $\Gamma^{(2,0)}$ to two-loop order come from the diagram D_2 shown in Fig. 1. Recall that

$$D_2=-\lambda^2\left[\frac{A\sigma_0}{6\epsilon}k_\alpha^4+\frac{B}{6\epsilon}k_\beta^2\right]+\text{regular terms}, \tag{A2}$$

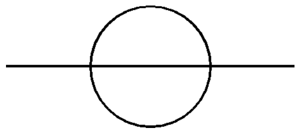

FIG. 1. Diagram D_2.

where $A\neq B$, and we have written down explicitly the combinatoric factor $\frac{1}{6}$ and the minus sign that multiplies all 1PI diagrams.

Following the prescriptions to obtain $\Gamma_R^{(2,0)}$ from $\Gamma^{(2,0)}$, given in Eq. (13), and recalling that $\sigma_0=Z_\sigma^{-1}\sigma$, we obtain

$$\Gamma_R^{(2,0)}=Z_\phi Z_\sigma^{-1}\sigma k_\alpha^4+Z_\phi k_\beta^2-\tilde{g}^2\left[\frac{A\sigma}{6\epsilon}k_\alpha^4+\frac{B}{6\epsilon}k_\beta^2\right]+\text{regular terms.} \tag{A3}$$

Note that, since D_2 is order λ^2, we can make the replacements $\sigma_0\to\sigma$ and $\lambda\to g$ in its contributions to $\Gamma_R^{(2,0)}$. The error is $O(g^4)$. Z_ϕ and Z_σ are chosen so that Eqs. (14) and (15) are satisfied. A simple calculation yields

$$Z_\phi=1+\frac{B}{6\epsilon}g^2,$$
$$Z_\sigma=1+\frac{(B-A)}{6\epsilon}g^2. \tag{A4}$$

Consider the diagrams shown in Fig. 2 that contribute to the connected four-point Green function $G_c^{(4,0)}$. Let us consider only the poles of the diagrams and neglect the regular parts. After expanding λ in terms of $\tilde{g}$, the primitive logarithmic divergences of the diagrams (C) and (D) are eliminated and the only divergence that remains comes from diagram (B) in Fig. 2. Diagram (B) results from the insertion of D_2 in the upper left leg of diagram (A). We have to insert D_2 in all legs of diagram (A). Thus, the singular part of $G_c^{(4,0)}$ is given by

$$\begin{aligned}G_c^{(4,0)}(k_1,k_2,k_3,k_4)=&-\tilde{g}G_0(k_1)G_0(k_2)G_0(k_3)G_0(k_4)\\&-\tilde{g}^3G_0^2(k_1)G_0(k_2)G_0(k_3)G_0(k_4)\\&\times\left[\frac{A\sigma_0}{6\epsilon}k_{1\alpha}^4+\frac{B}{6\epsilon}k_{1\beta}^2\right]-\dots\\&-\tilde{g}^3G_0(k_1)G_0(k_2)G_0(k_3)G_0^2(k_4)\\&\times\left[\frac{A\sigma_0}{6\epsilon}k_{4\alpha}^4+\frac{B}{6\epsilon}k_{4\beta}^2\right],\end{aligned} \tag{A5}$$

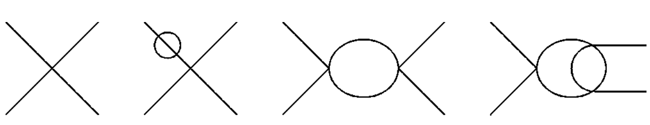

FIG. 2. Diagrams that contribute to $G_c^{(4,0)}$.

and

$$G_0(k)=\frac{1}{\sigma_0 k_\alpha^4+k_\beta^2} \tag{A6}$$

is the free propagator. Note the absence of the minus sign in the $\Gamma^{(2,0)}$ insertion that is now a part of $G_c^{(4,0)}$ and as such should not be multiplied by -1. We have to show that $G_{cR}^{(4,0)}=Z_\phi^{-2}G_c^{(4,0)}$ is finite, after we make the replacements $\sigma_0\to Z_\sigma^{-1}\sigma$ and $\tilde{g}\to Z_\phi^{-2}g$ in $G_c^{(4,0)}$. The renormalization constants Z_ϕ and Z_σ are given in Eq. (A4). After expanding σ_0, the free propagator $G_0(k)$, to order $\tilde{g}^3$, becomes

$$G_0(k)=G(k)-\tilde{g}^2\frac{\sigma k_\alpha^4}{6\epsilon}(A-B)G^2(k), \tag{A7}$$

where

$$G(k)=\frac{1}{\sigma k_\alpha^4+k_\beta^2}. \tag{A8}$$

In the terms proportional to $\tilde{g}^3$ in Eq. (A5) we can make the substitutions $\sigma_0\to\sigma$, $\tilde{g}\to g$, $Z_\phi\to 1$, $Z_\sigma\to 1$, $G_0(k)\to G(k)$. The error is order $\tilde{g}^5$. After making all these replacements the terms proportional to A in Eq. (A5) cancel out. On the other hand, the terms proportional to B combine in such a way as to produce terms like $\sigma k_{i\alpha}^4+k_{i\beta}^2=G^{-1}(k_i)$, $i=1,2,3$, and 4, which eliminate one of the squared propagators in the order g^3 terms. In this way we obtain the finite result

$$\begin{aligned}G_c^{(4,0)}&(k_1,k_2,k_3,k_4)\\&=-\left(gZ_\phi^{-4}+4g^3\frac{B}{6\epsilon}\right)G(k_1)G(k_2)G(k_3)G(k_4)\\&=-gG(k_1)G(k_2)G(k_3)G(k_4),\end{aligned} \tag{A9}$$

where we have used the definition of Z_ϕ given in Eq. (A4) to cancel out the singularities proportional to B.

To summarize: the expansion of σ_0 in the propagators of the lower order diagram, without $\Gamma^{(2,0)}$ insertions, cancels the poles proportional to A. The poles proportional to B combine to eliminate the extra propagator on the line where $\Gamma^{(2,0)}$ was inserted. In this way all terms become proportional to the lower order diagram. Finally, the Z_ϕ constant, which comes from the definition of the renormalized Green functions ($G_{cR}^{(N,0)}=Z_\phi^{-N/2}G_c^{(N,0)}$, $\Gamma_R^{(N,0)}=Z_\phi^{N/2}\Gamma^{(N,0)}$) and from the renormalized coupling constant ($g=Z_\phi^{-2}\tilde{g}$), eliminates the singularities proportional to B. This mechanism generalizes to all orders of perturbation.

Examining the demonstration above, one realizes that the essential ingredient is the presence of a factor Z_ϕ^{-1} for each line of the diagram. Consider a diagram of order $\tilde{g}^v$ that contributes to $G_c^{(N,0)}$ containing I internal lines and N external lines. In this case we need a factor Z_ϕ^{-I-N} to eliminate the divergences. The coupling constant provides a factor

Z_{ϕ}^{-2v}, another factor $Z_{\phi}^{N/2}$ comes from the definition of $G_{cR}^{(N,0)}$. There is a global factor $Z_{\phi}^{N/2-2v}=Z_{\phi}^{-I-N}$, and in the last equality we used the fact that since each internal line is shared by two vertices, then $4v=2I+N$. Thus, Z_{ϕ} is raised to a power equal to the total number of lines of the diagram. The demonstration for $\Gamma_{R}^{(N,0)}$ is analogous. However, in this case the external lines are removed and one needs a global factor Z_{ϕ}^{-I}.

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